Cohomological invariants in positive characteristic

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1 Introduction

Étale cohomology works especially well with $\mathbf{Z}/l$ coefficients such that l is invertible in the base field. In the 1970s, however, Milne, Bloch, and Illusie defined variants of étale cohomology based on differential forms. Using these constructions, Kato defined groups $H^i_{\text{ét}}(k, \mathbf{Z}/m(j))$ for a field k and any positive integer m, even when m is not invertible in k [17, p. 219]. Kato's groups behave surprisingly well. For example, we have $H^1_{\text{ét}}(k, \mathbf{Z}/m(0)) \cong H^1_{\text{ét}}(k, \mathbf{Z}/m)$, the group classifying cyclic $\mathbf{Z}/m$-extensions of k, and $H^2_{\text{ét}}(k, \mathbf{Z}/m(1)) \cong \mathrm{Br}(k)[m]$, the m-torsion subgroup of the Brauer group, whether m is invertible in k or not.

Nowadays, there is an "explanation" for Kato's groups: Voevodsky's étale motivic cohomology groups $H^i_{\text{ét}}(X, A(j))$ of a scheme X over a field k are defined for any abelian group A. They agree with the familiar étale cohomology with coefficients in $\mu_m^{\otimes j}$ when A is $\mathbf{Z}/m$ with m invertible in k and $j \geq 0$, and they agree with Kato's groups when $X = \mathrm{Spec}(k)$ and $A = \mathbf{Z}/m$ for any m [23, Theorem 10.2], [10].

In this paper, we make some new calculations of mod p étale motivic cohomology in characteristic p. In particular, we compute the group of cohomological invariants (in Serre's sense) for some important affine group schemes, such as the symmetric groups (Theorem 10.2), the finite group schemes $(\mu_p)^a \times (\mathbf{Z}/p)^b$ (Theorem 8.4), and the orthogonal groups $O(n)$ and $SO(n)$ (Theorems 12.1, 13.1, 13.3, 14.1). These calculations were done in [9, Chapters VI and VII] for $\mathbf{Z}/l$ coefficients with $l \neq p$, and we carry out the case $l = p$. For the orthogonal groups, the interesting new case is where these groups are considered over a field of characteristic 2. In that case, our calculation amounts to determining the group of cohomological invariants for quadratic forms in characteristic 2.

One outcome of the calculations is that there are often fewer mod p cohomological invariants when the base field has characteristic p. For example, a basis for the mod 2 cohomological invariants for the orthogonal group $O(n)$ in characteristic not 2 is given by the Stiefel-Whitney classes $1 = w_0, w_1, \ldots, w_n$, whereas in characteristic 2 there are only analogs of w_1 and w_2, the discriminant (or Arf invariant) and the Clifford invariant. In particular, cohomological invariants are not enough to give the lower bounds for the essential dimension of $O(n)$ and $SO(n)$ in characteristic 2 proved by Babic and Chernousov [2]. The cohomological invariants of the spin groups $\mathrm{Spin}(n)$ in characteristic 2 (as in other characteristics) are not known, but for $n \leq 10$ there are enough invariants to give optimal lower bounds on the essential dimension [35].

We also determine all operations on the mod p étale motivic cohomology of fields (section 11), extending Vial's computation of the operations on the mod p Milnor K-theory of fields [36].

As far as I know, this paper gives the first calculations of all mod p cohomological invariants for a given affine group scheme in characteristic p. Merkurjev gave a geometric description of such invariants (see Theorem 2.1), but the only full calculations seem to be in low degrees. In particular, Lourdeaux described the degree 2 cohomological invariants for all smooth connected affine groups [22, Théorème 2.3.2]. Blinstein and Merkurjev found the degree 3 cohomological invariants for tori [3]. Esnault-Kahn-Levine-Viehweg and Merkurjev determined all degree 3 invariants for simply connected semisimple groups; they are generated by the Rost invariant in the mod p case, as in the mod l case [7, Appendix B], [9, Part 2, Theorem 9.11]. Laackman and Merkurjev also found the degree 3 invariants for some other classes of reductive groups [24, 21].

A key difference between étale motivic cohomology in the mod p case and the mod l case is that mod p étale motivic cohomology of schemes is not $\mathbf{A}^1$-invariant. (For example, for k algebraically closed of characteristic p, $H^1_{\text{ét}}(k, \mathbf{Z}/p)$ is zero, while $H^1_{\text{ét}}(\mathbf{A}^1_k, \mathbf{Z}/p)$ is not zero: there are many nontrivial étale $\mathbf{Z}/p$-coverings of the affine line.) This failure is related to the phenomenon of wild ramification (section 4), which does not occur in the mod l case. One goal of this paper is to show that, although the lack of $\mathbf{A}^1$-invariance means that some familiar arguments no longer apply, mod p étale motivic cohomology is still a useful and computable theory. A crucial ingredient of the proofs is an analysis of tame and wild ramification for classes in étale motivic cohomology, extending work of Izhboldin (Theorem 4.3).

Thanks to Vladimir Chernousov, Skip Garibaldi, Philippe Gille, Alexander Merkurjev, Niranjan Ramachandran, and the referees for their suggestions. This work was supported by NSF grant DMS-1701237.

Contents

2 Background on étale motivic cohomology

Building on earlier work of Bloch and Kato, Geisser and Levine proved the relation between Voevodsky's étale motivic cohomology and Kato's invariants of fields based on differential forms. Namely, let k be a field of characteristic $p > 0$ which is perfect, meaning that every element of k is a pth power, and let X be a smooth scheme over k. For $j \geq 0$, let $\Omega^j_{\log}$ be the subsheaf of Ω^j_X generated locally by logarithmic differentials $df_1/f_1 \wedge \cdots \wedge df_j/f_j$ for units $f_1, \ldots, f_j$. (This is a sheaf of $\mathbf{F}_p$-vector spaces, not of O_X-modules.) More generally, for $r > 0$, let $W_r\Omega^j_{\log}$ be the analogous subsheaf of logarithmic de Rham-Witt differentials [13]. Then Voevodsky's object $\mathbf{Z}/p^r(j)$ in the derived category of Zariski (or étale) sheaves on X is isomorphic to the shift $W_r\Omega^j_{\log}[-j]$ [10, Proposition 3.1, Theorem 8.3]. As a result, étale motivic cohomology, meaning the étale cohomology of X with coefficients in $\mathbf{Z}/p^r(j)$, can be rewritten in terms of differential forms:

$$H^i_{\text{ét}}(X, \mathbf{Z}/p^r(j)) \cong H^{i-j}_{\text{ét}}(X, W_r\Omega^j_{\log}).$$

This has consequences for any field k of characteristic p, not necessarily perfect. Indeed, such a field has étale p-cohomological dimension at most 1 [31, section II.2.2]. As a result, $H^i_{\text{ét}}(k, \mathbf{Z}/p^r(j))$ is zero except when i is j or $j+1$. When $i = j$, Bloch and Kato identified this group with the Milnor K-group $K^M_j(k)/p^r$, or also with the group $W_r\Omega^j_{\log,k}$ [4, Corollary 2.8]. There are several ways to describe the remaining mod p^r étale motivic cohomology groups of a field, when $i = j+1$; we concentrate on the case $r = 1$.

Write $H^{i,j}(k) = H^i_{\text{ét}}(k, \mathbf{Z}/p(j))$. One description of these groups is in terms of Galois cohomology. For a field k of characteristic $p > 0$, not necessarily perfect, let k_s be a separable closure of k. Let Ω^j_k be the group of (absolute) differential forms on k, which can be viewed as $\Omega^j_{k/\mathbf{Z}}$ or $\Omega^j_{k/\mathbf{F}_p}$. Write $\Omega^j_{\log,k}$ for the subgroup of Ω^j_k generated by elements $(da_1/a_1) \wedge \cdots \wedge (da_j/a_j)$ with $a_1, \ldots, a_j$ in k^*. Then

$$H^{i,j}(k) \cong \begin{cases} \Omega^j_{\log,k} \cong H^0_{\text{Gal}}(k, \Omega^j_{\log,k_s}) & \text{if } i = j \\ H^1_{\text{Gal}}(k, \Omega^j_{\log,k_s}) & \text{if } i = j+1 \\ 0 & \text{otherwise.} \end{cases}$$

The Galois group $\text{Gal}(k_s/k)$ of a field k of characteristic $p > 0$ has p-cohomological dimension at most 1 [31, section II.2.2], which explains why only H^0 and H^1 occur here.

For another description of these groups (Kato's original definition [18]), define a group homomorphism $\mathcal{P}\colon \Omega^j_k \to \Omega^j_k/d\Omega^{j-1}_k$ by

$$\mathcal{P}(a(db_1/b_1) \wedge \cdots \wedge (db_j/b_j)) = (a^p - a)(db_1/b_1) \wedge \cdots \wedge (db_j/b_j).$$

Then $H^{j+1,j}(k)$ is isomorphic to the cokernel of $\mathcal{P}$ [14, Corollary 6.5]. In fact, there is an exact sequence:

$$0 \to H^{j,j}(k) \to \Omega^j_k \xrightarrow[\mathcal{P}]{} \Omega^j_k/d\Omega^{j-1}_k \to H^{j+1,j}(k) \to 0.$$

Let k be a field of characteristic $p > 0$. For an element a of k, write $[a]$ for the class of a in $H^{1,0}(k) = k/\mathcal{P}(k)$, where $\mathcal{P}(a) = a^p - a$, as above. For $b_1, \ldots, b_j$ in k^*, the *symbol* $\{b_1, \ldots, b_j\}$ in $H^{j,j}(k)$ means the class of the differential form $(db_1/b_1) \wedge \cdots \wedge (db_j/b_j)$; this agrees with the standard notation in Milnor K-theory, via the isomorphism $H^{j,j}(k) \cong K^M_j(k)/p$. Finally, for $a \in k$ and $b_1, \ldots, b_j \in k^*$, the symbol

$$[a, b_1, \ldots, b_j\} \in H^{j+1,j}(k)$$

means the class of the differential form $a(db_1/b_1) \wedge \cdots \wedge (db_j/b_j)$. Both groups $H^{j,j}(k)$ and $H^{j+1,j}(k)$ are generated by symbols, by the descriptions above.

For a scheme X of characteristic p, étale motivic cohomology with $\mathbf{Z}/l(j)$ coefficients for $l \neq p$ and $j \geq 0$ can be identified with étale cohomology with the familiar coefficients $\mu_l^{\otimes j}$. (For X smooth over k, which is the only case we will need, this is [23, Theorem 10.2].) In particular, it follows that étale motivic cohomology with $\mathbf{Z}/l(j)$ coefficients for $l \neq p$ is $\mathbf{A}^1$-invariant, by one of Grothendieck's fundamental results [26, Corollary VI.4.20]. By contrast, mod p étale motivic cohomology is not $\mathbf{A}^1$-invariant. For a simple example, look at $H^{1,0}(X) \cong H^1_{\text{ét}}(X, \mathbf{Z}/p)$. We have the Artin-Schreier exact sequence of étale sheaves:

$$0 \to \mathbf{Z}/p \to O_X \xrightarrow[\mathcal{P}]{} O_X \to 0,$$

where $\mathcal{P}(a) = a^p - a$. For X affine, it follows that we have an exact sequence

$$O(X) \xrightarrow[\mathcal{P}]{} O(X) \to H^1_{\text{ét}}(X, \mathbf{Z}/p) \to 0.$$

For example, if k is an algebraically closed field, then $H^1_{\text{ét}}(k, \mathbf{Z}/p) = 0$, whereas one checks from this exact sequence that $H^1_{\text{ét}}(\mathbf{A}^1_k, \mathbf{Z}/p)$ is isomorphic to a countably infinite direct sum of copies of k.

Let G be an affine group scheme of finite type over a field k. This determines a functor from fields over k to sets by $F \mapsto H^1(F, G)$, the set of isomorphism classes of G-torsors over F. (Here G-torsors are defined in the most general sense, using the fppf topology; for G smooth over k, this is the same as G-torsors in the étale topology [26, Remark III.4.8].) The abelian group of *cohomological invariants* of G with values in $H^i_{\text{ét}}(\mathbf{Z}/m(j))$, written $\text{Inv}^i_k(G, \mathbf{Z}/m(j))$, means the set of natural transformations from $H^1(F, G)$ to $H^i_{\text{ét}}(F, \mathbf{Z}/m(j))$, on the category of fields F over k. When the positive integer m is invertible in k, the group of cohomological invariants was computed for several important groups G in [9, Chapters VI and VII]: the symmetric groups, elementary abelian groups, and the orthogonal groups. In this paper, we will make the analogous mod p calculations when p is the characteristic of k.

A cohomological invariant for a group scheme G over k is *normalized* if it is equal to zero on the trivial G-torsor. It is immediate that the group of invariants for G splits as the direct sum of the "constant" invariants and the normalized invariants:

$$\text{Inv}^i_k(G, \mathbf{Z}/m(j)) \cong H^i(k, \mathbf{Z}/m(j)) \oplus \text{NormInv}^i_k(G, \mathbf{Z}/m(j)).$$

Some insight into the group of cohomological invariants is provided by the existence of a *versal torsor*. Let G be an affine group scheme over an infinite field k, and let V be a k-vector space on which G acts by affine transformations. Suppose that G acts freely on a nonempty Zariski open subset U of V, with a quotient scheme U/G. Then every G-torsor over an extension field of k is pulled back from the G-torsor $U \to U/G$ [9, section I.5]. As a result, we have an injection

$$\mathrm{Inv}_k^i(G, \mathbf{Z}/m(j)) \hookrightarrow H^i(k(U/G), \mathbf{Z}/m(j))$$

(by Merkurjev, in this generality; see Theorem 2.1 below). Also, cohomological invariants always give cohomology classes on $k(U/G)$ that are unramified along all divisors in U/G.

