ENDOMORPHISMS OF VARIETIES AND BOTT VANISHING

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Abstract. We show that a projective variety with an int-amplified endomorphism of degree invertible in the base field satisfies Bott vanishing. This is a new way to analyze which varieties have nontrivial endomorphisms. In particular, we extend some classification results on varieties admitting endomorphisms (for Fano threefolds of Picard number one and several other cases) to any characteristic. The classification results in characteristic zero are due to Amerik-Rovinsky-Van de Ven, Hwang-Mok, Paranjape-Srinivas, Beauville, and Shao-Zhong. Our method also bounds the degree of morphisms into a given variety. Finally, we relate Bott vanishing to global F-regularity.

1. Introduction

There is a long-standing conjecture about smooth Fano varieties admitting non-invertible surjective endomorphisms.

Conjecture 1.1. *Let X be a smooth Fano variety of Picard number 1 over an algebraically closed field of characteristic zero. Suppose that X admits a non-invertible surjective endomorphism. Then X is isomorphic to projective space.*

Conjecture 1.1 has been proved when

(1) $\dim X = 3$ [ARVdV99], [HM03],
(2) $\dim X = 4$ and X has Fano index greater than 1 [SZ22],
(3) X is a hypersurface [PS89], [Bea01], or
(4) X is a homogeneous space [PS89].

The aim of this paper is to give a new approach to this problem and to generalize cases (1), (2), and (3) above to arbitrary characteristic.

Theorem A. *Let X be a smooth projective variety over an algebraically closed field k. Assume that X admits an endomorphism whose degree is greater than 1 and invertible in k. Suppose that one of the following holds.*

(1) *X is a smooth Fano threefold of Picard number 1.*
(2) *X is a smooth Fano fourfold of Picard number 1 and Fano index greater than 1.*
(3) *X is a hypersurface of dimension at least 3.*

Then X is isomorphic to projective space.

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Our method also gives information on morphisms other than endomorphisms. The following result was known in characteristic zero in cases (1) and (2) [ARVdV99, Theorem 0.2], [HM03, Theorem 2], [SZ22, Theorem 1.5], and for quadrics in characteristic zero in case (3) [Ame98, Remark 4.1.2]. Our proof is short and valid in arbitrary characteristic.

Theorem B. *Let X be one of the varieties in Theorem A. Let Y be a smooth projective variety over k of the same dimension that also has Picard number 1. If X is not isomorphic to projective space, then there is an upper bound on the degrees of all morphisms $Y \to X$ that have degree invertible in k.*

The following assertion is a key ingredient for Theorem A. An endomorphism $f\colon X \to X$ is said to be *int-amplified* if there is an ample Cartier divisor H on X such that $f^*H - H$ is ample [Men20, MZ20].

Theorem C. *Let X be a normal projective variety over a perfect field k. Suppose that X admits an int-amplified endomorphism whose degree is invertible in k. Then X satisfies Bott vanishing for ample Weil divisors. That is,*

$$H^i(X, \Omega_X^{[j]}(A)) = 0$$

for every $i > 0$, $j \geq 0$, and A an ample Weil divisor.

Remark 1.2. The assumption "int-amplified" is weaker than some related conditions on endomorphisms, such as *polarized*, meaning that there is an ample Cartier divisor H with $f^*H \sim qH$ for some integer $q \geq 2$. For example, the endomorphism $f(x,y) = (x^2, y^3)$ of $\mathbb{P}^1 \times \mathbb{P}^1$ is int-amplified, but no positive iterate of f is polarized. On the other hand, for a variety with Picard group $\mathbb{Z}$, every endomorphism of degree greater than 1 is polarized and hence int-amplified.

Remark 1.3. A smooth Fano variety that satisfies Bott vanishing is rigid, since $H^1(X, TX) = H^1(X, \Omega_X^{d-1}(-K_X)) = 0$, where d is the dimension of X. So Theorem C implies that only finitely many smooth complex Fano varieties in each dimension admit an int-amplified endomorphism.

Remark 1.4. In proving Theorem C for singular varieties, we develop some interesting tools. In particular, we prove the finiteness of flat cohomology $H^1(X, \mu_p)$ for smooth varieties X over an algebraically closed field of characteristic p (Lemma 2.1).

Finally, we show that Bott vanishing implies good behavior in characteristic p:

Theorem D. *A smooth Fano variety over a perfect field of characteristic $p > 0$ that satisfies Bott vanishing is globally F-regular.*

It is known that the mod p reductions of a smooth Fano variety in characteristic zero are globally F-regular for sufficiently large primes p. The point of Theorem D is that it holds even if p is small. In this respect, Fano varieties that satisfy Bott vanishing behave well, somewhat like toric varieties (and this applies to all Fano varieties with an int-amplified endomorphism, by Theorem C). For example, Petrov showed that the Hodge spectral sequence degenerates for all smooth projective varieties that are globally F-split (which follows from globally F-regular) [Bha22, Corollary 2.7.6].

For hypersurfaces of dimension at least 3 and degree at least 2, it is straightforward to see that Bott vanishing fails. This implies Theorem A(3) by Theorem C.

The proof of Theorem A is similar for cases (1) and (2). To describe case (1): we show that projective space is the only smooth Fano threefold of Picard number 1 that satisfies Bott vanishing (see also Remark 1.5). In characteristic zero, this is an easy consequence of the classification [IP99]. So assume that the characteristic p is greater than 0. Since we do not have such a complete classification in this case, we lift X to characteristic zero. By Theorem C, we can take a lift $\widetilde{X}_{\overline{K}}$ of X, which is a smooth Fano threefold of Picard number 1 over an algebraically closed field $\overline{K}$ of characteristic zero. However, lifting endomorphisms is difficult in general, and therefore we prove that $\widetilde{X}_{\overline{K}}$ satisfies Bott vanishing instead. Then $\widetilde{X}_{\overline{K}} \cong \mathbb{P}^3_{\overline{K}}$ by the argument in characteristic zero. Finally, observing that the Fano indices are preserved by lifting, we conclude that $X \cong \mathbb{P}^3_k$.

Remark 1.5. The paper [Tot23] determines which smooth Fano threefolds in characteristic zero satisfy Bott vanishing [Tot23, Theorem 0.1].

For smooth Fano threefolds in positive characteristic that are constructed in the same way as smooth Fano threefolds in characteristic zero, that paper also determines which ones satisfy Bott vanishing. On the other hand, it is still open whether the classification of smooth Fano threefolds has the same form in every characteristic (see [AWZ23, section 7 and Appendix A] for some discussion). Therefore, in positive characteristic, it is not immediate from [Tot23] that projective space is the only smooth Fano threefold of Picard number 1 that satisfies Bott vanishing, as we show here.

1.1. **Related results.**

1.1.1. *Two-dimensional case in positive characteristic.* Nakayama showed that a smooth projective rational surface in characteristic p that admits an endomorphism whose degree is greater than 1 and prime to p must be toric [Nak10, Proposition 4.4]. On the other hand, he found a smooth rational surface that admits a separable polarized endomorphism but is not toric [Nak10, Example 4.5]. We do not know whether Theorems A, B, and C hold if the assumption that $\deg(f)$ is prime to p is weakened to the assumption that f is separable.

1.1.2. *Three-dimensional case in arbitrary characteristic.* Normal projective $\mathbb{Q}$-Gorenstein threefolds that admit polarized endomorphisms in arbitrary characteristic were studied in detail by Cascini-Meng-Zhang [CMZ20, Theorem 1.8]. In particular, they proved that a smooth rationally chain connected threefold in characteristic p admitting a polarized endomorphism f has an f-equivariant minimal model program if $p > 5$ and the degree of the Galois closure f^{Gal} of f is prime to p.

1.1.3. *Fano threefolds of arbitrary Picard number in characteristic zero.* Meng-Zhang-Zhong [MZZ22] proved that a smooth Fano threefold over an algebraically closed field of characteristic zero that admits an int-amplified endomorphism is toric. Therefore, it is natural to ask the following question.

Question 1.6. Let X be a smooth Fano threefold over an algebraically closed field of characteristic $p > 0$. Suppose that X admits an int-amplified endomorphism whose degree is prime to p. Is X toric?

1.2. **Notation and terminology.** Unless otherwise mentioned, k is an algebraically closed field of characteristic $p \geq 0$ and a variety is defined over k.

A morphism of varieties $f\colon Y \to X$ over k is *separable* if it is dominant and $k(Y)$ is a separable field extension of $k(X)$. Equivalently (for k algebraically closed), the derivative of f is surjective at some smooth point of Y.

For a normal variety X over a field k and $i \geq 0$, the sheaf of *reflexive differentials* $\Omega_X^{[i]}$ is the double dual $(\Omega_X^i)^{**}$. More generally, for a Weil divisor D on X, we write $\Omega_X^{[i]}(D)$ for the reflexive sheaf $(\Omega_X^i \otimes \mathcal{O}_X(D))^{**}$. If X is smooth over k, then $\mathcal{O}_X(D)$ is a line bundle and $\Omega_X^{[i]}(D)$ is just the tensor product $\Omega_X^i \otimes \mathcal{O}_X(D)$.

2. Bott vanishing and endomorphisms

In this section, we prove Theorem C, relating Bott vanishing with endomorphisms. We also prove a general relation between Bott vanishing and morphisms into a given variety, not just endomorphisms (Proposition 2.7).

2.1. **Finiteness of flat cohomology.** The following lemma may be known, but we could not find a reference in this generality.

Lemma 2.1. *Let X be a smooth variety over an algebraically closed field k, and let s be a positive integer. Then the flat cohomology group $H^1(X, \mu_s)$ is finite.*

Proof. The problem reduces to the case where s is prime. If s is invertible in k, then this finiteness holds for cohomology in all degrees [Mil80, Corollary VI.5.5]. So we can assume that k has characteristic $p > 0$ and $s = p$. (The result here is special to H^1. Indeed, a supersingular K3 surface X over k has $H^2(X, \mu_p)$ infinite, containing the additive group k [Art74, Proposition 4.2].) It is straightforward to see that $H^1(X, \mu_p)$ injects into $H^1(k(X), \mu_p) = k(X)^*/(k(X)^*)^p$ [Kel22, Lemma 3.9].

Let $\overline{X}$ be a normal compactification of X. By de Jong, there is a separable alteration $f\colon \overline{Y} \to \overline{X}$ [dJ96, Theorem 4.1]. That is, f is generically étale and $\overline{Y}$ is a smooth projective variety over k. Let Y be the inverse image of X in $\overline{Y}$. Here $k(Y)$ is a finite separable extension of $k(X)$, and so $k(X)^*/(k(X)^*)^p$ injects into $k(Y)^*/(k(Y)^*)^p$. By the previous paragraph, it follows that $H^1(X, \mu_p)$ injects into $H^1(Y, \mu_p)$.

So it suffices to show that $H^1(Y, \mu_p)$ is finite. Consider the Kummer sequence $O(Y)^*/(O(Y)^*)^p \to H^1(Y, \mu_p) \to \operatorname{Pic}(Y)[p]$. The group of units $O(Y)^*$ is an extension of a finitely generated abelian group by k^*, and so $O(Y)^*/(O(Y)^*)^p$ is finite. So it suffices to show that $\operatorname{Pic}(Y)[p]$ is finite.

Since $\overline{Y}$ is smooth, we have $\operatorname{Pic}(Y) = \operatorname{Pic}(\overline{Y})/M$, where M is the subgroup generated by the codimension-1 subvarieties of $\overline{Y} - Y$. In particular, the abelian group M

is finitely generated. By tensoring the exact sequence

$$0 \to M \to \mathrm{Pic}(\overline{Y}) \to \mathrm{Pic}(Y) \to 0$$

over $\mathbb{Z}$ with $\mathbb{Z}/p$, we have an exact sequence $\mathrm{Pic}(\overline{Y})[p] \to \mathrm{Pic}(Y)[p] \to M/p$. So $\mathrm{Pic}(Y)[p]$ is finite if $\mathrm{Pic}(\overline{Y})[p]$ is finite. That is immediate from the structure of the Picard group of a smooth projective variety: $\mathrm{Pic}(\overline{Y})$ is an extension of a finitely generated abelian group $NS(\overline{Y})$ by the k-points of an abelian variety [BGI71, Théorème XIII.5.1]. Lemma 2.1 is proved. □

2.2. **Bott vanishing.** In this subsection, we prove Theorem C. We are generalizing Fujino's proof of Bott vanishing for toric varieties, based on the existence of suitable endomorphisms, beyond the toric setting.

Definition 2.2. Let X be a smooth projective variety over a field. We say that X satisfies *Bott vanishing* if we have

$$H^i(X, \Omega^j_X(A)) = 0$$

for every $i > 0$, $j \geq 0$, and A an ample Cartier divisor.

For singular varieties, we consider the following strong version of Bott vanishing. A Weil divisor (with integer coefficients) is called *ample* if some positive multiple is an ample Cartier divisor. Likewise for *nef*.

Definition 2.3. Let X be a normal projective variety over a field. We say that X satisfies *Bott vanishing for ample Weil divisors* if we have

$$H^i(X, \Omega^{[j]}_X(A) = 0$$

for every $i > 0$, $j \geq 0$, and A an ample Weil divisor.

Remark 2.4. All projective toric varieties satisfy Bott vanishing for ample Weil divisors, by Fujino [Fuj07, Proposition 3.2].

Proof of Theorem C. Since k is perfect, the normal variety X over k is geometrically normal [SPA23, Tag 038O]. Replacing k with its algebraic closure, we may assume that k is algebraically closed. For clarity, we first prove the theorem for X smooth. (That is enough for the applications in this paper.) So let X be a smooth projective variety over an algebraically closed field k with an endomorphism f and an ample Cartier divisor H on X such that $f^*H - H$ is ample. In particular, f^*H is ample, and so f does not contract any curves. Therefore, $f\colon X \to X$ is finite. We assume that the degree of f is invertible in k.

Let A be any ample Cartier divisor on X. Fix $i > 0$ and $j \geq 0$. We want to show that

$$H^i(X, \Omega^j_X(A)) = 0.$$

We will use Fujita's vanishing theorem [Fuj83, Theorem 1]:

Theorem 2.5. *Let X be a projective scheme over a field, H an ample Cartier divisor on X, and E a coherent sheaf on X. Then there is a positive integer m such that $H^i(X, E \otimes \mathcal{O}_X(mH + D)) = 0$ for every $i > 0$ and every nef Cartier divisor D on X.*