For m invertible in k, this injection is in fact an isomorphism to the group $H^0_{\mathrm{Zar}}(U/G, H^i_{\text{ét}})$ of unramified classes, under the mild extra assumption that $V - U$ has codimension at least 2 in V [9, Part 1, Appendix C]. However, that argument relies on $\mathbf{A}^1$-invariance. For $p = \mathrm{char}(k)$, where $\mathbf{A}^1$-invariance fails, one cannot expect to identify the mod p cohomological invariants of G with the unramified cohomology of a quotient variety U/G; consider the case of the trivial group G and vector spaces $U = V$ of various dimensions. However, Merkurjev provided a substitute: for any positive integer m, the group of cohomological invariants for G need not be the whole group $H^0(U/G, H^i(\mathbf{Z}/m(j)))$, but it is always the subgroup of *balanced* elements in $H^i(k(U/G), \mathbf{Z}/m(j))$, meaning the elements whose pullbacks via the two projections $(U \times U)/G \to U/G$ are equal. Balanced elements are always unramified over U/G, and so the group of cohomological invariants can also be described as the subgroup of balanced elements in unramified cohomology. Blinstein and Merkurjev had proved this earlier when G is smooth over k [3, Theorem A]. Here is the general statement [25, Theorem 6.3]:

Theorem 2.1. *Let G be an affine group scheme of finite type over a field k. Let U be a smooth k-variety with a free G-action such that there is a quotient scheme U/G. Suppose that U is G-equivariantly birational to an affine space over k on which G acts by affine transformations. Let m be a positive integer and $i, j \geq 0$. Then*

$$\begin{aligned}\mathrm{Inv}_k^i(G, \mathbf{Z}/m(j)) &\cong H^i(k(U/G), \mathbf{Z}/m(j))_{\mathrm{bal}} \\ &\cong H^0_{\mathrm{Zar}}(U/G, H^i(\mathbf{Z}/m(j)))_{\mathrm{bal}}.\end{aligned}$$

Theorem 2.1 should be attributed to Merkurjev, but we give detailed references in section 3, since Merkurjev's argument emphasizes the case of coefficients $\mathbf{Q}/\mathbf{Z}(j)$ rather than $\mathbf{Z}/m(j)$. (The same proof works.) Our calculation of the cohomological invariants of the group scheme μ_p (Proposition 6.1), on which the rest of the paper depends, relies on Theorem 2.1.

Finally, we recall the relation between mod p cohomological invariants and the essential dimension at p [35, Lemma 3.1]:

Proposition 2.2. *Let G be an affine group scheme of finite type over an algebraically closed field k of characteristic $p > 0$. Then $\mathrm{Inv}_k^{j+1,j}(G) = 0$ for all $j \geq \mathrm{ed}(G; p)$.*

3 Proof of Theorem 2.1

Proof. (Theorem 2.1) When G is smooth over k, this was shown by Blinstein and Merkurjev [3, Theorem A]. The step where they use smoothness is in showing that the natural homomorphism

$$\mathrm{Inv}_k^i(G, \mathbf{Z}/m(j)) \to H^i(k(U/G), \mathbf{Z}/m(j))$$

is injective [3, Appendix A-I]. However, Merkurjev gave a different argument which shows that this homomorphism is injective for G not necessarily smooth [25, Theorem 6.1].

The other steps of Blinstein-Merkurjev's argument work without smoothness of G (and in fact are stated that way; they explain that their "linear algebraic groups" are not assumed to be smooth). Namely, an invariant for G gives an element of $H^i(k(U/G), \mathbf{Z}/m(j))$ which is balanced, because the pullbacks of the G-torsor $U \to U/G$ by the two projections $U^2/G \to U/G$ are both isomorphic to $U^2 \to U^2/G$. Conversely, [3, Lemmas 3.2 and 3.3] show that a balanced element gives an invariant of G. Finally, a balanced element is unramified along all irreducible divisors in U/G by [3, Proposition A.9], which works with $\mathbf{Z}/m(j)$ coefficients as well as with $\mathbf{Q}/\mathbf{Z}(j)$ coefficients. $\square$

4 Ramification and residues

In this section, building on the work of Izhboldin, we describe étale motivic cohomology for a field with a discrete valuation. In particular, there are notions of tame and wild ramification for cohomology classes, and a residue homomorphism. The quotient of étale motivic cohomology by the unramified subgroup can be described very explicitly (Theorem 4.3). Finally, we state Izhboldin's calculation of the étale motivic cohomology of a rational function field (Theorem 4.4). All this is used for the basic calculations of the paper, the determination of the cohomological invariants for the group schemes μ_p and $\mathbf{Z}/p$ (Propositions 6.1 and 8.1).

Let F be a field with a discrete valuation v. Let O_F be the valuation ring $\{x \in F : v(x) \geq 0\}$, and let $k = O_F/\mathfrak{m}$ be the residue field. Define the subgroup of *unramified* classes in $H^i_{\text{ét}}(F, \mathbf{Z}/m(j))$ to be the image of $H^i_{\text{ét}}(O_F, \mathbf{Z}/m(j))$. (Blinstein and Merkurjev use the same notion of "unramified" [3, start of section 5].) More concretely, for $p = \mathrm{char}(k)$, in the description of $H^{n+1}_{\text{ét}}(F, \mathbf{Z}/p(n))$ as a quotient of Ω^n_F (section 2), the unramified subgroup is the subgroup generated by elements $a(db_1/b_1) \wedge \cdots \wedge (db_n/b_n)$ with $a_i \in O_F$ and $b_1, \ldots, b_n \in O_F^*$. If m is invertible in k, then the subgroup of unramified classes is the kernel of the *residue* homomorphism [9, Part 1, section 7.9]:

$$\partial_v \colon H^i_{\text{ét}}(F, \mathbf{Z}/m(j)) \to H^{i-1}_{\text{ét}}(k, \mathbf{Z}/m(j-1)).$$

If m is not invertible in k, what happens is more complicated, but still manageable. Let F be a field with a discrete valuation v. An extension field of F is called *tame* if it is a union of finite extensions of F for which the extension of residue fields is separable and the ramification degree is invertible in the residue field k. Let F_{tame} be the maximal tamely ramified extension of F (with respect to v) in a separable closure of F. Define the *tame* (or *tamely ramified*) subgroup by

$$H^i_{\mathrm{tame}}(F, \mathbf{Z}/m(j)) = \ker\left(H^i_{\mathrm{\acute{e}t}}(F, \mathbf{Z}/m(j)) \to H^i_{\mathrm{\acute{e}t}}(F_{\mathrm{tame}}, \mathbf{Z}/m(j))\right).$$

The whole group H^i is tamely ramified if m is invertible in k. For general m, the residue homomorphism is not defined on all of $H^i_{\mathrm{\acute{e}t}}(F, \mathbf{Z}/m(j))$, but only on the tamely ramified subgroup [15, Corollary 2.7]:

$$\partial_v \colon H^i_{\mathrm{tame}}(F, \mathbf{Z}/m(j)) \to H^{i-1}_{\mathrm{\acute{e}t}}(k, \mathbf{Z}/m(j-1))$$

(In the case $m = p = \mathrm{char}(F)$, the tame subgroup will be described in terms of differential forms in the proof of Theorem 4.3. The residue homomorphism is characterized by the property that

$$\partial_v\left(a\frac{dt}{t} \wedge \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{n-1}}{b_{n-1}}\right) = \overline{a}\frac{d\overline{b_1}}{\overline{b_1}} \wedge \cdots \wedge \frac{d\overline{b_{n-1}}}{\overline{b_{n-1}}}$$

where $a \in O_F$, $b_1, \ldots, b_{n-1} \in O_F^*$, and t is any uniformizer in O_F. We write $a \mapsto \overline{a}$ for the surjection $O_F \to k$.)

As a result, mod p étale motivic cohomology does not fit into the framework of Rost's cycle modules [30]. On the good side, Theorem 4.3 will say: (1) The unramified subgroup of étale motivic cohomology is the kernel of the residue on the tamely ramified subgroup. (2) There is a satisfactory description of the quotient of étale motivic cohomology by the tamely ramified subgroup.

Remark 4.1. When $m = \mathrm{char}(k)$, Izhboldin calls our "tamely ramified" subgroup of étale motivic cohomology the "unramified" subgroup [15]. That has the confusing consequence that the residue homomorphism is nontrivial on his "unramified" subgroup. Our use of "tamely ramified" follows Kato [17, Theorem 3] and Auel-Bigazzi-Böhning-von Bothmer [1, Remark 3.8]. It also agrees with the terminology used for the Brauer group [33, Proposition 6.63].

When the discretely valued field F is complete of characteristic $p > 0$, Izhboldin analyzed the "wild quotient" of $H^{n+1,n}(F) = H^{n+1}_{\mathrm{\acute{e}t}}(F, \mathbf{Z}/p(n))$; we generalize his result (not assuming completeness) as Theorem 4.3. To set this up, use the description of $H^{n+1,n}(F)$ as a quotient of Ω^n_F from section 2. Define an increasing filtration of $H^{n+1,n}(F)$ by: for $i \geq 0$, let U_i be the subgroup of $H^{n+1,n}(F)$ generated by elements of the form

$$a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}$$

with $a \in F$, $b_1, \ldots, b_n \in F^*$, and $v(a) \geq -i$. It is clear that

$$0 \subset U_0 \subset U_1 \subset \cdots,$$

with $\bigcup_{i\geq 0} U_i = H^{n+1,n}(F)$. Theorem 4.3 will show that U_0 is the tamely ramified subgroup of $H^{n+1,n}(F)$.

Let $t \in O_F$ be a uniformizer for v. If $j > 0$ and j is prime to p, define a homomorphism

$$\Omega^n_k \to U_j/U_{j-1}$$

by

$$\overline{a}\frac{\overline{db_1}}{\overline{b_1}} \wedge \cdots \wedge \frac{\overline{db_n}}{\overline{b_n}} \mapsto \frac{a}{t^j}\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} \pmod{U_{j-1}},$$

for $a \in O_F$ and $b_1, \ldots, b_n \in O_F^*$. Let Z_k^n be the subgroup of closed forms in Ω_k^n. If $j > 0$ and $p|j$, define a homomorphism

$$\Omega_k^n/Z_k^n \oplus \Omega_k^{n-1}/Z_k^{n-1} \to U_j/U_{j-1}$$

by (for the first summand)

$$\overline{a}\frac{\overline{db_1}}{\overline{b_1}} \wedge \cdots \wedge \frac{\overline{db_n}}{\overline{b_n}} \mapsto \frac{a}{t^j}\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} \pmod{U_{j-1}}$$

and (for the second summand)

$$\overline{a}\frac{\overline{db_1}}{\overline{b_1}} \wedge \cdots \wedge \frac{\overline{db_{n-1}}}{\overline{b_{n-1}}} \mapsto \frac{a}{t^j}\frac{dt}{t} \wedge \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{n-1}}{b_{n-1}} \pmod{U_{j-1}},$$

where $a \in O_F$ and $b_1, \ldots, b_n \in O_F^*$.

It is straightforward to check that the homomorphisms above are well-defined (although they depend on the choice of uniformizer t). First check that the element in U_j/U_{j-1} associated to given elements $\overline{a} \in k$ and $\overline{b_i} \in k^*$ is independent of the choice of lifts to O_F. (For example, in the case $j > 0$, $p \nmid j$, it is clear that changing the lift of $\overline{a}$ changes the result by an element of U_{j-1}. Changing the lift of $\overline{b_i}$ amounts to multiplying b_i by $1+e$ for some $e \in \mathfrak{m}$; since $d(1+e)/(1+e) = (e/(1+e))(de/e)$, where $e/(1+e)$ is in $\mathfrak{m}$, this change of lift changes the result by adding an element of U_{j-1}, as we want.) To finish showing that the homomorphisms above are well-defined, use Kato's presentation of Ω_k^n [16, section 1.3, Lemma 5]:

Proposition 4.2. *For any field k and natural number n, the group of differentials $\Omega_k^n = \Omega_{k/\mathbf{Z}}^n$ is the quotient of $k \otimes_{\mathbf{Z}} (k^*)^{\otimes n}$ by the relations:*

$$[a, b_1, \ldots, b_n\} = 0$$

if $a \in k$, $b_1, \ldots, b_n \in k^*$, and $b_i = b_j$ for some $i \neq j$; and

$$[u+v, u+v, b_2, \ldots, b_n\} = [u, u, b_2, \ldots, b_n\} + [v, v, b_2, \ldots, b_n\}$$

if $u, v, u+v \in k^*$. (The map from this quotient group to Ω_k^n takes the symbol $[a, b_1, \ldots, b_n\}$ to $a_1(db_1/b_1) \wedge \cdots \wedge (db_n/b_n)$.)

When $j > 0$ and $p \nmid j$, it is straightforward from Proposition 4.2 to check that we have a well-defined homomorphism $\Omega_k^n \to U_j/U_{j-1}$, above. When $j > 0$ and $p|j$, we can likewise see that we have a well-defined homomorphism $\Omega_k^n/Z_k^n \oplus \Omega_k^{n-1}/Z_k^{n-1} \to U_j/U_{j-1}$, using Cartier's theorem that, for k of characteristic $p > 0$, the subgroup Z_k^n of closed forms in Ω_k^n is generated by the exact forms together with the forms $a^p(db_1/b_1) \wedge \cdots \wedge (db_n/b_n)$ [15, Lemma 1.5.1].

Our generalization of Izhboldin's result is:

Theorem 4.3. *Let F be a field of characteristic $p > 0$ with a discrete valuation v and residue field k. Then $H^{n+1,n}(F)$ is the union of the increasing sequence of subgroups $U_0 \subset U_1 \subset \cdots$ defined above, with isomorphisms (depending on a choice of uniformizer in F):*

$$U_j/U_{j-1} \cong \begin{cases} \Omega^n_k & \text{if } j > 0 \text{ and } p \nmid j, \\ \Omega^n_k/Z^n_k \oplus \Omega^{n-1}_k/Z^{n-1}_k & \text{if } j > 0 \text{ and } p|j. \end{cases}$$

Moreover, U_0 is the tame subgroup $H^{n+1,n}_{\text{tame}}(F)$ defined above, and there is a well-defined residue homomorphism on U_0, yielding an exact sequence

$$0 \to H^{n+1,n}_{\text{nr}}(F) \to H^{n+1,n}_{\text{tame}}(F) \xrightarrow[\partial_v]{} H^{n,n-1}(k) \to 0,$$

where $H^{n+1,n}_{\text{nr}}(F)$ is the unramified subgroup with respect to v. Finally, if the field F is henselian (for example, complete) with respect to v, then $H^{n+1,n}_{\text{nr}}(F) \cong H^{n+1,n}(k)$.

Without making a choice of uniformizer, the argument gives the following canonical descriptions of U_j/U_{j-1}, which we will not need: writing $\mathfrak{m}$ for the maximal ideal in the valuation ring O_F,

$$U_j/U_{j-1} \cong \Omega^n_k \otimes_k (\mathfrak{m}/\mathfrak{m}^2)^{\otimes -j}$$

if $j > 0$, $p \nmid j$, and

$$0 \to (\Omega^n_k/Z^n_k) \otimes_k (\mathfrak{m}/\mathfrak{m}^2)^{\otimes -j} \to U_j/U_{j-1} \to (\Omega^{n-1}_k/Z^{n-1}_k) \otimes_k (\mathfrak{m}/\mathfrak{m}^2)^{\otimes -j} \to 0$$

if $j > 0$, $p|j$.

Proof. When F is complete, this was proved by Izhboldin [15, Theorem 2.5]. We address the henselian case at the end. For any discretely valued field F, write F_v for the completion of F with respect to v. For brevity, write $U_j = U_j(F)$ and $N_j = U_j(F_v)$; thus we know that N_j/N_{j-1} is isomorphic to Ω^n_k for $j > 0$, $p \nmid j$, and to $\Omega^n_k/Z^n_k \oplus \Omega^{n-1}_k/Z^{n-1}_k$ for $j > 0$, $p|j$. There are obvious homomorphisms $U_j \to N_j$. We want to show that the homomorphism $U_j/U_{j-1} \to N_j/N_{j-1}$ is an isomorphism for all $j > 0$.