Since f is finite and X is smooth over k, there is a *trace map* $\tau_f\colon f_*\Omega^j_X \to \Omega^j_X$ such that the composition

$$\Omega^j_X \xrightarrow{f^*} f_*\Omega^j_X \xrightarrow{\tau_f} \Omega^j_X$$

is multiplication by $\deg(f)$, by Garel and Kunz [Gar84], [Kun86, section 16], [SPA23, Tag 0FLC]. Thus $\frac{1}{\deg(f)}\tau_f$ gives a splitting of the pullback $f^*\colon \Omega^j_X \hookrightarrow f_*\Omega^j_X$. Taking the pushforward by f, we obtain a split injective map $f_*\Omega^j_X \hookrightarrow (f^2)_*\Omega^j_X$, and thus a split injective map $\Omega^j_X \hookrightarrow (f^2)_*\Omega^j_X$. Repeating this procedure, for every positive integer e, $(f^e)^*\colon \Omega^j_X \hookrightarrow (f^e)_*\Omega^j_X$ splits. Tensoring with $\mathcal{O}_X(A)$, we have a split injective map

$$\Omega^j_X(A) \to (f^e)_*(\Omega^j_X((f^e)^*A)).$$

Taking cohomology (and using that f^e is finite), we have a split injective map

$$H^i(X, \Omega^j_X(A)) \hookrightarrow H^i(X, \Omega^j_X((f^e)^*A)).$$

So it suffices to find an $e \geq 1$ such that the right hand side is zero. Let m be a positive integer associated to the given ample Cartier divisor H (with $f^*H - H$ ample) and the coherent sheaf $E = \Omega^i_X$, in Fujita vanishing (Theorem 2.5). Then it suffices to find an $e \geq 1$ such that $(f^e)^*A - mH$ is nef.

Since $f^*H - H$ is ample, there is a rational number $c > 1$ such that $f^*H - cH$ is ample. Here f^* takes nef divisors to nef divisors. So, for every $e \geq 1$, $(f^e)^*H - c^eH$ is nef.

Since A is ample, there is a rational number $u > 0$ such that $A - uH$ is ample. Using again that f^* takes nef divisors to nef divisors, we find that for every $e \geq 1$, $(f^e)^*A - u(f^e)^*H$ is nef. It follows that $(f^e)^*A - uc^eH$ is nef. There is a positive integer e such that $uc^e \geq m$. Then $(f^e)^*A - mH$ is nef, as we want. Bott vanishing is proved.

More generally, assume that X is normal (rather than smooth). Since f is a finite morphism between normal varieties, reflexive differentials $\Omega^{[j]}_X = (\Omega^j_X)^{**}$ pull back under f. (We can pull differential forms back outside $X^{\mathrm{sing}} \cup f(X^{\mathrm{sing}})$, and that gives a pullback map on reflexive differentials since the complement has codimension at least 2.) That is, we have a pullback map $\Omega^{[j]}_X \to f_*\Omega^{[j]}_X$. Likewise, the trace map $\tau_f\colon f_*\Omega^{[j]}_X \to \Omega^{[j]}_X$ is defined outside $X^{\mathrm{sing}} \cup f(X^{\mathrm{sing}})$, so it extends to all of X since the sheaf $\Omega^{[j]}_X$ is reflexive. Given this, the proof above works without change for an ample Cartier divisor A.

Finally, to prove the full theorem, assume that X is normal and A is an ample Weil divisor. For this, we need a version of Fujita vanishing for $\mathbb{Q}$-Cartier Weil divisors:

Lemma 2.6. *Let X be a normal projective variety over an algebraically closed field k, H an ample Cartier divisor on X, E a reflexive sheaf on X, and s a positive integer. Then there is a positive integer m such that $H^i(X, E(mH + D)) = 0$ for every $i > 0$ and every nef Weil divisor D such that sD is Cartier.*

Proof. Since X is a normal projective variety over k, the Picard group $\mathrm{Pic}(X)$ is an extension of a finitely generated abelian group by the k-points of an abelian variety [BGI71, Théorème XIII.5.1], [Kle05, Theorem 5.4]. Therefore, $\mathrm{Pic}(X)/s$ is finite. Let

U be the smooth locus of X, so that the divisor class group $\mathrm{Cl}(X)$ is isomorphic to $\mathrm{Pic}(U)$. By Lemma 2.1, $H^1(U,\mu_s)$ is finite. By the Kummer sequence $H^1(U,\mu_s)\to H^1(U,G_m)\xrightarrow{s}H^1(U,G_m)$, the s-torsion subgroup $\mathrm{Cl}(X)[s]=\mathrm{Pic}(U)[s]$ is also finite. By tensoring the exact sequence $0\to\mathrm{Pic}(X)\to\mathrm{Cl}(X)\to\mathrm{Cl}(X)/\mathrm{Pic}(X)\to 0$ over $\mathbb{Z}$ with $\mathbb{Z}/s$, we have an exact sequence $\mathrm{Cl}(X)[s]\to(\mathrm{Cl}(X)/\mathrm{Pic}(X))[s]\to\mathrm{Pic}(X)/s$. So $(\mathrm{Cl}(X)/\mathrm{Pic}(X))[s]$ is finite.

Let $D_1,\ldots,D_r$ be Weil divisors with sD_j Cartier that represent every element of the group $(\mathrm{Cl}(X)/\mathrm{Pic}(X))[s]$. By subtracting a suitable multiple of H from each D_j, we can assume that $-D_j$ is nef for each j. Then, for every nef Weil divisor D on X with sD Cartier, there is a $1\le j\le r$ such that $D-D_j$ is Cartier. Apply Fujita's theorem (Theorem 2.5) to the coherent sheaf $\oplus_{j=1}^r E(D_j)$. (By definition, $E(D_j)$ means the reflexive sheaf $(E\otimes\mathcal{O}_X(D_j))^{**}$.) This gives that there is a positive integer m such that $H^i(X,E(mH+D_j+N))=0$ for every $i>0$, every $1\le j\le r$, and every nef Cartier divisor N. Since $-D_j$ is nef for each j, it follows that for every nef Weil divisor D with sD Cartier, we have $H^i(X,E(mH+D))=0$ for every $i>0$. Lemma 2.6 is proved. □

We now complete the proof of Theorem C for X normal over an algebraically closed field k and an ample Weil divisor A. This follows by the argument above (where A is an ample Cartier divisor), using Lemma 2.6 in place of Fujita's theorem. There is a positive integer s such that sA is Cartier. Since f is a finite morphism between normal varieties, $(f^e)^*A$ is a Weil divisor (with integer coefficients) for every positive integer e, and $s(f^e)^*A$ is Cartier. This is what we need in order to apply Lemma 2.6. Theorem C is proved. □

2.3. **Bounding the degree of morphisms.**

Proposition 2.7. *Let X and Y be normal projective varieties of the same dimension over a field k, and suppose that both have Picard number 1. If there are morphisms from Y to X with arbitrarily large degree such that the degree is invertible in k, then X must satisfy Bott vanishing for ample Weil divisors.*

Proof. Let $f\colon Y\to X$ be a morphism whose degree is invertible in k. Let H be an ample Cartier divisor on X. Then f^*H has positive degree on some curve, hence is ample since Y has Picard number 1. It follows that $f\colon Y\to X$ is finite.

Let A be an ample Weil divisor on X, and fix $i>0$ and $j\ge 0$. We need to prove that

$$H^i(X,\Omega_X^{[j]}(A))=0.$$

Since $\deg(f)$ is invertible in k, we have a split injection

$$\Omega_X^{[j]}(A)\hookrightarrow f_*(\Omega_Y^{[j]}(f^*A)),$$

as in the proof of Theorem C. Taking cohomology (and using that f is finite), we have a split injection

$$H^i(X,\Omega_X^{[j]}(A))\hookrightarrow H^i(Y,\Omega_Y^{j}(f^*A)).$$

If there are morphisms $f\colon Y \to X$ with arbitrarily large degree such that the degree is invertible in k, then f^*A becomes arbitrarily large in the ample cone of Y (here just one ray). So Fujita vanishing for Weil divisors (Lemma 2.6) gives that, for f of sufficiently large degree, $H^i(Y, \Omega_Y^{[j]}(f^*A)) = 0$. By the previous paragraph, it follows that $H^i(X, \Omega_X^{[j]}(A)) = 0$. Bott vanishing is proved. □

3. Bott vanishing for specific classes of varieties

3.1. **Hypersurfaces.** In this subsection, we prove that projective space is the only smooth hypersurface of dimension at least 3 that satisfies Bott vanishing.

Lemma 3.1. *Let $X \subset \mathbb{P}^{d+1}$ be a smooth hypersurface of degree $d+n$ over a field. Suppose that $d > 1$ and $n > 0$. Then $H^{d-1}(X, \Omega_X^1(n)) \neq 0$, and in particular, X does not satisfy Bott vanishing.*

Proof. By adjunction, the dualizing sheaf ω_X is isomorphic to $\mathcal{O}_X(n-2)$. So Serre duality gives that $H^d(X, \mathcal{O}_X(-d)) \cong H^0(X, \mathcal{O}_X(d+n-2))^* \neq 0$. By the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{d+1}} \to \mathcal{O}_{\mathbb{P}^{d+1}}(1)^{\oplus d+2} \to T\mathbb{P}^{d+1} \to 0$$

we have $H^0(\mathbb{P}^{d+1}, T\mathbb{P}^{d+1}(-2)) = 0$. By the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^{d+1}}(-d) \to \Omega^1_{\mathbb{P}^{d+1}}(n) \to \Omega^1_{\mathbb{P}^{d+1}}(n)|_X \to 0,$$

we have $H^d(X, \Omega^1_{\mathbb{P}^{d+1}}(n)|_X) = 0$. Here we used that $H^{d+1}(\mathbb{P}^{d+1}, \Omega^1_{\mathbb{P}^{d+1}}(-d)) \cong H^0(\mathbb{P}^{d+1}, T\mathbb{P}^{d+1}(-2))^* = 0$ and Bott vanishing on $\mathbb{P}^{d+1}$. By the exact sequence

$$0 \to \mathcal{O}_X(-d) \to \Omega^1_{\mathbb{P}^{d+1}}(n)|_X \to \Omega^1_X(n) \to 0,$$

we have $H^{d-1}(X, \Omega^1_X(n)) \neq 0$, as desired. □

Proposition 3.2. *Let $X \subset \mathbb{P}^{d+1}$ be a smooth hypersurface of dimension at least 2 over an algebraically closed field. If X satisfies Bott vanishing, then $X \cong \mathbb{P}^d$ or X is the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. A Fano variety that satisfies Bott vanishing is rigid, since $H^1(X, TX) = H^1(X, \Omega_X^{d-1}(-K_X)) = 0$, where $d = \dim(X)$. That excludes all Fano hypersurfaces of degree at least 3, and Bott vanishing also fails for quadrics of dimension at least 3 [AWZ21, Example 3.2.6]. Finally, Lemma 3.1 shows that Bott vanishing fails for all non-Fano hypersurfaces of dimension at least 2. (This is mostly relevant for the Calabi-Yau case. Indeed, Bott vanishing fails for all varieties of positive dimension d with ample canonical class, since $H^d(X, \omega_X) = k \neq 0$.) □

3.2. **Fano threefolds.** In this subsection, we prove that projective space is the only Fano threefold of Picard number 1 that satisfies Bott vanishing, in any characteristic. Throughout this subsection, we use the following convention.

Convention 3.3. Let k be an algebraically closed field of positive characteristic. We denote $W(k)$ the ring of Witt vectors and K the field of fractions of $W(k)$. For a scheme X over k, we say that a scheme $\widetilde{X}$ over $W(k)$ is a *lift of X* if $\widetilde{X} \otimes_{W(k)} k \cong X$

and $\widetilde{X}$ is flat over $W(k)$. For a lift $\widetilde{X}$ and a Cartier divisor $\widetilde{A}$ on $\widetilde{X}$, we denote the geometric generic fiber of $\widetilde{X} \to \operatorname{Spec} W(k)$ by $\widetilde{X}_{\overline{K}}$ and the pullback of $\widetilde{A}$ to $\widetilde{X}_{\overline{K}}$ by $\widetilde{A}_{\overline{K}}$.

Definition 3.4. Let X be a smooth Fano variety over an algebraically closed field. The *Fano index* $r(X) \in \mathbb{Z}_{>0}$ of X is the largest integer $n \in \mathbb{Z}_{>0}$ such that $-K_X \sim nA$ for some Cartier divisor A.

Theorem 3.5. *Let X be a smooth Fano variety of dimension d over an algebraically closed field. If $r(X) \geq d+1$, then $X \cong \mathbb{P}^d$.*

Proof. The assertion follows from [KK00, Corollary 2 and 3]. □

Proposition 3.6. *Let X be a smooth Fano variety over an algebraically closed field k of positive characteristic. Suppose there exists a lift $\widetilde{X}$ over $W(k)$ of X. In addition, assume that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Then the following hold.*

(1) *The specialization map* $\mathrm{sp}\colon \operatorname{Pic}(\widetilde{X}_{\overline{K}}) \to \operatorname{Pic}(X)$ *is an isomorphism of abelian groups.*
(2) $r(X) = r(\widetilde{X}_{\overline{K}})$.
(3) *Suppose that the Picard number of X is 1. If X satisfies Bott vanishing, then so does $\widetilde{X}_{\overline{K}}$.*

Remark 3.7. For a smooth Fano threefold X, Shepherd-Barron [SB97, Corollary 1.5] proved that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ (see also [Kaw21, Corollary 3.7] for the case where $p = 2$ or 3). Also, for a smooth Fano variety X of any dimension that satisfies Bott vanishing (hence Kodaira vanishing), we have $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. See [GJ18, Theorem 1.1, Proposition 6.3] for what is known about the Fano index in families without assuming that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.

Proof. First, we prove (1). Since smoothness and ampleness are open properties in a family, the geometric generic fiber $\widetilde{X}_{\overline{K}}$ is a smooth Fano variety and $\operatorname{Pic}(\widetilde{X}_{\overline{K}}) \cong \operatorname{NS}(\widetilde{X}_{\overline{K}})$ is a free $\mathbb{Z}$-module, where $\operatorname{NS}(X)$ denotes the Néron-Severi group. Let $\mathrm{sp}\colon \operatorname{Pic}(\widetilde{X}_{\overline{K}}) \to \operatorname{Pic}(X)$ be the specialization map (see the proof of [MP12, Proposition 3.3] for the construction). Since $H^1(X, \mathcal{O}_X) = 0$, we have $\operatorname{Pic}(X) = \operatorname{NS}(X)$ by [FGI+05, Theorem 9.5.11]. Since $\operatorname{Pic}(\widetilde{X}_{\overline{K}})$ is torsion-free, it follows from [MP12, Proposition 3.6] that $\mathrm{sp}\colon \operatorname{Pic}(\widetilde{X}_{\overline{K}}) \to \operatorname{Pic}(X)$ is injective. Since $H^2(X, \mathcal{O}_X) = 0$, there exists a lift $\widetilde{D} \in \operatorname{Pic}(\widetilde{X})$ for every Cartier divisor D on X by [FGI+05, Corollary 8.5.6]. By the construction of the specialization, it follows that $\mathrm{sp}(\widetilde{D}_{\overline{K}}) = D$, and thus the specialization map sp is surjective. Thus, (1) holds.