First, suppose that $j > 0$ and $p \nmid j$. Fix a uniformizer t for F. From before the theorem, we have homomorphisms

$$\Omega^n_k \to U_j/U_{j-1} \to N_j/N_{j-1}$$

whose composition is an isomorphism by Izhboldin. To show that these homomorphisms are isomorphisms, it suffices to show that our homomorphism $\Omega^n_k \to U_j/U_{j-1}$ is surjective. Because $F^* = t^{\mathbf{Z}} \times O^*_F$, U_j/U_{j-1} is generated by two types of elements: $(a/t^j)(db_1/b_1) \wedge \cdots \wedge (db_n/b_n)$ with $a \in O_F$ and $b_1, \ldots, b_n \in O^*_F$, and elements $(a/t^j)(dt/t) \wedge (db_2/b_2) \wedge \cdots \wedge (db_n/b_n)$ with $a \in O_F$ and $b_2, \ldots, b_n \in O^*_F$. The first elements are clearly in the image of Ω^n_k, by our construction. For the second type of element, use that $p \nmid j$, so that $d(-1/(jt^j)) = (1/t^j)dt/t$. Therefore, for $a \in O_F$, which we can assume is not zero, and $b_2, \ldots, b_n \in O^*_F$,

$$d\Big(-\frac{a}{jt^j}\frac{db_2}{b_2} \wedge \cdots \wedge \frac{db_n}{b_n}\Big) = -\frac{a}{jt^j}\frac{da}{a} \wedge \frac{db_2}{b_2} \wedge \cdots \wedge \frac{db_n}{b_n} + \frac{a}{t^j}\frac{dt}{t} \wedge \frac{db_2}{b_2} \wedge \cdots \wedge \frac{db_n}{b_n}.$$

Since exact forms represent zero in $H^{n+1,n}(F)$, it follows that the element $(a/t^j)(dt/t)\wedge(db_2/b_2)\wedge\cdots\wedge(db_n/b_n)$ in U_j/U_{j-1} that we are considering is equal to an element $(a/(jt^j))(da/a)\wedge(db_2/b_2)\wedge\cdots\wedge(db_n/b_n)$. If a is in O_F^*, then this element is in the image of Ω^n_k, as we want. On the other hand, if $a\in\mathfrak{m}$, then our element is in U_{j-1}, hence zero in U_j/U_{j-1}. This completes the proof that $U_j/U_{j-1}\cong\Omega^n_k$ for $j>0$, $p\nmid j$.

For $j>0$, $p|j$, we defined homomorphisms (before the theorem)

$$\Omega^n_k/Z^n_k\oplus\Omega^{n-1}_k/Z^{n-1}_k\to U_j/U_{j-1}\to N_j/N_{j-1}$$

whose composition is an isomorphism. To show that these homomorphisms are isomorphisms, it suffices to show that $\Omega^n_k/Z^n_k\oplus\Omega^{n-1}_k/Z^{n-1}_k\to U_j/U_{j-1}$ is surjective. That is immediate from the definition of this homomorphism. Indeed, since $F^*=t^{\mathbf{Z}}\times O_F^*$, U_j/U_{j-1} is generated by two types of elements: $(a/t^j)(db_1/b_1)\wedge\cdots\wedge(db_n/b_n)$ with $a\in O_F$ and $b_1,\ldots,b_n\in O_F^*$, and $(a/t^j)(dt/t)\wedge(db_2/b_2)\wedge\cdots\wedge(db_n/b_n)$ with $a\in O_F$ and $b_2,\ldots,b_n\in O_F^*$.

Thus we have determined the structure of U_j/U_{j-1} for all $j>0$. It is clear that $H^{n+1,n}(F)=\bigcup_{j\geq 0}U_j$.

Next, let us show that U_0 is the tame subgroup of $H^{n+1,n}(F)$, defined earlier in this section as the kernel of $H^{n+1,n}(F)\to H^{n+1,n}(F_{\text{tame}})$. By definition, U_0 is spanned by elements

$$a\frac{db_1}{b_1}\wedge\cdots\wedge\frac{db_n}{b_n}$$

with $a\in O_F$ and $b_1,\ldots,b_n\in F^*$. Such an element maps to zero in $H^{n+1,n}$ of the extension $F[x]/(x^p-x-a)$ of F, which is unramified and hence tame over F. Conversely, let u be an element of $H^{n+1,n}(F)$ that maps to zero in $H^{n+1,n}(F_{\text{tame}})$. A totally ramified tame finite extension has degree prime to p, and so u must map to zero in $H^{n+1,n}$ of the maximal unramified extension F_{nr} of F. If u is not in U_0, then it has nonzero image in U_i/U_{i-1} for some $i>0$. The residue field of F_{nr} is the separable closure k_s of k, and the maps $\Omega^j_k\to\Omega^j_{k_s}$ are injective for all j; so our description of U_i/U_{i-1} implies that the map $U_i/U_{i-1}\to U_i(F_{\text{nr}})/U_{i-1}(F_{\text{nr}})$ is injective. This contradicts that u maps to zero in $H^{n+1,n}(F_{\text{nr}})$. So in fact u is in U_0. We have shown that $U_0=H^{n+1,n}_{\text{tame}}(F)$.

Next, we show that the obvious homomorphism

$$H^{n+1,n}_{\text{tame}}(F)/H^{n+1,n}_{\text{nr}}(F)\to H^{n,n-1}_{\text{tame}}(F_v)/H^{n+1,n}_{\text{nr}}(F_v)\cong H^{n,n-1}(k)$$

is an isomorphism. Here we define the unramified subgroup $H^{n+1,n}_{\text{nr}}(F)$ as the image of $H^{n+1,n}(O_F)$, or more concretely as the subgroup generated by differential forms $a(db_1/b_1)\wedge\cdots\wedge(db_n/b_n)$ with $a\in O_F$ and $b_1,\ldots,b_n\in O_F^*$.

First, we define a homomorphism $H^{n,n-1}(k)\to H^{n+1,n}_{\text{tame}}(F)/H^{n+1,n}_{\text{nr}}(F)$; it will be clear that the composition

$$H^{n,n-1}(k)\to H^{n+1,n}_{\text{tame}}(F)/H^{n+1,n}_{\text{nr}}(F)\to H^{n,n-1}(k)$$

is the identity. Namely, we map

$$\overline{a}\frac{d\overline{b_1}}{\overline{b_1}}\wedge\cdots\wedge\frac{d\overline{b_{n-1}}}{\overline{b_{n-1}}}\mapsto a\frac{dt}{t}\wedge\frac{db_1}{b_1}\wedge\cdots\wedge\frac{db_{n-1}}{b_{n-1}}\pmod{H^{n+1,n}_{\text{nr}}(F)},$$

where $a \in O_F$ and $b_1, \ldots, b_{n-1} \in O_F^*$.

As in previous arguments, it is straightforward to check that the resulting element of $H^{n+1,n}_{\text{tame}}(F)/H^{n+1,n}_{\text{nr}}(F)$ does not depend on the choice of lifts of $\overline{a} \in k$ and $\overline{b_1}, \ldots, \overline{b_{n-1}} \in k^*$ to O_F. For brevity, we just write this out for $\overline{a}$. Namely, changing the lift of $\overline{a}$ changes the element of $H^{n+1,n}(F)$ by an expression of the form $ct(dt/t) \wedge (db_1/b_1) \wedge \cdots \wedge (db_{n-1}/b_{n-1})$ with $c \in O_F$ and $b_1, \ldots, b_{n-1} \in O_F^*$. We rewrite that in $H^{n+1,n}(F)$ as:

$$\begin{aligned} ct\frac{dt}{t} \wedge \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{n-1}}{b_{n-1}} &= d\left[ct\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{n-1}}{b_{n-1}}\right] - t\, dc \wedge \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{n-1}}{b_{n-1}} \\ &= -ct\frac{dc}{c} \wedge \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{n-1}}{b_{n-1}}, \end{aligned}$$

using that exact forms represent zero in $H^{n+1,n}(F)$. If c is in O_F^*, then this element is unramified. Since O_F is additively generated by O_F^*, we find that the element above is always unramified, as we want.

Thus we have a well defined function from $k \times (k^*)^{n-1}$ to the quotient group $H^{n+1,n}_{\text{tame}}(F)/H^{n+1,n}_{\text{nr}}(F)$. It is clearly multilinear, and so it gives a homomorphism from the abelian group $k \otimes_{\mathbf{Z}} (k^*)^{\otimes n}$ to the latter quotient group. By Proposition 4.2, the homomorphism factors through Ω^{n-1}_k if the following elements map to zero:

$$[\overline{a}, \overline{b_1}, \ldots, \overline{b_{n-1}}\}$$

with $\overline{b_i} = \overline{b_j} \in k^*$ for some $i \neq j$, and

$$[\overline{u}+\overline{v}, \overline{u}+\overline{v}, \overline{b_2}, \ldots, \overline{b_{n-1}}\} - [\overline{u}, \overline{u}, \overline{b_2}, \ldots, \overline{b_{n-1}}\} - [\overline{v}, \overline{v}, \overline{b_2}, \ldots, \overline{b_{n-1}}\}$$

if $\overline{u}, \overline{v}, \overline{u}+\overline{v} \in k^*$. It is easy to check that these elements map to zero, by choosing suitable lifts (for example, take b_i to be equal to b_j when $\overline{b_i} = \overline{b_j}$ for some $i \neq j$).

Thus we have a well-defined homomorphism from Ω^{n-1}_k to the quotient group $H^{n+1,n}_{\text{tame}}(F)/H^{n+1,n}_{\text{nr}}(F)$. To show that the homomorphism vanishes on exact $(n-1)$-forms, it suffices to show that each element of the form $[\overline{a}, \overline{a}, \overline{b_2}, \ldots, \overline{b_{n-1}}\}$ maps to zero. Those elements map to zero by definition of the homomorphism, using that exact n-forms represent zero in $H^{n+1,n}(F)$. Finally, to show that the homomorphism factors through the quotient $H^{n,n-1}(k)$ of Ω^{n-1}_k, it suffices to show that $[\overline{a}^p - \overline{a}, \overline{b_1}, \ldots, \overline{b_{n-1}}\}$ maps to zero. That holds because forms $(a^p - a)(dt/t) \wedge (db_1/b_1) \wedge \cdots \wedge (db_{n-1}/b_{n-1})$ represent zero in $H^{n+1,n}(F)$.

Thus we have a well-defined homomorphism φ from $H^{n,n-1}(k)$ to the quotient group $H^{n+1,n}_{\text{tame}}(F)/H^{n+1,n}_{\text{nr}}(F)$. Composing this with the residue homomorphism from the latter group to $H^{n,n-1}(k)$ (discussed earlier) gives the identity. Therefore, φ is an isomorphism if it is surjective. To prove surjectivity, use that $H^{n+1,n}_{\text{tame}}(F)$ is generated by elements $a(db_1/b_1) \wedge \cdots \wedge (db_n/b_n)$ with $a \in O_F$ and $b_1, \ldots, b_n \in F^*$. Since $F^* = t^{\mathbf{Z}} \times O_F^*$, $H^{n+1,n}_{\text{tame}}(F)$ is in fact generated by elements $a(db_1/b_1) \wedge \cdots \wedge (db_n/b_n)$ and $a(dt/t) \wedge (db_2/b_2) \wedge \cdots \wedge (db_n/b_n)$ with $a \in O_F$ and $b_i \in O_F^*$. Elements of the first type are unramified, hence zero in $H^{n+1,n}_{\text{tame}}(F)/H^{n+1,n}_{\text{nr}}(F)$, and elements of the second type are in the image of φ. Thus φ is an isomorphism.

Finally, when F is henselian with respect to v, we want to show that $H^{n+1,n}_{\text{nr}}(F) \cong H^{n+1,n}(k)$. Here $H^{n+1,n}_{\text{nr}}(F)$ is the subgroup of $H^{n+1,n}(F)$ generated by elements of the form

$$a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}$$

with $a \in O_F$ and $b_1, \ldots, b_n \in O_F^*$. We want to show that the map $H^{n+1,n}(k) \to H^{n+1,n}_{\text{nr}}(F)$ given by the formula

$$\overline{a}\frac{d\overline{b_1}}{\overline{b_1}} \wedge \cdots \wedge \frac{d\overline{b_n}}{\overline{b_n}} \mapsto a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n},$$

for $a \in O_F$ and $b_1, \ldots, b_n \in O_F^*$, is defined and an isomorphism.

We first show that given $\overline{a} \in k$ and $\overline{b_1}, \ldots, \overline{b_n} \in k^*$, the choice of lifts to O_F does not affect the right side in $H^{n+1,n}_{\text{nr}}(F)$. The choice of lift a does not matter, because every element of $\mathfrak{m} \subset O_F$ can be written as $u^p - u$ for some $u \in F$, using that O_F is henselian [26, Theorem I.4.2(d')]. Next, the choice of lift b_1 (say) does not matter, because for $c_1 \neq 0 \in \mathfrak{m}$ and $b_1 = 1 + c_1$, with elements $a \in O_F$ and $b_2, \ldots, b_n \in O_F^*$,

$$a\frac{d(1+c_1)}{1+c_1} \wedge \frac{db_2}{b_2} \wedge \cdots \wedge \frac{db_n}{b_n} = \frac{ac_1}{1+c_1}\frac{dc_1}{c_1} \wedge \frac{db_2}{b_2} \wedge \cdots \wedge \frac{db_n}{b_n},$$

which is zero in $H^{n+1,n}_{\text{nr}}(F)$ because $ac_1/(1+c_1)$ is in $\mathfrak{m}$ and hence can be written as $u^p - u$ for some $u \in F$.

Thus the formula above gives a well-defined homomorphism $\varphi\colon k \otimes_{\mathbf{Z}} (k^*)^{\otimes n} \to H^{n+1,n}_{\text{nr}}(F)$. Since $(du/u) \wedge (du/u) = 0$ and

$$(u+v)\frac{d(u+v)}{u+v} = u\frac{du}{u} + v\frac{dv}{v}$$

for $u, v, u+v \in F^*$, Proposition 4.2 gives that φ passes to a homomorphism $\Omega^n_k \to H^{n+1,n}_{\text{nr}}(F)$. Clearly φ takes exact forms to exact forms, hence to zero in $H^{n+1,n}(F) = \text{coker}(\mathcal{P}\colon \Omega^n_F \to \Omega^n_F/d\Omega^{n-1}_F)$. Thus φ passes to a homomorphism $\Omega^n_k/d\Omega^{n-1}_k \to H^{n+1,n}_{\text{nr}}(F)$. Finally, φ takes differential forms in the image of $\mathcal{P}$ over k to differential forms in the image of $\mathcal{P}$ over F, and so φ passes to a well-defined homomorphism $H^{n+1,n}(k) \to H^{n+1,n}_{\text{nr}}(F)$. This is surjective by definition. Injectivity follows from Izhboldin's result that the composed map to $H^{n+1,n}_{\text{nr}}$ of the completion F_v is an isomorphism [15, Corollary 2.7]. ∎

Finally, we state Izhboldin's calculation of the mod p étale motivic cohomology of the rational function field in one variable over any field of characteristic p [15, Theorem 4.5]. For example, this result gives the p-torsion in the Brauer group of $k(t)$, generalizing the Faddeev exact sequence (which addresses the special case where k is perfect) [11, Corollary 6.4.6]. Our terminology is slightly different from Izhboldin's, but the translation is straightforward.