Next, we prove (2). By the definition of $r(\widetilde{X}_{\overline{K}})$, we can take an ample Cartier divisor $\overline{A}$ on $\widetilde{X}_{\overline{K}}$ such that $-K_{\widetilde{X}_{\overline{K}}} \sim r(\widetilde{X}_{\overline{K}})\overline{A}$. Then we have

$$-K_X \sim \mathrm{sp}(-K_{\widetilde{X}_{\overline{K}}}) \sim \mathrm{sp}(r(\widetilde{X}_{\overline{K}})\overline{A}) = r(\widetilde{X}_{\overline{K}})\mathrm{sp}(\overline{A}),$$

which shows that $r(\widetilde{X}_{\overline{K}}) \leq r(X)$.

By the definition of $r(X)$, we can take an ample Cartier divisor A on X such that $-K_X \sim r(X)A$. Let $\widetilde{A} \in \mathrm{Pic}(\widetilde{X})$ be a lift of A. Then we obtain $-K_{\widetilde{X}_{\overline{K}}} \sim r(X)\widetilde{A}_{\overline{K}}$, which shows that $r(X) \leq r(\widetilde{X}_{\overline{K}})$. Thus, (2) holds.

Finally, we prove (3). Take an ample Cartier divisor $\overline{A}$ on $\widetilde{X}_{\overline{K}}$ and fix $i > 0$ and $j \geq 0$. We prove

$$H^i(\widetilde{X}_{\overline{K}}, \Omega^j_{\widetilde{X}_{\overline{K}}}(\overline{A})) = 0.$$

Let $A \coloneqq \mathrm{sp}(\overline{A})$. Then $\overline{A} \sim \widetilde{A}_{\overline{K}}$ for a lift $\widetilde{A}$ of A by the argument in (1). Since $\overline{A}$ is ample, we can take $m \gg 0$ such that $h^0(\widetilde{X}_{\overline{K}}, \mathcal{O}_{\widetilde{X}_{\overline{K}}}(m\widetilde{A}_{\overline{K}})) > 1$, and upper semi-continuity ([Har77, Theorem III.12.8]) shows that $h^0(X, \mathcal{O}_X(mA)) > 1$. Since the Picard number of X is 1, it follows that A is ample. By assumption, we have

$$H^i(X, \Omega^j_X(A)) = 0,$$

and upper semi-continuity shows the desired vanishing in characteristic zero. Thus, (3) holds. $\square$

We use Proposition 3.6 to reduce the following Proposition to the case of characteristic zero.

Proposition 3.8. *Let X be a smooth Fano threefold of Picard rank 1 over an algebraically closed field k such that X satisfies Bott vanishing. Then X is isomorphic to projective space.*

Proof. **Step 1** (characteristic zero case). Assume that k has characteristic zero. By Bott vanishing, we have $H^i(X, TX) = H^i(X, \Omega^2_X(-K_X)) = 0$ for all $i > 0$. By the Hirzebruch–Riemann–Roch theorem (cf. [AWZ23, Proof of Theorem 7.4]), we have

$$0 \leq \dim_k H^0(X, TX) = \chi(X, TX) = \frac{1}{2}(-K_X)^3 - 18 + \rho(X) - \dim_k H^1(X, \Omega^2_X),$$

where $\rho(X)$ is the Picard number of X. Then by [IP99, Tables 12.2], it follows that X is isomorphic to the quintic del Pezzo threefold V_5 (a smooth codimension-3 linear section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$), the quadric threefold Q, or $\mathbb{P}^3$. Since V_5 and Q do not satisfy Bott vanishing by [AWZ23, Lemma 7.10] and [BTLM97, subsection 4.1] (or Proposition 3.2), it follows that $X \cong \mathbb{P}^3$.

Step 2 (positive characteristic case). Assume that k has positive characteristic. Since X is Fano and satisfies Bott vanishing, we have

$$H^i(X, TX) = H^i(X, \mathcal{O}_X) = 0$$

for all $i > 0$. By [FGI+05, Theorem 9.5.11], we can take a lift $\widetilde{X}$ over $W(k)$ of X, and the geometric generic fiber $\widetilde{X}_{\overline{K}}$ is a smooth Fano threefold. By Proposition 3.6 (1) and (3), the Picard number of $\widetilde{X}_{\overline{K}}$ is 1 and $\widetilde{X}_{\overline{K}}$ satisfies Bott vanishing. Thus, by Step 1, we have $\widetilde{X}_{\overline{K}} \cong \mathbb{P}^3_{\overline{K}}$. By Proposition 3.6(2), we have $r(X) = r(\widetilde{X}_{\overline{K}}) = 4$, and we conclude that $X \cong \mathbb{P}^3_k$ by Theorem 3.5. $\square$

3.3. **Images of toric varieties.** Let $f\colon Y \to X$ be a morphism from a projective toric variety Y onto a smooth projective variety X of Picard number 1. In characteristic zero, generalizing Lazarsfeld's result on images of projective space [Laz84, Theorem 4.1], Occhetta-Wiśniewski [OW02, Theorem 1.1] proved that X is isomorphic to projective space.

This result does not extend to characteristic $p > 0$ in full generality. For example, in characteristic 2, there is a finite purely inseparable morphism from $\mathbb{P}^3$ onto a smooth quadric threefold [Eke87, Proposition 2.5]. However, Occhetta-Wiśniewski's proof does work without change for *separable* morphisms in positive characteristic. That is:

Theorem 3.9. *Let X be a smooth projective variety of Picard number 1 over an algebraically closed field. Let Y be a proper toric variety. If there is a separable morphism $Y \to X$, then X is isomorphic to projective space.*

For possible future use, let us show how a special case of Theorem 3.9 follows from our arguments with Bott vanishing. One can also prove a version of Proposition 3.10 using Achinger-Witaszek-Zdanowicz's results on images of F-liftable varieties [AWZ21, Theorem 4.4.1]. For example, they showed that a smooth complex surface that is an image of a proper toric variety must be toric [AWZ21, Theorems 2 and 3].

Proposition 3.10. *Let X be a smooth projective threefold of Picard number 1 over an algebraically closed field k. Let Y be a proper toric variety of the same dimension. If there is a morphism $Y \to X$ of degree invertible in k, then X is isomorphic to projective space.*

Proof. Let $Y \to Y' \to X$ be the Stein factorization. Replacing Y by Y', we may assume that f is finite and Y is a normal toric variety, by [Tan22, Theorem 1.1].

Let A be an ample Cartier divisor on X, and fix $i > 0$ and $j \geq 0$. By the same proof as in Proposition 2.7 (pushing forward differential forms), we have a split injection

$$H^i(X, \Omega^j_X(A)) \hookrightarrow H^i(Y, \Omega^{[j]}_Y(f^*A)),$$

where $\Omega^{[j]}_Y$ is the sheaf of reflexive differentials, $(\Omega^j_Y)^{**}$. Since f^*A is ample, Bott vanishing for toric varieties gives that the group on the right is zero, and so the group on the left is zero. That is, X satisfies Bott vanishing.

Since f is separable, we have $f^*(-K_X) = -K_Y + R$ for an effective divisor R, the *ramification divisor* [Kol13, equation 2.41.2]. Since $-K_Y$ is big and the Picard number of X is 1, it follows that $-K_X$ is ample. Therefore, $X \cong \mathbb{P}^3$ by Proposition 3.8. $\square$

3.4. **Fano fourfolds of index greater than 1.** In this subsection, we prove that projective space is the only Fano fourfold of Picard number 1 and Fano index greater than 1 that satisfies Bott vanishing.

Proposition 3.11. *Let X be a smooth Fano fourfold of Picard number 1 and Fano index greater than 1 over an algebraically closed field k. If X satisfies Bott vanishing, then X is isomorphic to projective space.*

Proof. If k has characteristic $p > 0$, then (as in the proof of Proposition 3.8) X lifts to characteristic 0, and the lift also satisfies Bott vanishing. So it suffices to prove the proposition for k of characteristic zero. (That will imply that the lift has Fano index 5, so X in characteristic p has Fano index 5 and satisfies Kodaira vanishing; so it is isomorphic to $\mathbb{P}^4_k$.)

So assume that k has characteristic zero. Then the smooth Fano fourfolds of Picard number 1 and index greater than 1 were classified by Fujita and Mukai [KP22], [Muk89]. Since we assume that X satisfies Bott vanishing, it is in particular rigid. By the classification, X is either $\mathbb{P}^4$, the quadric 4-fold Q, a codimension-2 linear section of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$, or a codimension-2 linear section of the Lagrangian Grassmannian $\mathrm{LGr}(3,6) \subset \mathbb{P}^{13}$.

In each case other than $\mathbb{P}^4$, a Riemann-Roch calculation suffices to disprove Bott vanishing. Let $Y \subset \mathbb{P}^N$ be a smooth projective variety over an algebraically closed field k of characteristic zero, and let X be a smooth hyperplane section. Assume that Y is Fano of index $b \geq 2$, meaning that $-K_Y = \mathcal{O}_Y(b)$.

Lemma 3.12. (1) *We have*

$$\chi(X, TX(-1)) = \chi(Y, TY(-1)) - \chi(Y, TY(-2)) - 1.$$

(2) *For* $2 \leq a \leq b-1$, *we have*

$$\chi(X, TX(-a)) = \chi(Y, TY(-a)) - \chi(Y, TY(-a-1)).$$

Proof. We have exact sequences of coherent sheaves: $0 \to TY(-1) \to TY \to TY|_X \to 0$ and $0 \to TX \to TY|_X \to \mathcal{O}_X(1) \to 0$. For any integer a, it follows that $\chi(X, TX(-a)) = \chi(Y, TY(-a)) - \chi(Y, TY(-a-1)) - \chi(X, \mathcal{O}_X(1-a))$. Since $-K_Y = \mathcal{O}_Y(b)$, the last term is easy to compute by Kodaira vanishing: $\chi(X, \mathcal{O}_X(1-a))$ is zero for $2 \leq a \leq b-1$, and it is 1 for $a = 1$. ☐

We return to the proof of Proposition 3.11. We know that Bott vanishing fails for the quadric fourfold, as mentioned in Proposition 3.2. (In fact, Lemma 3.12 implies for any $n \geq 3$ that the quadric n-fold has $\chi(Q, \Omega^{n-1}_Q(n-2)) = \chi(Q, TQ(-2)) = -1$.)

It remains to disprove Bott vanishing for the other two fourfolds above. By applying Lemma 3.12 twice, for Y Fano of index $b \geq 3$, a codimension-2 linear section X has

$$\chi(X, TX(-1)) = \chi(Y, TY(-1)) - 2\chi(Y, TY(-2)) + \chi(Y, TY(-3)) - 2.$$

Snow showed (in characteristic zero, as here) that if Y is a Grassmannian $\mathrm{Gr}(s,t)$ other than projective space or $\mathrm{Gr}(2,4)$, then $-K_Y = O_Y(t)$ and $TY(-a)$ has zero cohomology in all degrees for $1 \leq a \leq t-1$ [Sno86, Theorem 3.4(3)]. Likewise, if Y is the Lagrangian Grassmannian $\mathrm{LGr}(s,2s)$ with $s \geq 2$ and $s \neq 4$, then $-K_Y = O_Y(s+1)$ and $TY(-a)$ has zero cohomology in all degrees for $1 \leq a \leq s$ [Sno88, Theorem 2.4(3)]. By the formula above, it follows that both a codimension-2 linear section of $\mathrm{Gr}(2,5)$ and a codimension-2 linear section of $\mathrm{LGr}(3,6)$ have $\chi(X, TX(-1)) = -2$, disproving Bott vanishing. (Here $TX(-1) = \Omega^3_X(2)$ in the first case, and $TX(-1) = \Omega^3_X(1)$ in the second case.) Proposition 3.11 is proved. ☐

4. Proof of Theorem A and Theorem B

Proof of Theorems A and B. In the situation of Theorem A (resp. B), X satisfies Bott vanishing by Theorem C (resp. Proposition 2.7), and the assertion follows from Propositions 3.2, 3.8, and 3.11. □

5. Global F-regularity of smooth Fano varieties satisfying Bott vanishing

In this section, we prove that smooth Fano varieties in characteristic $p > 0$ that satisfy Bott vanishing are globally F-regular (Theorem D). In particular, this applies to every smooth Fano variety with an int-amplified endomorphism whose degree is invertible in k, by Theorem C. More generally, we prove the same implication for Fano varieties with toric singularities (Corollary 5.6). It would be interesting to prove this implication for more singular Fano varieties.

Definition 5.1. Let X be a normal variety over a perfect field k of characteristic $p > 0$. The variety X is *globally F-regular* if for every effective Weil divisor D on X, there exists an integer $e \geq 1$ such that the composite map

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$$

of the e-times iterated Frobenius map $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ and the natural inclusion $F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$ splits as an $\mathcal{O}_X$-module homomorphism. (Note that $F^e_*\mathcal{O}_X(D)$ means $F^e_*(\mathcal{O}_X(D))$.)

Let X be a smooth variety over a perfect field of characteristic $p > 0$. The Frobenius pushforward of the de Rham complex

$$F_*\Omega^\bullet_X : F_*\mathcal{O}_X \overset{F_*d}{\to} F_*\Omega_X \overset{F_*d}{\to} \cdots$$

is a complex of $\mathcal{O}_X$-module homomorphisms. Define locally free $\mathcal{O}_X$-modules as follows.

$$B^i_X := \operatorname{im}(F_*d : F_*\Omega^{i-1}_X \to F_*\Omega^i_X),$$
$$Z^i_X := \ker(F_*d : F_*\Omega^i_X \to F_*\Omega^{i+1}_X).$$

By definition, we have an exact sequence

$$0 \to Z^i_X \to F_*\Omega^i_X \overset{F_*d}{\to} B^{i+1}_X \to 0. \tag{5.1.1}$$

We also have the exact sequence arising from the Cartier isomorphism (see [BK05, Theorem 1.3.4], for example),

$$0 \to B^i_X \to Z^i_X \overset{C^i}{\to} \Omega^i_X \to 0. \tag{5.1.2}$$

Theorem 5.2. *Let X be a smooth Fano variety over a perfect field of characteristic $p > 0$. If X satisfies Bott vanishing, then X is globally F-regular.*

Proof. We first prove that global F-regularity is equivalent to global F-splitting for smooth Fano varieties, using the results of Schwede and Smith. Take $e \gg 0$ so that $(1-p^e)K_X$ is very ample and take an effective member $C \in |(1-p^e)K_X|$. Since $X \setminus C$

is smooth and affine, it is globally F-regular. By [SS10, Theorem 3.9], to prove that X is globally F-regular, it suffices to show that

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X(C) = F^e_*\mathcal{O}_X((1-p^e)K_X)$$

splits. Since X is globally F-split, we have a split surjection $F^e_*\mathcal{O}_X \to \mathcal{O}_X$. Let d be the dimension of X. Tensoring with the dualizing sheaf $\omega_X = \Omega^d_X$, we have a split surjection

$$\mathcal{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X((1-p^e)K_X), \omega_X) \cong F^e_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X((1-p^e)K_X), \omega_X) \cong F^e_*\omega_X^{p^e} \to \omega_X,$$

where the first isomorphism follows from Grothendieck duality [SS10, Section 4.1]. By tensoring the splitting map with ω_X^* and dualizing, we obtain the desired splitting $F^e_*\mathcal{O}_X((1-p^e)K_X) \to \mathcal{O}_X$.