Theorem 4.4. *Let k be a field of characteristic $p > 0$, and let n be a natural number. Let S be the set of closed points in $\mathbf{P}^1_k$. For $v \in S$, write $k(t)_v$ for the completion of the field $k(\mathbf{P}^1) = k(t)$ at v. Then:*

(1) The natural homomorphism

$$H^{n+1,n}(k(t)) \to \bigoplus_{v \in S} H^{n+1,n}(k(t)_v)/H^{n+1,n}_{\text{tame}}(k(t)_v)$$

is surjective. The wild quotients on the right are described by Theorem 4.3.

(2) The kernel of that surjection, which we call $H^{n+1,n}_{\text{tame over }\mathbf{P}^1}(k(t))$, fits into an exact sequence:

$$0 \to H^{n+1,n}(k) \to H^{n+1,n}_{\text{tame over }\mathbf{P}^1}(k(t)) \to \bigoplus_{v\in S} H^{n,n-1}(k(v)) \to H^{n,n-1}(k) \to 0.$$

Here $k(v)$ denotes the residue field of $\mathbf{P}^1_k$ at a closed point v, and the homomorphism to $H^{n,n-1}(k(v))$ is the residue defined above.

5 Finite étale group schemes

Reichstein and Vistoli showed that every finite étale group scheme G over a field k of characteristic $p > 0$ has essential dimension at p at most 1 [28, Theorem 1]. By Proposition 2.2, it follows that the mod p cohomological invariants of G are nearly trivial when k is algebraically closed. In this section, we generalize that statement to perfect base fields (Theorem 5.2).

In particular, this result applies to an abstract finite group, viewed as a group scheme over k. By contrast, more general finite group schemes can have richer mod p cohomological invariants. It would be interesting to find out how far the results of this section extend to imperfect fields; see section 10 for the case of the symmetric groups.

Theorem 5.1. *Let G be a smooth affine group over a field k of characteristic $p > 0$. For any $m \geq 0$, all invariants of G over k with values in $H^{m,m}$ are constant. That is, $\mathrm{Inv}^{m,m}_k(G) = H^{m,m}(k)$.*

Proof. Let α be a normalized invariant for G of degree (m,m). Let E be any G-torsor over a field F/k; we want to show that $\alpha(E) = 0$. Since G_F is smooth over F, E becomes trivial over the separable closure F_s. Therefore, the image of $\alpha(E)$ in $H^{m,m}(F_s)$ is zero. But $H^{m,m}(F) \to H^{m,m}(F_s)$ is injective, by Bloch and Kato's isomorphism $H^{m,m}(F) \cong \Omega^m_{\log,F} \subset \Omega^m_F$ (discussed in section 2). So $\alpha(E) = 0$, as we want. □

In particular, finite étale group schemes have no normalized mod p cohomological invariants of bidegree (m,m). We now check that this is also true (over a perfect base field) in the other possible bidegrees, $(m+1,m)$, except for bidegree $(1,0)$ (which is described in Theorem 15.1).

Theorem 5.2. *Let G be a finite étale group scheme over a perfect field k of characteristic $p > 0$. Then $\mathrm{Inv}^{m+1,m}_k(G) = 0$ for all $m \geq 1$.*

Proof. For a scheme X over k, write F_1 for the relative Frobenius $X \to X_1$, which is a morphism of k-schemes. Since G is finite étale over k, $F_1 \colon G \to G_1$ is an isomorphism of k-group schemes. So every invariant u in $\mathrm{Inv}^{m+1,m}_k(G)$ is the pullback via F_1 of some invariant v in $\mathrm{Inv}^{m+1,m}_k(G_1)$.

Let V be a faithful representation of G over k. Then the open subset U of V on which G acts freely is nonempty, and there is a quotient variety U/G over k. Consider the relative Frobenius morphism $F_1 \colon U/G \to (U/G)_1 = U_1/G_1$. A point of U/G over a field E/k determines a G-torsor over E, and the image of that point in U_1/G_1 determines the associated G_1-torsor over E. By the previous paragraph, the given element $u \in H^{m+1,m}(k(U/G))$ is the pullback of $v \in H^{m+1,m}(k(U_1/G_1))$.

Since k is perfect, we can identify $F_1 \colon U/G \to U_1/G_1$ (as a morphism of schemes, not of schemes over k) with the absolute Frobenius morphism on U/G. As a result, the pullback $F_1^* \colon k(U_1/G_1) \to k(U/G)$ sends $k(U_1/G_1)^*$ into $(k(U/G)^*)^p$. Therefore, F_1^* acts by zero on $H^{m+1,m}(F)$ for all $m \geq 1$, by the interpretation in terms of differential forms (section 2): the pullback of a form db/b is of the form $d(c^p)/c^p = 0$.

Since $m \geq 1$, it follows that the element u (being a pullback from U_1/G_1) is zero in $H^{m+1,m}(k(U/G))$. Since the G-torsor over U/G is versal (section 2), we have shown that every element of $\mathrm{Inv}_k^{m+1,m}(G)$ is zero. □

6 Invariants of μ_p in $H^{n+1,n}$

In this section, we use Theorem 4.3 to compute the cohomological invariants in $H^{n+1,n}$ of the group scheme μ_p of pth roots of unity over any field of characteristic p. This is crucial for the rest of the paper. More generally, we find the invariants for the product of μ_p with any group scheme.

Proposition 6.1. *Let k be a field of characteristic $p > 0$, and let n be a natural number. Then*

$$\mathrm{Inv}_k^{n+1,n}(\mu_p) \cong H^{n+1,n}(k) \oplus H^{n,n-1}(k).$$

In more detail, every invariant for μ_p over k with values in $H^{n+1,n}$ has the form

$$u(\alpha) = v + w\alpha$$

for some $v \in H^{n+1,n}(k)$ and $w \in H^{n,n-1}(k)$. Here α denotes any μ_p-torsor over a field F/k, and we use the identification $H^1(F, \mu_p) \cong H^{1,1}(F)$.

Proof. Let u be an invariant for μ_p over k with values in $H^{n+1,n}$. Let $\{t\}$ denote the μ_p-torsor over the field $k(t)$ associated to $t \in k(t)^*/(k(t)^*)^p \cong H^1(k(t), \mu_p)$. Then u gives an element $u(\{t\}) \in H^{n+1,n}(k(t))$. Here $\{t\}$ is a versal torsor for μ_p, corresponding to the μ_p-torsor $W \to W/\mu_p$ where $W = \mathbf{A}_k^1 - 0$, by section 2. So u is determined by the element $u(\{t\})$ in $H^{n+1,n}(k(t))$.

We know that $u(\{t\})$ is unramified on $W/\mu_p \cong \mathbf{A}_k^1 - 0$ by Theorem 2.1. Let us show that it is also tamely ramified at $t = 0$ in $\mathbf{P}_k^1$; the same argument gives that $u(\{t\})$ is tamely ramified at $t = \infty$. If $u(\{t\})$ is not tamely ramified at $t = 0$, then $u(\{t\})$ is in $U_j - U_{j-1}$ for some $j > 0$, with respect to the valuation $t = 0$ on $k(t)$, in the notation of section 4. Suppose first that $p \nmid j$; then, by Theorem 4.3, we can write

$$u(\{t\}) = \sum \frac{a}{t^j} \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} \pmod{U_{j-1}}$$

with a in the local ring $O_{\mathbf{A}^1,0}$ and $b_1, \ldots, b_n \in O^*_{\mathbf{A}^1,0}$. (The expression is meant to indicate a finite sum with the value of b_1 in one summand not necessarily equal to that of b_1 in another summand, and likewise for all the variables $a, b_1, \ldots, b_n$.)

We know that $u(\{t\})$ is balanced, meaning that its pullback by the two morphisms $(W \times W)/\mu_p \to W/\mu_p$ are equal (Theorem 2.1). We can identify the function field of $(W \times W)/\mu_p$ with the rational function field $k(x, y)$, and balancedness means that $u(\{x^py\}) = u(\{y\})$. (This is clear directly, since x^py and y define isomorphic μ_p-torsors over $k(x,y)$.) So we must have

$$\sum \frac{a(x^py)}{x^{pj}y^j}\frac{db_1(x^py)}{b_1(x^py)} \wedge \cdots \wedge \frac{db_n(x^py)}{b_n(x^py)} = \sum \frac{a(y)}{y^j}\frac{db_1(y)}{b_1(y)} \wedge \cdots \wedge \frac{db_n(y)}{b_n(y)}$$

in $H^{n+1,n}(k(x,y))$. The element on the right is clearly unramified along the divisor $x=0$ in $\mathbf{A}^2_k = \text{Spec } k[x,y]$, and so the element on the left is also unramified along $x=0$. That element is visibly in U_{pj} with respect to the valuation $x=0$, and so its class in U_{pj}/U_{pj-1} must be zero. Since the residue field for that valuation on $k(x,y)$ is $k(y)$, Theorem 4.3 gives that the form

$$\sum \frac{a(0)}{y^j}\frac{db_1(0)}{b_1(0)} \wedge \cdots \wedge \frac{db_n(0)}{b_n(0)}$$

in $\Omega^n_{k(y)}$ is closed. That is,

$$0 = \sum \frac{1}{y^j} da(0) \wedge \frac{db_1(0)}{b_1(0)} \wedge \cdots \wedge \frac{db_n(0)}{b_n(0)} - j\sum \frac{a(0)}{y^{j+1}} dy \wedge \frac{db_1(0)}{b_1(0)} \wedge \cdots \wedge \frac{db_n(0)}{b_n(0)}$$

in $\Omega^{n+1}_{k(y)}$. The differential forms on $k(y)$ are easy to describe:

$$\Omega^{n+1}_{k(y)} \cong \left[k(y) \otimes_k \Omega^{n+1}_k\right] \oplus \left[dy \cdot k(y) \otimes_k \Omega^n_k\right].$$

So both sums in the expression above must be zero. Since we are assuming that $p \nmid j$, it follows that both $\sum da(0) \wedge (db_1(0)/b_1(0)) \wedge \cdots$ in Ω^{n+1}_k and $\sum a(0)(db_1(0)/b_1(0)) \wedge \cdots$ in Ω^n_k are zero. The second statement means that the element $u(\{t\}) \in U_j = U_j(k(t))$ is actually in U_{j-1}, contradicting our assumption.

Now suppose that $u(\{t\})$ is in $U_j - U_{j-1}$ (with respect to the valuation $t=0$ on $k(t)$) with $j > 0$ and $p | j$. Because $k(t)^* = t^{\mathbf{Z}} \times O^*_{\mathbf{A}^1,0}$, we can write $u(\{t\})$ as a sum of two types of terms:

$$u(\{t\}) = \sum \frac{a}{t^j}\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} + \sum \frac{e}{t^j}\frac{dt}{t} \wedge \frac{dc_1}{c_1} \wedge \cdots \wedge \frac{dc_n}{c_{n-1}}$$

with $a(t)$ and $e(t)$ in $O_{\mathbf{A}^1,0}$ and $b_i(t)$ and $c_i(t)$ in $O^*_{\mathbf{A}^1,0}$.

As in the previous argument, the elements x^py and y in $k(x,y)^*$ determine isomorphic μ_p-torsors over $k(x,y)$, and so the pullbacks of $u(\{t\})$ to $H^{n+1,n}(k(x,y))$ by $t=y$ and $t=x^py$ must be equal. The first pullback is clearly unramified along the divisor $x=0$ in $\mathbf{A}^2_k = \text{Spec } k[x,y]$, and so the second pullback must also be. That is,

$$\sum \frac{a(x^py)}{x^{pj}y^j}\frac{db_1(x^py)}{b_1(x^py)} \wedge \cdots \wedge \frac{db_n(x^py)}{b_n(x^py)} + \sum \frac{e(x^py)}{x^{pj}y^j}\frac{dy}{y} \wedge \frac{dc_1(x^py)}{c_1(x^py)} \wedge \cdots \wedge \frac{dc_{n-1}(x^py)}{c_{n-1}(x^py)}$$

in $H^{n+1,n}(k(x,y))$ is unramified along $x=0$. It is visibly in U_{pj} with respect to the valuation $x=0$, and so its class in U_{pj}/U_{pj-1} must be zero. By Theorem 4.3, this means that the form

$$\sum \frac{a(0)}{y^j}\frac{db_1(0)}{b_1(0)} \wedge \cdots \wedge \frac{db_n(0)}{b_n(0)} + \sum \frac{e(0)}{y^j}\frac{dy}{y} \wedge \frac{dc_1(0)}{c_1(0)} \wedge \cdots \wedge \frac{dc_{n-1}(0)}{c_{n-1}(0)}$$

in $\Omega^n_{k(y)}$ is closed. That is, using that $p|j$,

$$0 = \sum \frac{1}{y^j} da(0) \wedge \frac{db_1(0)}{b_1(0)} \wedge \cdots \wedge \frac{db_n(0)}{b_n(0)} + \sum \frac{1}{y^j} de(0) \wedge \frac{dy}{y} \wedge \frac{dc_1(0)}{c_1(0)} \wedge \cdots \wedge \frac{dc_{n-1}(0)}{c_{n-1}(0)}$$

in $\Omega^{n+1}_{k(y)}$. Since

$$\Omega^{n+1}_{k(y)} \cong \left[k(y) \otimes_k \Omega^{n+1}_k \right] \oplus \left[dy \cdot k(y) \otimes_k \Omega^n_k \right],$$

it follows that the form $\sum da(0) \wedge (db_1(0)/b_1(0)) \wedge \cdots$ is zero in Ω^{n+1}_k and $\sum de(0) \wedge (dc_1(0)/c_1(0)) \wedge \cdots$ is zero in Ω^n_k. That is, $\sum a(0) \wedge (db_1(0)/b_1(0)) \wedge \cdots$ in Ω^n_k is closed, and $\sum e(0) \wedge (dc_1(0)/c_1(0)) \wedge \cdots$ in Ω^{n-1}_k is closed. Since $p|j$, this says exactly (by Theorem 4.3) that the element

$$u(\{t\}) = \sum \frac{a}{t^j} \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} + \sum \frac{e}{t^j} \frac{dt}{t} \wedge \frac{dc_1}{c_1} \wedge \cdots \wedge \frac{dc_{n-1}}{c_{n-1}}$$

in $H^{n+1,n}(k(t))$ is zero in U_j/U_{j-1}, contradicting our assumption.