We now show that X is globally F-split, i.e., the exact sequence

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B^1_X \to 0$$

splits. It suffices to show that $\mathrm{Ext}^1(B^1_X, \mathcal{O}_X) = 0$. Since we have a perfect pairing $B^1_X \times B^d_X \to \omega_X$ (see the proof of [MS87, Lemma 1.1]), it follows that

$$\mathrm{Ext}^1(B^1_X, \mathcal{O}_X) = H^1(X, B^d_X(-K_X)).$$

Since $-K_X$ is ample, this last group is zero by the following lemma. $\square$

Lemma 5.3. *Let X be a smooth projective variety over a perfect field of characteristic $p > 0$. If X satisfies Bott vanishing, then*

$$H^i(X, B^j_X(A)) = 0$$

and

$$H^i(X, Z^j_X(A)) = 0$$

for every $i > 0$, $j \geq 0$, and A an ample Cartier divisor.

Proof. Let A be an ample Cartier divisor. We know that $\Omega^j_X(A)$ and $(F_*\Omega^j_X)(A) = F_*(\Omega^j_X(pA))$ have zero cohomology in positive degrees. Let us prove by induction on j that $B^j_X(A)$ and $Z^j_X(A)$ have zero cohomology in positive degrees. This is clear for $B^0_X = 0$. The exact sequence (5.1.2) shows that this vanishing for B^j_X implies it for Z^j_X. The exact sequence (5.1.1) shows that this vanishing for Z^j_X implies it for B^{j+1}_X. That completes the proof by induction. $\square$

Remark 5.4. By [SS10, Theorem 1.2], mod p reductions of a log Fano projective variety in characteristic zero are globally F-regular for sufficiently large prime numbers p. The proof of Theorem 5.2 gives a simple proof of this implication for smooth Fano varieties. Indeed, Bott vanishing can be replaced by Serre vanishing when $p \gg 0$.

We now generalize Theorem 5.2 to allow toric singularities. A normal variety has *toric singularities* if it is étale-locally isomorphic to a toric variety near each point.

Lemma 5.5. *Let X be a variety with toric singularities over a perfect field of characteristic $p > 0$. Let m be an integer. Then we have natural exact sequences*

$$0 \to Z^{[i]}_X(mK_X) \to F_*(\Omega^{[i]}_X(pmK_X)) \to B^{[i+1]}_X(mK_X) \to 0$$

and

$$0 \to B_X^{[i]}(mK_X) \to Z_X^{[i]}(mK_X) \to \Omega_X^{[i]}(mK_X) \to 0.$$

Proof. For a toric variety Y, Blickle showed that the Cartier operator $\widehat{C}\colon \mathcal{H}^i(F_*\Omega_Y^{[\bullet]}) \to \Omega_Y^{[i]}$ is an isomorphism [Bli01, Theorem 3.6]. The same proof shows that $\mathcal{H}^i(F_*(\Omega_Y^{[\bullet]}(pE))) \to \Omega_Y^{[i]}(E)$ is an isomorphism for any toric Weil divisor E. Writing D for the sum of the irreducible toric divisors, we have $-K_Y \sim D$, and so the Cartier operator $\mathcal{H}^i(F_*(\Omega_Y^{[\bullet]}(pmK_Y))) \to \Omega_Y^{[i]}(mK_Y)$ is an isomorphism for each integer m. This statement can be checked locally in the étale topology, and so it holds more generally for any variety X with toric singularities.

Let $\widetilde{Z}^i_{X,m} = \ker(F_*(\Omega_X^{[i]}(pmK_X)) \to F_*(\Omega_X^{[i+1]}(pmK_X)))$ and $\widetilde{B}^i_{X,m} = \operatorname{im}(F_*(\Omega_X^{[i-1]}(pmK_X)) \to F_*(\Omega_X^{[i]}(pmK_X)))$, and write $Z_X^{[i]}(mK_X)$ and $B_X^{[i]}(mK_X)$ for their double duals. Since $\widetilde{Z}^i_{X,m}$ is the kernel of a map between reflexive sheaves, it is reflexive; that is, $Z_X^{[i]}(mK_X) = \widetilde{Z}^i_{X,m}$. By definition, we have an exact sequence

$$0 \to \widetilde{B}^i_{X,m} \to \widetilde{Z}^i_{X,m} \to \mathcal{H}^i(F_*(\Omega_X^{[\bullet]}(pmK_X)) \to 0,$$

and $\mathcal{H}^i(F_*(\Omega_X^{[\bullet]}(pmK_X))$ is isomorphic to $\Omega_X^{[i]}(mK_X)$, as we have said. So $\widetilde{B}^i_{X,m}$ is also the kernel of a map between reflexive sheaves, and so it is reflexive. That is, $B_X^{[i]}(mK_X) = \widetilde{B}^i_{X,m}$.

Therefore, we have an exact sequence

$$0 \to B_X^{[i]}(mK_X) \to Z_X^{[i]}(mK_X) \to \Omega_X^{[i]}(mK_X) \to 0.$$

Since $Z_X^{[i]}(mK_X) = \widetilde{Z}^i_{X,m}$ and $B_X^{[i]}(mK_X) = \widetilde{B}^i_{X,m}$, we also have an exact sequence $0 \to Z_X^{[i]}(mK_X) \to F_*(\Omega^i_X(pmK_X)) \to B_X^{[i+1]}(mK_X) \to 0$. □

Corollary 5.6. *Let X be a Fano variety with toric singularities over a perfect field of characteristic $p > 0$. If X satisfies Bott vanishing for ample Weil divisors, then X is globally F-regular.*

In particular, a Fano variety with toric singularities in characteristic $p > 0$ that admits an int-amplified endomorphism of degree prime to p is globally F-regular.

Proof. (Corollary 5.6) Using Lemma 5.5, the proof of Theorem 5.2 applies. In more detail: by Schwede-Smith, since X is Fano and locally F-regular, global F-regularity is equivalent to global F-splitting [SS10, Theorems 3.9 and 4.3]. By Lemma 5.5, we have an exact sequence

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B_X^{[1]} \to 0$$

on X, which we want to show is split. Since X is locally F-split, this sequence is locally split, and so it corresponds to an element of $H^1(X, \mathcal{H}om(B_X^{[1]}, \mathcal{O}_X))$. Since we have a perfect pairing $B^1_X \times B^d_X \to \omega_X$ on the smooth locus of X, the sheaf $\mathcal{H}om(B_X^{[1]}, \mathcal{O}_X)$ is the reflexive sheaf $B^d_X(-K_X)$. So it suffices to show that $H^1(X, B^d_X(-K_X)) = 0$. This follows from Bott vanishing by the inductive proof of Lemma 5.3, in view of Lemma 5.5. □

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References

[Ame98] Ekaterina Amerik. On a problem of Noether-Lefschetz type. *Compositio Math.*, 112(3):255–271, 1998. 2

[Art74] M. Artin. Supersingular $K3$ surfaces. *Ann. Sci. École Norm. Sup. (4)*, 7:543–567 (1975), 1974. 4

[ARVdV99] E. Amerik, M. Rovinsky, and A. Van de Ven. A boundedness theorem for morphisms between threefolds. *Ann. Inst. Fourier (Grenoble)*, 49(2):405–415, 1999. 1, 2

[AWZ21] Piotr Achinger, Jakub Witaszek, and Maciej Zdanowicz. Global Frobenius liftability I. *J. Eur. Math. Soc. (JEMS)*, 23(8):2601–2648, 2021. 8, 11