Thus we have shown that $u(\{t\})$ in $H^{n+1}(k(t))$ is tamely ramified at $t = 0$ in $\mathbf{P}^1_k$. By the same argument, it is tamely ramified at $t = \infty$. By Theorem 4.4, the subgroup of elements of $H^{n+1,n}(k(t))$ that are unramified on $\mathbf{A}^1_k - 0$ and tamely ramified at 0 and ∞ is isomorphic to $H^{n+1,n}(k) \oplus H^{n,n-1}(k)$. Thus $\mathrm{Inv}^{n+1,n}_k(\mu_p)$ injects into that direct sum. Since we already know invariants for μ_p that give all elements of that direct sum, we have

$$\mathrm{Inv}^{n+1,n}_k(\mu_p) = H^{n+1,n}(k) \oplus H^{n,n-1}(k).$$

∎

Proposition 6.2. *Let H be an affine group scheme of finite type over a field k, and let n be a natural number. Then*

$$\mathrm{Inv}^{n+1,n}_k(\mu_p \times H) \cong \mathrm{Inv}^{n+1,n}_k(H) \oplus \mathrm{Inv}^{n,n-1}_k(H).$$

Explicitly, every invariant for $\mu_p \times H$ over k with values in $H^{n+1,n}$ has the form $u(\alpha, \beta) = v(\beta) + w(\beta)\alpha$ for some invariants v of H in $H^{n+1,n}$ and w of H in $H^{n,n-1}$. Here α denotes any μ_p-torsor over a field F/k, and we use the identification $H^1(F, \mu_p) \cong H^{1,1}(F)$.

Proof. Let V be a k-vector space on which H acts by affine transformations, and suppose that H acts freely on a nonempty open subset U of V and the quotient scheme U/H exists. (Such pairs (V, U) do exist [34, Remark 2.7].)

Let u be an invariant of $\mu_p \times H$ over k with values in $H^{n+1,n}$. For any field L over k and any H-torsor β over L, we get an invariant u_β of μ_p over L with values in $H^{n+1,n}$ by defining

$$u_\beta(\alpha) = u(\alpha, \beta)$$

for any μ_p-torsor α over an extension field of L. By Proposition 6.1, there are unique elements $v \in H^{n+1,n}(L)$ and $w \in H^{n,n-1}(L)$ such that $u_\beta(\alpha) = v + w\alpha$ for all μ_p-torsors α over fields over L. Here we are identifying $H^1(E, \mu_p)$ with $H^{1,1}(E)$, for fields E over L.

By that uniqueness, v and w are invariants of H-torsors β on fields over k. These invariants satisfy (and are characterized uniquely by): for every $(\mu_p \times H)$-torsor (α, β) on a field E over k,

$$u(\alpha, \beta) = v(\beta) + w(\beta)\alpha.$$

Thus every invariant for $\mu_p \times H$ has this form, with the invariants v and w uniquely determined. Conversely, for any invariants v and w for H over k, the formula above defines an invariant for $\mu_p \times H$. Thus we have shown that

$$\mathrm{Inv}_k^{n+1,n}(\mu_p \times H) \cong \mathrm{Inv}_k^{n+1,n}(H) \oplus \mathrm{Inv}_k^{n,n-1}(H).$$

□

7 Invariants of μ_p in $H^{n,n}$

As part of Vial's determination of the operations on Milnor K-theory of fields, he computed the invariants for μ_p with values in $H^{n,n}$. In this section, we use Vial's result to compute the invariants of $\mu_p \times H$ in $H^{n,n}$ for any group scheme H.

Vial's result is as follows [36, Theorem 3.4].

Theorem 7.1. *Let k be a field of characteristic $p > 0$, and let n be a natural number. Then*

$$\mathrm{Inv}_k^{n,n}(\mu_p) \cong H^{n,n}(k) \oplus H^{n-1,n-1}(k).$$

In more detail, for $v \in H^{n,n}(k)$ and $w \in H^{n-1,n-1}(k)$, the corresponding invariant of a μ_p-torsor α over a field F/k is $v + w\alpha$, where $\alpha \in H^1(F, \mu_p) = H^{1,1}(F)$.

From there, we can compute the invariants of $\mu_p \times H$ in $H^{n,n}$ for any group scheme H.

Proposition 7.2. *Let H be an affine group scheme of finite type over a field k. Then*

$$\mathrm{Inv}_k^{n,n}(\mu_p \times H) \cong \mathrm{Inv}_k^{n,n}(H) \oplus \mathrm{Inv}_k^{n-1,n-1}(H).$$

Explicitly, every invariant for $\mu_p \times H$ over k with values in $H^{n,n}$ has the form $u(\alpha, \beta) = v(\beta) + w(\beta)\alpha$ for some invariants v of H in $H^{n,n}$ and w of H in $H^{n-1,n-1}$. Here α denotes any μ_p-torsor over a field F/k, and we use the identification $H^1(F, \mu_p) \cong H^{1,1}(F)$.

We omit the proof, as it is identical to that of Proposition 6.2, using Theorem 7.1 on invariants of μ_p in $H^{n,n}$ in place of Proposition 6.1 on invariants on μ_p in $H^{n+1,n}$.

8 Invariants of $\mathbf{Z}/p$ in $H^{n+1,n}$

In this and the next section, we find the cohomological invariants of $\mathbf{Z}/p$. When k is perfect, this was mostly done in Theorem 5.2. Here we consider any field of characteristic p, as is needed for inductive arguments. More generally, we find the invariants for the product of $\mathbf{Z}/p$ with any group scheme. Combining this with Proposition 6.2, we determine all invariants of the group scheme $(\mathbf{Z}/p)^r \times (\mu_p)^s$ (Theorem 8.4).

Proposition 8.1. *Let k be a field of characteristic $p > 0$, and let n be a natural number. Then*

$$\mathrm{Inv}_k^{n+1,n}(\mathbf{Z}/p) \cong H^{n+1,n}(k) \oplus H^{n,n}(k).$$

In more detail, every invariant for $\mathbf{Z}/p$ over k with values in $H^{n+1,n}$ has the form $u(\alpha) = v + w\alpha$ for some $v \in H^{n+1,n}(k)$ and $w \in H^{n,n}(k)$. Here α denotes the class of any $\mathbf{Z}/p$-torsor over a field F/k, and we use the identification $H^1(F, \mathbf{Z}/p) \cong H^{1,0}(F)$.

Proof. Let $G = \mathbf{Z}/p$ act freely on the affine line U over k by translations. Then $U \to U/G \cong \mathbf{A}^1$ is a versal torsor ξ for G. Let u be a normalized cohomological invariant for G over k with values in $H^{n+1,n}$; then u is determined by $u(\xi)$ in $H^{n+1,n}(k(U/G)) = H^{n+1,n}(k(t))$.

Since the G-torsor ξ over U/G pulls back to a trivial torsor over U, $u(\xi)$ in $H^{n+1,n}(k(U/G))$ pulls back to zero in $H^{n+1,n}(k(U))$. We now use the following result of Izhboldin's [14, Theorem B].

Theorem 8.2. *Let F be a field of characteristic $p > 0$, and let E/F be a cyclic extension of degree p. Then the sequence*

$$H^{n,n}(F) \to H^{n+1,n}(F) \to H^{n+1,n}(E)$$

is exact. Here the second homomorphism is the obvious pullback, and the first homomorphism is the product with the class of E/F in $H^{1,0}(F)$.

It follows that $u(\xi) = [t]v$ for some v in $H^{n,n}(k(t)) \cong \Omega^n_{\log,k(t)}$. (Here we use that the $\mathbf{Z}/p$-covering $U \to U/G \cong \mathbf{A}^1_k$ corresponds to the element $t \in k(t)/\mathcal{P}(k(t)) \cong H^{1,0}(k(t))$.) In the description of $H^{n+1,n}(k(t))$ by differential forms (section 2), it follows that $u(\xi)$ is a sum $\sum t(da_1/a_1) \wedge \cdots \wedge (da_n/a_n)$ with $a_1, \ldots, a_n \in k(t)^*$. In coordinates $y = 1/t$, this says that $u(\xi) = \sum(1/y)(da_1/a_1) \wedge \cdots \wedge (da_n/a_n)$ with $a_i \in k(y)^*$. Because $1/y$ has only a simple pole at $y = 0$, we can bound how ramified $u(\xi)$ is at the point $y = 0$ (corresponding to $t = \infty$) in $\mathbf{P}^1_k$. Namely, in the notation of section 4, $u(\xi)$ is in U_1 with respect to the valuation $y = 0$ on $k(y) = k(t)$.

Using that $k(y)^* = y^{\mathbf{Z}} \times O^*_{\mathbf{A}^1_y,0}$, we can rewrite $u(\xi)$ as

$$u(\xi) = \sum \frac{1}{y}\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} + \sum \frac{1}{y}\frac{dy}{y} \wedge \frac{dc_1}{c_1} \wedge \cdots \wedge \frac{dc_{n-1}}{c_n},$$

with b_i, c_i units at $y = 0$. The forms in the second sum here are exact, being equal to $d(-(1/y)(dc_1/c_1) \wedge \cdots \wedge (dc_{n-1}/c_{n-1}))$. So $u(\xi)$ in $H^{n+1,n}(k(y))$ is represented by the form $\sum(1/y)(db_1/b_1) \wedge \cdots \wedge (db_n/b_n)$ with $b_i \in O^*_{\mathbf{A}^1_y,0}$. By the formula for the isomorphism $U_1/U_0 \cong \Omega^1_k$ (Theorem 4.3) associated to the choice of uniformizer y, it follows that the class of $u(\xi)$ in $U_1/U_0 \cong \Omega^n_k$ is in $\Omega^n_{\log,k} \cong H^{n,n}(k)$.

We know that each element u of $H^{n,n}(k)$ gives a normalized cohomological invariant for $G = \mathbf{Z}/p$ over k with values in $H^{n+1,n}$, by the product $H^1_{\text{ét}}(k, G) \times H^{n,n}(k) \to H^{n+1,n}(k)$. It is immediate that $u(\xi)$ in $H^{n+1,n}(k(t))$ has class (with respect to the valuation $t = \infty$) in U_1/U_0 equal to u. So, by subtracting off an invariant of this form, we can assume that our normalized invariant u has the property that $u(\xi)$ in $H^{n+1,n}(k(t))$ has class in U_1/U_0 (at $t = \infty$) equal to zero. Equivalently, $u(\xi)$ is tamely ramified at $t = \infty$. We want to show that a normalized invariant with this property is zero.

We know that $u(\xi)$ in $H^{n+1,n}(k(t))$ is unramified over $U/G = \mathbf{A}^1_k = \mathrm{Spec}\ k[t]$, by Theorem 2.1. By Theorem 4.4, since $u(\xi)$ is unramified on $\mathbf{A}^1_k$ and tamely ramified at $t = \infty$, it is in fact unramified on all of $\mathbf{P}^1_k$ and comes from an element of $H^{n+1,n}(k)$. But we took u to be a normalized invariant, and so $u(\xi)$ pulls back to zero in $H^{n+1,n}(k(U))$, whereas pullback to $H^{n+1,n}(k(U))$ has trivial kernel on the subgroup $H^{n+1,n}(k) \subset H^{n+1,n}(k(U/G))$. So $u(\xi) = 0$ and hence $u = 0$. Thus the only invariants for $G = \mathbf{Z}/p$ are those listed. $\square$

Proposition 8.3. *Let H be an affine group scheme of finite type over a field k of characteristic $p > 0$. Then*

$$\mathrm{Inv}^{n+1,n}(\mathbf{Z}/p \times H) \cong \mathrm{Inv}^{n+1,n}_k(H) \oplus \mathrm{Inv}^{n,n}_k(H).$$

Explicitly, every invariant for $\mathbf{Z}/p \times H$ over k with values in $H^{n+1,n}$ has the form $u(\alpha, \beta) = v(\beta) + w(\beta)\alpha$ for some invariants v of H in $H^{n+1,n}$ and w of H in $H^{n,n}$. Here α denotes the class of any $\mathbf{Z}/p$-torsor over a field F/k, and we use the identification $H^1(F, \mathbf{Z}/p) \cong H^{1,0}(F)$.

We omit the proof, as it is identical to that of Proposition 6.2, using Proposition 8.1 on invariants of $\mathbf{Z}/p$ in place of Proposition 6.1 on invariants on μ_p.

Combining several earlier results, we now compute all cohomological invariants of the group scheme $(\mathbf{Z}/p)^r \times (\mu_p)^s$.

Theorem 8.4. *Let k be a field of characteristic $p > 0$, and let r, s, n be natural numbers. Then every cohomological invariant for $(\mathbf{Z}/p)^r \times (\mu_p)^s$ over k with values in $H^{n+1,n}$ is of the form*

$$u([a_1], \ldots, [a_r], \{b_1\}, \ldots, \{b_s\}) = \sum_{I \subset \{1,\ldots,s\}} c_I \prod_{i \in I} \{b_i\} + \sum_{j=1}^{r} [a_j] \sum_{I \subset \{1,\ldots,s\}} e_{j,I} \prod_{i \in I} \{b_i\}$$

for some (unique) elements c_I in $H^{n-|I|+1,n-|I|}(k)$ and $e_{j,I}$ in $H^{n-|I|,n-|I|}(k)$. That is,

$$\mathrm{Inv}_k((\mathbf{Z}/p)^r \times (\mu_p)^s) \cong \bigoplus_{I \subset \{1,\ldots,s\}} H^{n-|I|+1,n-|I|}(k) \oplus \bigoplus_{j=1}^{r} \bigoplus_{I \subset \{1,\ldots,s\}} H^{n-|I|,n-|I|}(k).$$

Proof. The group scheme $(\mathbf{Z}/p)^r$ is smooth over k, and so all its invariants in $H^{n,n}$ are constant (Theorem 5.1). Applying Proposition 8.3 (on products with $\mathbf{Z}/p$), we find that

$$\mathrm{Inv}^{n+1,n}_k((\mathbf{Z}/p)^r) \cong H^{n+1,n}(k) \oplus \bigoplus_{j=1}^{r} H^{n,n}(k).$$

Applying Proposition 6.2 (on products with μ_p) gives the invariants for $(\mathbf{Z}/p)^r \times (\mu_p)^s$. $\square$

9 Invariants of $\mathbf{Z}/p$ times any group in $H^{n,n}$

By Theorem 5.1, the normalized invariants of any smooth k-group with values in $H^{n,n}$ are zero. In particular, this applies to $\mathbf{Z}/p$ as a group over k. In this section, we compute the invariants in $H^{n,n}$ for the product of $\mathbf{Z}/p$ with any group scheme H. More generally, we can handle $G \times H$ for any smooth k-group G.

Proposition 9.1. *Let k be a field of characteristic $p > 0$. Let G and H be affine k-group schemes of finite type over k with G smooth over k. Then*

$$\mathrm{Inv}_k^{n,n}(G \times H) \cong \mathrm{Inv}_k^{n,n}(H).$$

The proof is identical to that of Proposition 6.2, starting from the fact that $\mathrm{Inv}_F^{n,n}(G) \cong H^{n,n}(F)$ for every field F over k (Theorem 5.1).

10 Symmetric groups

For all finite groups (as opposed to more general finite group schemes), it may be possible to determine the mod p cohomological invariants over all fields of characteristic p, not just perfect fields as in Theorem 5.2. Perhaps all invariants come from the abelianization of the group. We now prove this in the case of the symmetric groups.

Equivalently, we determine the cohomological invariants of étale algebras in characteristic 2. There are analogies with Serre's calculation in characteristic not 2. Regardless of the characteristic, all invariants of étale algebras with odd-primary coefficients are constant (by Theorem 10.1 and [9, section 24]). Over a field k of characteristic not 2, $\mathrm{Inv}_k^*(S_n, \mathbf{F}_2)$ is a free module over $H^*(k, \mathbf{F}_2)$ with basis $1 = w_0, w_1, \ldots, w_m$, where $m = \lfloor n/2 \rfloor$ [9, Theorem 25.13]. Here the elements w_i are the Stiefel-Whitney classes of the trace form $\mathrm{tr}(xy)$ associated to an étale algebra. For k of characteristic 2, Theorem 10.2 says that there is only an analog of w_1.

Theorem 10.1. *Let k be a field of characteristic $p > 2$, and let n be a positive integer. For each integer r, every invariant of the symmetric group S_n over k with values in mod p étale motivic cohomology ($H^{r,r}$ or $H^{r+1,r}$) is constant.*

Proof. For $H^{r,r}$, this follows from Theorem 5.1. So let u be an invariant for S_n over k with values in $H^{r+1,r}$. Let V be the standard representation of S_n, of dimension n over k. Then u gives an element of $H^{r+1,r}(k(V/S_n))$, and u is determined by this element, by Theorem 2.1.

The action of S_n of V extends to the permutation action of S_n on $X = (\mathbf{P}^1)^n$ over k, with $X/S_n \cong \mathbf{P}^n$. The group S_n acts freely on X outside the union of the $\binom{n}{2}$ irreducible divisors $x_i = x_j$ in X, where $1 \leq i < j \leq n$. These divisors are permuted transitively by S_n, and so the morphism $X \to X/S_n = \mathbf{P}^n$ is ramified only over one irreducible divisor, the discriminant $\Delta \subset \mathbf{P}^n$.

By Theorem 2.1, using that X is a compactification of a representation of S_n, the element $u \in H^{r+1,r}(k(X/S_n))$ is unramified outside the divisor Δ. Likewise, the alternating group A_n acts freely on $X = (\mathbf{P}^1)^n$ outside a closed subset of codimension at least 2, and so the pullback of u to $H^{r+1,r}(k(X/A_n))$ is unramified along every irreducible divisor in X/A_n.

Since p is odd and the class u pulls back to an unramified class by the double cover $X/A_n \to X/S_n$, u is in fact unramified along every irreducible divisor in $X/S_n \cong \mathbf{P}^n$. (This follows from the description of $H^{r+1,r}(k(X/S_n))/H^{r+1,r}_{\mathrm{nr}}(k(X/S_n))$ in Theorem 4.3, where "nr" denotes the subgroup of classes unramified along Δ. Use that a uniformizer t (the discriminant polynomial) in $k(X/S_n)$ along Δ pulls back in $k(X/A_n)$ to u^2, for some uniformizer u along the inverse image of Δ.)

By Theorem 4.4, every unramified cohomology class in $H^{r+1,r}$ of the function field of $\mathbf{P}^1$ over a field k_0 is pulled back from a unique class on k_0. Applying this repeatedly gives the same statement on the function field of $\mathbf{P}^n$. It follows that the class u is pulled back from $H^{r+1,r}(k)$. Thus u is constant as an invariant of S_n. $\square$

Theorem 10.2. *Let k be a field of characteristic 2, and let $n \geq 2$. For each integer r, every invariant of the symmetric group S_n over k with values in $H^{r,r}$ is constant. Also, the group of invariants with values in $H^{r+1,r}$ is $H^{r+1,r}(k) \oplus H^{r,r}(k)$. Every invariant for S_n over k in $H^{r+1,r}$ has the form*

$$u(x) = c + \operatorname{disc}(x)e$$

for some (unique) $c \in H^{r+1,r}(k)$ and $e \in H^{r,r}(k)$. Here $\operatorname{disc}(x)$ is the invariant of S_n in $H^{1,0}$ corresponding to the sign homomorphism $S_n \to \mathbf{Z}/2$.

Proof. Every invariant of S_n with values in $H^{r,r}$ is constant by Theorem 5.1. We now consider invariants in $H^{r+1,r}$. I claim that the restriction

$$\operatorname{NormInv}_k^{r+1,r}(S_n) \to \operatorname{NormInv}_k^{r+1,r}(S_2 \times S_{n-2})$$

is injective. Indeed, let u be a normalized invariant for S_n that restricts to 0 as an invariant of $S_2 \times S_{n-2}$. As in the proof of Theorem 10.1, consider the action of $G = S_n$ on $X = (\mathbf{P}^1)^n$. We know that u is determined by its class in $H^{r+1,r}(k(X/S_n))$, and that this class is unramified outside the discriminant divisor Δ in $X/S_n \cong \mathbf{P}^n$.

We are given that u pulls back to 0 in $H^{r+1,r}(k(X/(S_2 \times S_{n-2})))$. The point is that S_n acts transitively on the set of divisors $x_i = x_j$ in $X = (\mathbf{P}^1)^n$, and the stabilizer subgroup of the divisor $x_1 = x_2$ is $S_2 \times S_{n-2}$. As a result, the map $X/(S_2 \times S_{n-2}) \to X/S_n$ splits completely over Δ; that is, the completions of the two function fields along the corresponding divisors are isomorphic. It follows that $u \in H^{r+1,r}(k(X/S_n))$ is unramified along Δ. Since u is also unramified along all other irreducible divisors in $\mathbf{P}^n_k$, u is pulled back from $H^{r+1,r}(k)$. Since u pulls back to 0 as an invariant of $S_2 \times S_{n-2}$, u is equal to 0 in $H^{r+1,r}(k)$, as we want.

By Proposition 8.3, we have $\operatorname{NormInv}_k^{r+1,r}(S_2 \times S_{n-2}) \cong \operatorname{NormInv}_k^{r+1,r}(S_{n-2}) \oplus \operatorname{Inv}_k^{r,r}(S_{n-2})$. Since S_{n-2} is smooth over k, $\operatorname{Inv}_k^{r,r}(S_{n-2})$ is isomorphic to $H^{r,r}(k)$ by Theorem 5.1. So $\operatorname{NormInv}_k^{r+1,r}(S_2 \times S_{n-2}) \cong \operatorname{NormInv}_k^{r+1,r}(S_{n-2}) \oplus H^{r,r}(k)$. Let $m = \lfloor n/2 \rfloor$. Repeatedly applying the isomorphism just mentioned together with the previous paragraph's result, we find that restricting from S_n to its subgroup $(\mathbf{Z}/2)^m$ gives an injection

$$\varphi\colon \operatorname{NormInv}_k^{r+1,r}(S_n) \hookrightarrow \bigoplus_{i=1}^{m} H^{r,r}(k).$$

Since the normalizer of $(\mathbf{Z}/2)^m$ in S_n contains S_m, the image of φ must be fixed by S_m. So we have an injection

$$\operatorname{NormInv}_k^{r+1,r}(S_n) \hookrightarrow H^{r,r}(k).$$

That is, every normalized invariant u of S_n is determined by its restriction to the subgroup $H = \langle (12) \rangle \cong \mathbf{Z}/2 \subset S_n$, where it has the form $u([a]) = [a]e$ for some $e \in H^{r,r}(k)$, writing $[a]$ for an element of $H^{1,0}$.

Conversely, for any $e \in H^{r,r}(k)$, there is a normalized invariant of S_n that restricts to the invariant $u([a]) = [a]e$ on the subgroup H; namely, the pullback of $e \in \mathrm{NormInv}_k^{r+1,r}(\mathbf{Z}/2) \cong H^{r,r}(k)$ via the sign homomorphism $S_n \to \mathbf{Z}/2$. (Here we use that the composition $\langle(12)\rangle \subset S_n \to \mathbf{Z}/2$ is the identity.) Thus we have shown that $\mathrm{NormInv}_k^{r+1,r}(S_n) \cong H^{r,r}(k)$. $\square$

11 Operations on étale motivic cohomology of fields

Vial found all operations on Milnor K-theory mod l of fields over a given field k [36, Theorem 1]. Roughly speaking, all operations are spanned by Kahn and Rost's divided power operations. (By contrast, Steenrod operations are essentially trivial on the motivic cohomology of fields.) Here l may be equal to the characteristic of k, and so Vial's result describes all operations on the mod p étale motivic cohomology groups $H^{n,n}$ of fields of characteristic p.

We now find all operations on the mod p étale motivic cohomology groups, both $H^{m,m}$ and $H^{m+1,m}$, in characteristic p. (Think of $H^{m,m}$ or $H^{m+1,m}$ as a functor from fields over k to sets; then an "operation" means a natural transformation from one such functor to another. In particular, operations are not assumed to be additive.) In short, only the known operations exist. The proofs use the computation of the cohomological invariants of the group scheme $(\mathbf{Z}/p)^r \times (\mu_p)^s$ (Theorem 8.4).

We state four theorems, describing operations from $H^{m,m}$ or $H^{m+1,m}$ to $H^{r,r}$ or $H^{r+1,r}$. First, here is Vial's theorem on operations from $H^{m,m}$ to $H^{r,r}$, in the case of mod p cohomology for fields of characteristic p. If $p = 2$ and $m \geq 2$, or if p is odd and $m \geq 2$ is even, then (by Kahn and Rost) there are *divided power* operations $\gamma_i \colon H^{m,m}(F) \to H^{im,im}(F)$ for all $i \geq 0$ and all fields F of characteristic p, defined on a sum of symbols $s_j = \{b_{j1}, \ldots, b_{jm}\}$ by

$$\gamma_i\left(\sum_{j=1}^{n} s_j\right) = \sum_{|T|=i} \prod_{j \in T} s_j,$$

where the sum runs over all subsets T of $\{1, \ldots, n\}$ of order i. These are typically not additive operations; instead, they satisfy $\gamma_i(x + y) = \sum_{j=0}^{i} \gamma_j(x)\gamma_{i-j}(y)$ [36, Properties 2.3].

Theorem 11.1. *(Vial) For a field k of characteristic $p > 0$, the group of operations $H^{m,m} \to H^{r,r}$ on fields over k is of the form:*

(1) if $m = 0$: $H^{r,r}(k)^{\oplus p}$;

(2) if $p = 2$ and $m = 1$, or p is odd and $m \geq 1$ is odd: $H^{r,r}(k) \oplus H^{r-m,r-m}(k)$, with every operation of the form $u(x) = c + ex$ for some (unique) c and e;

(3) if $p = 2$ and $m \geq 2$, or p is odd and $m \geq 2$ is even: every operation has the form $u(x) = \sum_{i \geq 0} c_i \gamma_i(x)$ for some (unique) elements c_i in $H^{r-im,r-im}(k)$.

We now state the other three theorems on operations.

Theorem 11.2. *For a field k of characteristic $p > 0$, the group of operations $H^{m,m} \to H^{r+1,r}$ on fields over k is as listed in Theorem 11.1, but with the coefficients c, e, and so on in $H^{j+1,j}$ rather than $H^{j,j}$.*

Theorem 11.3. *For a field k of characteristic $p > 0$, every operation $H^{m+1,m} \to H^{r,r}$ on fields over k is constant. In particular, every normalized operation is zero.*

Theorem 11.4. *For a field k of characteristic $p > 0$ and a natural number m, every operation $H^{m+1,m} \to H^{r+1,r}$ on fields over k is of the form $u(x) = c + ex$ for some (unique) elements $c \in H^{r+1,r}(k)$ and $e \in H^{r-m,r-m}(k)$. In particular, every normalized operation is additive.*

Proof. (Theorem 11.2) In the notation of section 2, every element of $H^{m+1,m}(F)$ (for a field F over k) can be written as a finite sum of symbols

$$x = \sum_{i=1}^{n} [a_i, b_{i1}, \ldots, b_{im}\}$$

with $a_i \in F$ and $b_{ij} \in F^*$. Moreover, this expression in $H^{m+1,m}(F)$ only depends on the classes of a_i in $F/\mathcal{P}(F) = H^{1,0}(F)$ and b_{ij} in $(F^*)/(F^*)^p = H^{1,1}(F)$. Here $\mathcal{P}(a) = a^p - a$.

Let u be an operation from $H^{m,m}$ to $H^{r+1,r}$ on fields over k. The case $m = 0$ is easy, since $H^{0,0}(F) \cong \mathbf{F}_p$ for every field F over k. So assume that m is positive. If $m = 1$, then an operation from $H^{1,1}$ to $H^{r+1,r}$ is the same as an invariant of the group scheme μ_p over k with values in $H^{r+1.r}$, and these are described in Proposition 6.1. So we can assume that m is at least 2.

Let n be a positive integer, and write $\vec{n}$ for the set $\{1, \ldots, n\}$. Applying the operation u to sums of n symbols gives an invariant of the group scheme $(\mu_p)^{mn}$ over k with values in $H^{r+1,r}$. By Theorem 8.4, this has the form, for $x = \sum_{i=1}^n \{b_{i1}, \ldots, b_{im}\}$ with $b_{ij} \in F^*$ for a field F/k:

$$u(x) = \sum_{T \subset \vec{n} \times \vec{m}} c_T \prod_{(i,j) \in T} \{b_{ij}\}$$

for some (unique) elements $c_T \in H^{r-|T|+1, r-|T|}(k)$.

If $b_{ij} = 1 \in F^*$ for some pair (i, j), then $\{b_{i1}, \ldots, b_{im}\} = 0$, and so the operation above must be independent of b_{il} for all $l \neq j$. By the uniqueness in Theorem 8.4, it follows that u must have the form:

$$u\bigg(\sum_{i=1}^{n} \{b_{i1}, \ldots, b_{im}\}\bigg) = \sum_{T \subset \vec{n}} c_T \prod_{i \in T} \{b_{i1}, \ldots, b_{im}\}.$$

Also, the operation must be independent of the order of the n summands in x. If $p = 2$, or if $p > 2$ and m is even, then multiplication of elements of $H^{m,m}$ is commutative. In that case, u must have the form:

$$u\bigg(\sum_{i=1}^{n} \{b_{i1}, \ldots, b_{im}\}\bigg) = \sum_{j=0}^{n} c_j \sum_{\substack{T \subset \vec{n} \\ |T| = j}} \prod_{i \in T} \{b_{i1}, \ldots, b_{im}\}.$$

Thus every operation is a linear combination (with coefficients in $H^{*+1,*}(k)$) of divided power operations. Conversely, divided power operations are well-defined under our assumptions (that $m \geq 2$ and, if p is odd, then m is even), by Theorem 11.1. Here we have considered operations on elements of $H^{m,m}$ written as a sum of a fixed number of symbols, but (since we can take one symbol to be zero) these descriptions must be compatible as the number of symbols varies. This completes the proof under the assumptions mentioned.