[AWZ23] Piotr Achinger, Jakub Witaszek, and Maciej Zdanowicz. Global Frobenius liftability II: surfaces and Fano threefolds. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 24:329–366, 2023. 3, 10

[Bea01] Arnaud Beauville. Endomorphisms of hypersurfaces and other manifolds. *Internat. Math. Res. Notices*, 2001(1):53–58, 2001. 1

[BGI71] Pierre Berthelot, Alexander Grothendieck, and Luc Illusie. *Théorie des intersections et théorème de Riemann-Roch (SGA 6)*. Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin-New York, 1971. 5, 6

[Bha22] Bhargav Bhatt. Prismatic F-gauges. *Princeton lecture notes, https://www.math.ias.edu/~bhatt/teaching/mat549f22/lectures.pdf*, 2022. 3

[BK05] Michel Brion and Shrawan Kumar. *Frobenius splitting methods in geometry and representation theory*, volume 231 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2005. 13

[Bli01] Manuel Blickle. Cartier isomorphism for toric varieties. *J. Algebra*, 237(1):342–357, 2001. 15

[BTLM97] Anders Buch, Jesper F. Thomsen, Niels Lauritzen, and Vikram Mehta. The Frobenius morphism on a toric variety. *Tohoku Math. J. (2)*, 49(3):355–366, 1997. 10

[CMZ20] Paolo Cascini, Sheng Meng, and De-Qi Zhang. Polarized endomorphisms of normal projective threefolds in arbitrary characteristic. *Math. Ann.*, 378(1-2):637–665, 2020. 3

[dJ96] A. J. de Jong. Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.*, 83:51–93, 1996. 4

[Eke87] Torsten Ekedahl. Foliations and inseparable morphisms. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 139–149. Amer. Math. Soc., Providence, RI, 1987. 11

[FGI+05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. 9, 10

[Fuj83] Takao Fujita. Vanishing theorems for semipositive line bundles. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 519–528. Springer, Berlin, 1983. 5

[Fuj07] Osamu Fujino. Multiplication maps and vanishing theorems for toric varieties. *Math. Z.*, 257(3):631–641, 2007. 5

[Gar84] Emmanuelle Garel. An extension of the trace map. *J. Pure Appl. Algebra*, 32(3):301–313, 1984. 6

[GJ18] Frank Gounelas and Ariyan Javanpeykar. Invariants of Fano varieties in families. *Mosc. Math. J.*, 18(2):305–319, 2018. 9

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. 10

[HM03] Jun-Muk Hwang and Ngaiming Mok. Finite morphisms onto Fano manifolds of Picard number 1 which have rational curves with trivial normal bundles. *J. Algebraic Geom.*, 12(4):627–651, 2003. 1, 2

[IP99] V. A. Iskovskikh and Yu. G. Prokhorov. Fano varieties. In *Algebraic geometry, V*, volume 47 of *Encyclopaedia Math. Sci.*, pages 1–247. Springer, Berlin, 1999. 3, 10

[Kaw21] Tatsuro Kawakami. On Kawamata-Viehweg type vanishing for three dimensional Mori fiber spaces in positive characteristic. *Trans. Amer. Math. Soc.*, 374(8):5697–5717, 2021. 9

[Kel22] Timo Keller. On the p-torsion of the Tate-Shafarevich group of abelian varieties over higher dimensional bases over finite fields. *J. Théor. Nombres Bordeaux*, 34(2):497–513, 2022. 4

[KK00] Yasuyuki Kachi and János Kollár. Characterizations of $\mathbf{P}^n$ in arbitrary characteristic. *Asian J. Math.*, 4(1):115–121, 2000. 9

[Kle05] Steven L. Kleiman. The Picard scheme. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 235–321. Amer. Math. Soc., Providence, RI, 2005. 6

[Kol13] János Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács. 11

[KP22] Alexander Kuznetsov and Yuri Prokhorov. On higher-dimensional del Pezzo varieties. *arXiv preprint arXiv:2206.01549, to appear in Izv. Ross. Akad. Nauk Ser. Mat.*, 2022. 12

[Kun86] Ernst Kunz. *Kähler differentials*. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1986. 6

[Laz84] Robert Lazarsfeld. Some applications of the theory of positive vector bundles. In *Complete intersections (Acireale, 1983)*, volume 1092 of *Lecture Notes in Math.*, pages 29–61. Springer, Berlin, 1984. 11

[Men20] Sheng Meng. Building blocks of amplified endomorphisms of normal projective varieties. *Math. Z.*, 294(3-4):1727–1747, 2020. 2

[Mil80] James S. Milne. *Étale cohomology*. Princeton Mathematical Series, No. 33. Princeton University Press, Princeton, N.J., 1980. 4

[MP12] Davesh Maulik and Bjorn Poonen. Néron-Severi groups under specialization. *Duke Math. J.*, 161(11):2167–2206, 2012. 9

[MS87] V. B. Mehta and V. Srinivas. Varieties in positive characteristic with trivial tangent bundle. *Compositio Math.*, 64(2):191–212, 1987. With an appendix by Srinivas and M. V. Nori. 14

[Muk89] Shigeru Mukai. Biregular classification of Fano 3-folds and Fano manifolds of coindex 3. *Proc. Nat. Acad. Sci. U.S.A.*, 86(9):3000–3002, 1989. 12

[MZ20] Sheng Meng and De-Qi Zhang. Normal projective varieties admitting polarized or int-amplified endomorphisms. *Acta Math. Vietnam.*, 45(1):11–26, 2020. 2

[MZZ22] Sheng Meng, De-Qi Zhang, and Guolei Zhong. Non-isomorphic endomorphisms of Fano threefolds. *Math. Ann.*, 383(3-4):1567–1596, 2022. 3

[Nak10] Noboru Nakayama. Separable endomorphisms of surfaces in positive characteristic. In *Algebraic geometry in East Asia—Seoul 2008*, volume 60 of *Adv. Stud. Pure Math.*, pages 301–330. Math. Soc. Japan, Tokyo, 2010. 3

[OW02] Gianluca Occhetta and Jarosław A. Wiśniewski. On Euler-Jaczewski sequence and Remmert-van de Ven problem for toric varieties. *Math. Z.*, 241(1):35–44, 2002. 11

[PS89] K. H. Paranjape and V. Srinivas. Self-maps of homogeneous spaces. *Invent. Math.*, 98(2):425–444, 1989. 1

[SB97] N. I. Shepherd-Barron. Fano threefolds in positive characteristic. *Compositio Math.*, 105(3):237–265, 1997. 9

[Sno86] Dennis M. Snow. Cohomology of twisted holomorphic forms on Grassmann manifolds and quadric hypersurfaces. *Math. Ann.*, 276(1):159–176, 1986. 12

[Sno88] Dennis M. Snow. Vanishing theorems on compact Hermitian symmetric spaces. *Math. Z.*, 198(1):1–20, 1988. 12

[SPA23] The Stacks Project Authors. The Stacks Project. https://stacks.math.columbia.edu/, 2023. 5, 6

[SS10] Karl Schwede and Karen E. Smith. Globally F-regular and log Fano varieties. *Adv. Math.*, 224(3):863–894, 2010. 14, 15

[SZ22] Feng Shao and Guolei Zhong. Boundedness of finite morphisms onto Fano manifolds with large Fano index. *arXiv preprint arXiv:2211.16380*, 2022. 1, 2

[Tan22] Hiromu Tanaka. Contraction images of toric varieties. *arXiv preprint arXiv:2208.01454*, 2022. 11

[Tot23] Burt Totaro. Bott vanishing for Fano 3-folds. *arXiv preprint arXiv:2302.08142*, 2023. 3

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