There remains the case where $p > 2$ and m is odd. Here multiplication of elements of $H^{m,m}$ is anti-commutative. In this case, since the operation $u(x)$ must be unchanged after switching two summands of T, we must have (in the notation above) $c_T = -c_T$ for every set $T \subset \{1, \ldots, n\}$ of order at least 2. Since c_T is an element of an $\mathbf{F}_p$-vector space with p odd, that means that $c_T = 0$ if T has order at least 2. So u has the form

$$u\Big(\sum_{i=1}^{n} \{b_{i1}, \ldots, b_{im}\}\Big) = c + \sum_{i=1}^{n} e_i \{b_{i1}, \ldots, b_{im}\}.$$

Using again that u is unchanged by permuting the summands, we have $e_1 = \cdots = e_n$. So u has the form

$$u(x) = c + ex$$

for some $c, e \in H^{*+1,*}(k)$. □

Proof. (Theorem 11.3) Let u be an operation from $H^{m+1,m}$ to $H^{r,r}$ on fields over k. Applying u to sums of n symbols,

$$u\Big(\sum_{i=1}^{n} [a_i, b_{i1}, \ldots, b_{im}\}\Big)$$

gives an invariant of the group scheme $(\mathbf{Z}/p)^n \times (\mu_p)^{mn}$ over k with values in $H^{r,r}$. By Proposition 9.1, such an invariant must be independent of $a_1, \ldots, a_n \in H^{1,0}(k)$. But if we take those elements to be zero, then the element $\sum_{i=1}^{n} [a_i, b_{i1}, \ldots, b_{im}\}$ in $H^{m+1,m}$ is zero. So every operation from $H^{m+1,m}$ to $H^{r,r}$ is constant. □

Proof. (Theorem 11.4) Let u be any operation from $H^{m+1,m}$ to $H^{r+1,r}$ on fields over k. For a positive integer n, restricting u to sums of n symbols gives an invariant of the group scheme $(\mathbf{Z}/p)^n \times (\mu_p)^{mn}$ over k with values in $H^{r+1,r}$. By Theorem 8.4, we can write u on an element $x = \sum_{i=1}^{n} [a_i, b_{i1}, \ldots, b_{im}\}$ as

$$u(x) = \sum_{T \subset \vec{n} \times \vec{m}} c_T \prod_{(i,j) \in T} \{b_{ij}\} + \sum_{l=1}^{n} [a_l] \sum_{T \subset \vec{n} \times \vec{m}} e_{l,T} \prod_{(i,j) \in T} \{b_{ij}\}$$

for some (unique) elements c_T in $H^{r-|T|+1, r-|T|}(k)$ and $e_{l,T}$ in $H^{r-|T|, r-|T|}(k)$.

In fact, all coefficients c_T with T nonempty are zero, because the input x in $H^{r+1,r}$ is zero if all a_i are zero, no matter what the $b_{i,j}$ are. Thus u can be written as:

$$u\Big(\sum_{i=1}^{n} [a_i, b_{i1}, \ldots, b_{im}\}\Big) = c + \sum_{l=1}^{n} [a_l] \sum_{T \subset \vec{n} \times \vec{m}} e_{l,T} \prod_{(i,j) \in T} \{b_{ij}\}.$$

Next, let $1 \leq i \leq m$. Note that the term $[a_i, b_{i1}, \ldots, b_{im}\}$ in x is zero if a_i is zero or if any of $b_{i1}, \ldots, b_{im}$ is 1. So, if a_i is 0, then $u(x)$ must be independent of $b_{i1}, \ldots, b_{im}$; and if some b_{ij} is equal to 1, then $u(x)$ must be independent of a_i. Using the uniqueness of the coefficients (from Theorem 8.4) again, it follows that $e_{s,T}$ is zero for all $T \neq \{s\} \times \vec{m}$. That is, u can be written as:

$$u\Big(\sum_{i=1}^{n} [a_i, b_{i1}, \ldots, b_{im}\}\Big) = c + \sum_{i=1}^{n} [a_i, b_{i1}, \ldots, b_{im}\} e_i$$

for some elements c in $H^{r+1,r}(k)$ and e_i in $H^{r-m,r-m}(k)$.

Finally, the operation u must be unchanged if we permute the n summands in the input. It follows that $e_1 = \cdots = e_n$. That is, the operation u is given on sums of n symbols by

$$u(x) = c + xe$$

for some (uniquely determined) c in $H^{r+1,r}(k)$ and e in $H^{r-m,r-m}(k)$. Since we can take one symbol to be zero, these elements c and e must be unchanged if we change the number n of symbols in x. That is, the operation u is given by $u(x) = c + xe$ on all of $H^{m+1,m}(F)$, for fields F over k. $\square$

12 Invariants of $O(2n)$ in characteristic 2

Define a quadratic form q_0 on a vector space V over a field k to be *nonsingular* if the orthogonal complement $V^\perp \subset V$ has dimension at most 1 and q_0 is nonzero at each nonzero element of $V^\perp$. *Quadratic forms will be understood to be nonsingular in this paper.* One reason for the importance of this class of quadratic forms is that the simple algebraic groups of type B_n and D_n over any field are essentially automorphism groups of nonsingular quadratic forms. Note that if k has characteristic 2, then the bilinear form $b_0(x, y) = q_0(x + y) - q_0(x) - q_0(y)$ associated to q_0 is alternating. So $V^\perp$ has dimension 0 if q_0 has even dimension and dimension 1 if q_0 has odd dimension.

Let q_0 be a quadratic form of even dimension over a field k of characteristic 2. In Theorems 12.1 and 14.1, we compute the cohomological invariants for the orthogonal group $O(q_0)$ and its identity component, which we call $SO(q_0)$ (even though $O(2n)$ is contained in $SL(2n)$ in characteristic 2). We consider the invariants with values in $H^{m+1,m}$; since these group schemes are smooth, their invariants in $H^{m,m}$ are constant by Theorem 5.1. In short, the fundamental invariants are the discriminant (or Arf invariant) in $H^{1,0}$ and the Clifford invariant in $H^{2,1}$, the class of the Clifford algebra in the Brauer group [6, section 14].

Theorem 12.1. *Let k be a field of characteristic 2, m a natural number. Let q_0 be a quadratic form of dimension $2n$ over k with $n \geq 1$. Then*

$$\mathrm{Inv}_k^{m+1,m}(O(q_0)) \cong H^{m+1,m}(k) \oplus H^{m,m}(k) \oplus H^{m-1,m-1}(k).$$

Explicitly, we can view the invariants for $O(q_0)$ as the invariants of quadratic forms of dimension $2n$ over fields F/k. Every invariant has the form

$$u(q) = c + \mathrm{disc}(q)e + \mathrm{clif}(q)f$$

for some (uniquely determined) $c \in H^{m+1,m}(k)$, $e \in H^{m,m}(k)$, and $f \in H^{m-1,m-1}(k)$.

Note the contrast with Serre's calculation in characteristic not 2: for a quadratic form q_0 of dimension m over a field k of characteristic not 2, $\mathrm{Inv}_k^*(O(q_0), \mathbf{F}_2)$ is a free module over $H^*(k, \mathbf{F}_2)$ with basis the Stiefel-Whitney classes $1 = w_0, w_1, w_2, \ldots, w_m$ [9, Theorem 17.3]. (In characteristic not 2, the weight makes no difference; that is, the étale motivic cohomology group $H^i_{\text{ét}}(F, \mathbf{Z}/2(j))$ is the same for all $j \geq 0$.)

In characteristic 2, Theorem 12.1 says that there are analogs of w_1 (the discriminant or Arf invariant in $H^{1,0}$) and w_2 (the Clifford invariant in $H^{2,1}$) for quadratic forms, but no analogs of the higher Stiefel-Whitney classes. (By contrast, symmetric bilinear forms of any characteristic have Stiefel-Whitney classes w_i in $H^{i,i}$ [6, section 5].)

As a result, cohomological invariants are not enough to prove the lower bound in Babic and Chernousov's computations of the essential dimension in characteristic 2, such as $\mathrm{ed}(O(2n)) = n+1$ [2]. Chernousov explained to me that their arguments in fact apply to the essential dimension at 2, so that we have $\mathrm{ed}(O(2n); 2) = n+1$. (Use that every odd-degree extension of a field K of characteristic 2 is separable, and that every discrete valuation v of K has an extension w to L for which the ramification index $e(w/v)$ and the residue degree $f(w/v)$ are odd.)

This is a bit disappointing, but note that even in characteristic not 2, the first two Stiefel-Whitney classes of a quadratic form are far more important than the higher ones. For example, if F is a field of characteristic not 2 in which -1 is a square, let $W(F)$ be the Witt ring and $I \subset W(F)$ the ideal of even-dimensional forms. Then w_1 and w_2 give isomorphisms $w_1 \colon I/I^2 \to H^1(F, \mathbf{Z}/2)$ and $w_2 \colon I^2/I^3 \to H^2(F, \mathbf{Z}/2)$, whereas all Stiefel-Whitney classes of positive degree vanish on I^3 [6, Exercise 5.14]. Thus, for $j \geq 3$, the isomorphism $I^j/I^{j+1} \cong H^j(F, \mathbf{Z}/2)$ proved by Orlov-Vishik-Voevodsky [27] does not come from an invariant defined on all quadratic forms of a given dimension, but only from an invariant of some subclass of forms.

This line of thought suggests looking at the invariants of the connected group $SO(q)$ and its double cover $\mathrm{Spin}(q)$ in characteristic 2. In this paper, we find the invariants only for $SO(q)$. We know that $\mathrm{Spin}(q)$ will have a nontrivial invariant in $H^{3,2}$ by Kato's isomorphism

$$I_q^{m+1}(k)/I_q^{m+2}(k) \cong H^{m+1,m}(k),$$

applied in the case $m = 2$ [18]. (We use the notation of [6, section 9.B]: $I_q(k)$ is the quadratic Witt group, which is a module over the bilinear Witt ring $W(k)$, and $I_q^m(k) := I^{m-1} I_q(k)$ for $m \geq 1$. For the hyperbolic form $q_0 = nH$, torsors for $\mathrm{Spin}(q_0)$ over k give quadratic forms in $I_q^3(k)$.) This invariant for $\mathrm{Spin}(q)$ was generalized by Esnault-Kahn-Levine-Viehweg and Merkurjev to the Rost invariant of any simply connected group [7, Appendix B], [9, Part 2, Theorem 9.11]. For $n \leq 14$, some higher-degree invariants of $\mathrm{Spin}(n)$ have been constructed by Rost and Garibaldi in characteristic not 2 and by the author in characteristic 2 [8, section 23], [35]. It would be interesting to construct invariants for spin groups of higher dimensions.

Proof. (Theorem 12.1) For any field F over k, $H^1(F, O(q_0))$ can be identified with the set of isomorphism classes of quadratic forms over F of dimension $2n$ [20, equation 29.28]. Therefore, computing the invariants for $O(q_0)$ amounts to computing the invariants for quadratic forms of dimension $2n$. In particular, this description shows that the invariants of $O(q_0)$ are the same for all quadratic forms q_0 of dimension $2n$ over k. So we can assume that q_0 is the simplest quadratic form, $q_0 = nH$, the orthogonal direct sum of n copies of the hyperbolic plane $q_H(x, y) = xy$.

The group scheme $\mathbf{Z}/2$ is contained in $O(H)$ by switching x and y, and this commutes with the action of the group scheme μ_2 by scalar multiplication. So we have a subgroup $\mathbf{Z}/2 \times \mu_2$ in $O(H)$, and hence a subgroup $(\mathbf{Z}/2)^n \times (\mu_2)^n$ in $O(nH)$. Let F be a field over k. For elements $a \in F$ and $b \in F^*$, which give a $\mathbf{Z}/2$-torsor $[a]$ and a μ_2-torsor $\{b\}$ over F, the associated 2-dimensional quadratic form (given by

$H^1(F, \mathbf{Z}/2 \times \mu_2) \to H^1(F, O(H))$ can be written as $b\langle\langle a]] = b[1, a] = bx^2 + bxy + aby^2$. Every quadratic form of dimension 2 over F arises this way; that is, every form of dimension 2 is a scalar multiple of a 1-fold Pfister form [6, section 9.B]. Moreover, every quadratic form over F of dimension $2n$ is an orthogonal direct sum of 2-dimensional forms [6, Corollary 7.32], and so

$$H^1(F, (\mathbf{Z}/2)^n \times (\mu_2)^n) \to H^1(F, O(nH))$$

is surjective.

As a result, for the quadratic form $q_0 = nH$, the restriction

$$\mathrm{Inv}_k^{m+1,m}(O(q_0)) \to \mathrm{Inv}_k^{m+1,m}((\mathbf{Z}/2)^n \times (\mu_2)^n)$$

is injective. By Theorem 8.4, every invariant for $O(q_0)$ over k with values in $H^{m+1,m}$ can be written as:

$$u\biggl(\sum_{i=1}^n b_i\langle\langle a_i]]\biggr) = \sum_{I\subset\{1,\ldots,n\}} c_I \prod_{i\in I}\{b_i\} + \sum_{j=1}^n [a_j] \sum_{I\subset\{1,\ldots,n\}} e_{j,I} \prod_{i\in I}\{b_i\}$$

for some (uniquely determined) $c_I \in H^{m-|I|+1,m-|I|}(k)$ and $e_{j,I} \in H^{m-|I|,m-|I|}(k)$.

If $a_1 = \cdots = a_n = 0$, then the quadratic form $\sum_i b_i\langle\langle a_i]]$ is hyperbolic. So the invariant above is constant (independent of $b_1, \ldots, b_n \in k^*$) in that case. By the uniqueness in Theorem 8.4, it follows that $c_I = 0$ for all $I \neq \emptyset$.

Next, if $a_j = 0$, then the quadratic form $b_j\langle\langle a_j]]$ is hyperbolic, and so the invariant above is independent of $b_j \in k^*$. So $e_{l,I} = 0$ unless I is empty or $I = \{l\}$. Thus the invariant has the form

$$u\biggl(\sum_{i=1}^n b_i\langle\langle a_i]]\biggr) = c + \sum_{j=1}^n [a_j]e_j + \sum_{j=1}^n [a_j, b_j\}f_j,$$

for some (uniquely determined) $c \in H^{m+1,m}(k)$, $e_j \in H^{m,m}(k)$, and $f_j \in H^{m-1,m-1}(k)$.

The invariant u must be invariant under permuting the n pairs $(a_1, b_1), \ldots, (a_n, b_n)$. It follows that $e_1 = \cdots = e_n$ and $f_1 = \cdots = f_n$. That is,

$$u\biggl(\sum_{i=1}^n b_i\langle\langle a_i]]\biggr) = c + \biggl[\sum_{j=1}^n [a_j]\biggr]e + \biggl[\sum_{j=1}^n [a_j, b_j\}\biggr]f$$

for some (uniquely determined) $c \in H^{m+1,m}(k)$, $e \in H^{m,m}(k)$, and $f \in H^{m-1,m-1}(k)$.

The *discriminant* (or *Arf invariant*) $\mathrm{disc}(q)$ of the quadratic form $q = \sum_{i=1}^n b_i\langle\langle a_i]]$ is $\sum_{j=1}^n a_j \in k/\mathcal{P}(k) = H^{1,0}(k)$ [6, Example 13.5]. Also, the *Clifford invariant* $\mathrm{clif}(q)$ is $\sum_{j=1}^n [a_j, b_j\} \in H^{2,1}(k) = \mathrm{Br}(k)[2]$ [6, section 14]. Since these are known to be invariants of quadratic forms, we have determined all the invariants for $O(q_0)$. $\square$

13 Invariants of $O(2n+1)$ and $SO(2n+1)$

Let q_0 be a quadratic form on a vector space V of dimension $2n+1$ over a field k of characteristic 2. (Quadratic forms are understood to be nonsingular in the sense of section 12.) Then the orthogonal group $O(q_0)$ is not smooth over k; it is a product $\mu_2 \times SO(q_0)$, with $SO(q_0)$ smooth and connected over k. In this section, we determine the invariants for $O(q_0)$ and $SO(q_0)$. Note a difference between even- and odd-dimensional quadratic forms in characteristic 2: the discriminant of an odd-dimensional quadratic form lies in $H^{1,1}(k) = H^1(k, \mu_2)$, not in $H^{1,0}(k) = H^1(k, \mathbf{Z}/2)$.

Theorem 13.1. *Let k be a field of characteristic 2, n a positive integer. Let q_0 be a quadratic form of dimension $2n+1$ over k. For any natural number m,*

$$\mathrm{Inv}_k^{m+1,m}(O(q_0)) \cong H^{m+1,m}(k) \oplus H^{m,m-1}(k) \oplus H^{m-1,m-1}(k) \oplus H^{m-2,m-2}(k).$$

Explicitly, we can view the invariants of $O(q_0)$ as the invariants for quadratic forms q of dimension $2n+1$ over fields F/k. Every invariant has the form

$$u(q) = c + \mathrm{disc}(q)e + \mathrm{clif}(q)f + \mathrm{clif}(q)\,\mathrm{disc}(q)g$$

for some (uniquely determined) $c \in H^{m+1,m}(k)$, $e \in H^{m,m-1}(k)$, $f \in H^{m-1,m-1}(k)$, and $g \in H^{m-2,m-2}(k)$.

Proof. Regardless of the choice of form q_0, the $O(q_0)$-torsors over a field F/k can be identified (up to isomorphism) with the quadratic forms of dimension $2n+1$ over F. Every nonsingular quadratic form on a vector space V of dimension $2n+1$ over a field F/k can be written as the orthogonal direct sum of the 1-dimensional form $V^{\perp}$, described by an element of $H^1(F, \mu_2)$, and a nonsingular form of dimension $2n$. For an element b_0 in F^*, we write $\langle b_0\rangle$ for the 1-dimensional quadratic form $q(x) = b_0x^2$. So we can write $q_0 = \langle b_0\rangle + q_1$ for some b_0 in k^* and some nonsingular quadratic form q_1 over k of dimension $2n$. (Here q_1 is not uniquely determined by q_0.) Since every quadratic form of dimension $2n+1$ over a field F/k can be similarly decomposed as $\langle b\rangle + r$, the map $H^1(F, \mu_2 \times O(q_1)) \to H^1(F, O(q_0))$ is surjective.

It follows that the restriction

$$\mathrm{Inv}_k^{m+1,m}(O(q_0)) \to \mathrm{Inv}_k^{m+1,m}(\mu_2 \times O(q_1))$$

is injective. By Theorems 6.2 and 12.1, it follows that every invariant for $O(q_0)$ has the form

$$u(\langle b\rangle + r) = c + \mathrm{disc}(r)e + \mathrm{clif}(r)f + \{b\}g + \mathrm{disc}(r)\{b\}h + \mathrm{clif}(r)\{b\}l$$

for some (unique) $c \in H^{m-1,m}(k)$, $e \in H^{m,m}(k)$, $f \in H^{m-1,m-1}(k)$, $g \in H^{m,m-1}(k)$, $h \in H^{m-1,m-1}(k)$, and $l \in H^{m-2,m-2}(k)$.

For a field F over k and any b in F^* and a_1 in F, the quadratic form $\langle b\rangle + b\langle\langle a_1]]$ is isotropic, by inspection, and so it is isomorphic to $\langle b\rangle + H$, where H is the hyperbolic plane. (This is a known failure of cancellation for quadratic forms in characteristic 2 [6, equation 8.7].) So the given invariant u must take the same value on $\langle b\rangle + b\langle\langle a_1]] + (n-1)H$ as on $\langle b\rangle + nH$. That is,

$$c + \{b\}g = c + [a_1]e + [a_1, b\}f + \{b\}g + [a_1, b\}h$$

as invariants of $\mu_2 \times \mathbf{Z}/2$ (where we used that $\{b, b\} = 0$ in $H^{2,2}$). By the description of the invariants for $\mu_2 \times \mathbf{Z}/2$ in Theorem 8.4, it follows that $e = 0$ and $f = h$. Thus the invariant u has the form

$$u(\langle b\rangle + r) = c + (\mathrm{clif}(r) + \mathrm{disc}(r)\{b\})f + \{b\}g + \mathrm{clif}(r)\{b\}l$$

for some (unique) $c \in H^{m-1,m}(k)$, $f \in H^{m-1,m-1}(k)$, $g \in H^{m,m-1}(k)$, and $l \in H^{m-2,m-2}(k)$.

Here $\{b\}$ in $H^{1,1}$ is an invariant of $q = \langle b \rangle + r$, the *discriminant* $\operatorname{disc}(q)$ (called the "half-discriminant" in [19, IV.3.1.3]). (It is clear that this is an invariant of q, because it describes the restriction of q to the 1-dimensional subspace $V^{\perp} \subset V$.) The other known invariant of odd-dimensional quadratic forms in characteristic 2 is the *Clifford invariant* in the Brauer group $H^{2,1}$, given by [19, Corollary IV.7.3.2]:

$$\begin{aligned} \operatorname{clif}(\langle b \rangle + r) &= \operatorname{clif}(br) \\ &= \operatorname{clif}(r) + \operatorname{disc}(r)\{b\}. \end{aligned}$$

Since $\operatorname{clif}(r)\{b\}$ is equal to $\operatorname{clif}(q) \operatorname{disc}(q)$, that is also an invariant of q. Thus we have found all the invariants of q. □

Since $O(2n+1)$ is not a smooth group scheme, its invariants in $H^{m,m}$ are not immediate from Theorem 5.1, but they are easy to compute:

Proposition 13.2. *Let k be a field of characteristic 2, n a positive integer, q_0 a quadratic form of dimension $2n+1$ over k. Then*

$$\operatorname{Inv}_k^{m,m}(O(q_0)) \cong H^{m,m}(k) \oplus H^{m-1,m-1}(k)$$

for every natural number m. Explicitly, we can view the invariants for $O(q_0)$ as the invariants of quadratic forms q of dimension $2n+1$ over fields F/k. Every invariant in $H^{m,m}$ has the form

$$u(q) = c + \operatorname{disc}(q)e$$

for some (uniquely determined) $c \in H^{m,m}(k)$ and $e \in H^{m-1,m-1}(k)$.

Proof. Since $O(q_0)$ is isomorphic to $\mu_2 \times SO(q_0)$ and $SO(q_0)$ is smooth over k, the conclusion is immediate from Proposition 7.2 and Theorem 5.1. □

Now we turn to the smooth connected group $SO(2n+1)$. Since it is smooth, its invariants in $H^{m,m}$ are all constant (Theorem 5.1). Here are its invariants in $H^{m+1,m}$.

Theorem 13.3. *Let k be a field of characteristic 2, m a natural number, n a positive integer, q_0 a quadratic form of dimension $2n+1$ over k. Then the group of cohomological invariants for $SO(q_0)$ is given by*

$$\operatorname{Inv}_k^{m+1,m}(SO(q_0)) \cong H^{m+1,m}(k) \oplus H^{m-1,m-1}(k).$$

Concretely, writing $[d] = \operatorname{disc}(q_0) \in H^{1,1}(k)$, we can view the invariants for $SO(q_0)$ as the invariants of quadratic forms q of dimension $2n+1$ and discriminant $[d]$. Every invariant has the form

$$u(q) = c + \operatorname{clif}(q)f$$

for some (uniquely determined) $c \in H^{m+1,m}(k)$ and $f \in H^{m-1,m-1}(k)$.

Proof. Over any field F/k, the torsors for $SO(q_0)$ can be identified (up to isomorphism) with the quadratic forms q of dimension $2n+1$ and discriminant $[d]$. We have $\mathrm{disc}(aq) = \{a\} + \mathrm{disc}(q)$ in $H^{1,1}$, and so the invariants of quadratic forms with discriminant $[d]$ are in fact independent of $[d]$.

So we can assume that q_0 has discriminant $1 \in (k^*)/(k^*)^2 \cong H^{1,1}(k)$. Then q_0 can be written as $\langle 1 \rangle + q_1$ for some nonsingular quadratic form q_1 over k of dimension $2n$. The inclusion $O(q_1) \subset SO(q_0)$ gives a surjection $H^1(F, O(q_1)) \to H^1(F, SO(q_0))$, since every form q of dimension $2n+1$ with trivial discriminant over a field F/k can be written as an orthogonal sum $\langle 1 \rangle + r$ for some nonsingular quadratic form r of dimension $2n$ (not unique). So $\mathrm{Inv}_k^{m+1,m}(SO(q_0))$ injects into $\mathrm{Inv}_k^{m+1,m}(O(q_1))$. By Theorem 12.1, every invariant u for $SO(q_0)$ can be written as

$$u(\langle 1 \rangle + r) = c + \mathrm{disc}(r)e + \mathrm{clif}(r)f$$

for some (unique) $c \in H^{m+1,m}(k)$, $e \in H^{m,m}(k)$, and $f \in H^{m-1,m-1}(k)$.

We use a special case of the isomorphism from the proof of Theorem 13.1: for any field F/k and $a_1 \in F$, the quadratic form $\langle 1 \rangle + \langle\langle a_1]] + (n-1)H$ is isomorphic to $\langle 1 \rangle + nH$. So the invariant $u(q)$ must take the same value on these two forms. That is,

$$c + [a_1]e = c,$$

and so $[a_1]e$ is equal to zero as an invariant of $\mathbf{Z}/p$ (thinking of $a_1 \in F$ as an element of $H^1(F, \mathbf{Z}/p)$ for fields F/k). By the description of the invariants for $\mathbf{Z}/p$ (Proposition 8.1), it follows that $e = 0$. Thus the invariant u has the form

$$u(\langle 1 \rangle + r) = c + \mathrm{clif}(r)f$$

for some (unique) $c \in H^{m+1,m}(k)$ and $f \in H^{m-1,m-1}(k)$.

Here $\mathrm{clif}(\langle 1 \rangle + r) = \mathrm{clif}(r)$, by the description of the Clifford invariant for odd-dimensional forms in the proof of Theorem 13.1. So $\mathrm{clif}(r)$ is an invariant of $q = \langle 1 \rangle + r$. Thus we have found all the invariants for q. □

14 Invariants of $SO(2n)$

Let k be a field of characteristic 2, n a positive integer, q_0 a quadratic form of dimension $2n$ over k. The orthogonal group $O(q_0)$ is smooth over k, with two connected components. We write $SO(q_0)$ for the identity component, even though the whole group $O(2n)$ is contained in $SL(2n)$ in characteristic 2. Since $SO(q_0)$ is smooth, its invariants in $H^{m,m}$ are constant (Theorem 5.1). Here are its invariants in $H^{m+1,m}$.

Theorem 14.1. *Let k be a field of characteristic 2, n a positive integer, q_0 a quadratic form of dimension $2n$ over k. Let $[d]$ be the discriminant of q_0 in $H^{1,0}(k)$. Let m be a natural number. Then the group of cohomological invariants for $SO(q_0)$ with values in $H^{m+1,m}$ is given by*

$$\begin{cases} H^{m+1,m}(k) \oplus [d]H^{m-1,m-1}(k) & \text{if } n = 1 \\ H^{m+1,m}(k) \oplus H^{m-1,m-1}(k) \oplus \{\lambda \in H^{m-2,m-2}(k) : [d]\lambda = 0\} & \text{if } n = 2 \\ H^{m+1,m}(k) \oplus H^{m-1,m-1}(k) & \text{if } n \geq 3. \end{cases}$$

We can equivalently view the invariants for $SO(q_0)$ as the invariants of quadratic forms q of dimension $2n$ and discriminant $[d]$ over fields F/k. For $n \geq 3$, every invariant has the form

$$u(q) = c + \mathrm{clif}(q)f$$

for some (uniquely determined) $c \in H^{m+1,m}(k)$ and $f \in H^{m-1,m-1}(k)$. For $n = 2$, a 4-dimensional quadratic form q with discriminant $[d]$ has an invariant $b_\lambda(q)$ in $H^{m+1,m}$ for each $\lambda \in H^{m-2,m-2}(k)$ with $[d]\lambda = 0$, as well as the Clifford invariant in $H^{2,1}$.

The invariants for 4-dimensional quadratic forms with given discriminant are analogous to those found by Serre in all even dimensions at least 4 when the characteristic is not 2 [9, Proposition 20.1]. Likewise, the invariants for 2-dimensional quadratic forms with given discriminant are analogous to those found by Serre in dimension 2 when the characteristic is not 2 [9, Exercise 20.9].

Every 1-dimensional torus over k is of the form $SO(q_0)$ for some 2-dimensional quadratic form q_0, and so Theorem 14.1 describes all mod p cohomological invariants for every 1-dimensional torus. Blinstein and Merkurjev described the cohomological invariants in degrees at most 3 for tori of any dimension [3, Theorem 4.3].

Proof. The map $H^1(F, SO(q_0)) \to H^1(F, O(q_0))$ is injective, with image the set of isomorphism classes of $2n$-dimensional quadratic forms over F with discriminant $[d]$ [20, equation 29.29]. So we can think of the invariants for $SO(q_0)$ as the invariants for quadratic forms (on fields over k) of dimension $2n$ with discriminant $[d]$.

Every such form q over a field F/k can be written as $q = r + b_1\langle\langle \mathrm{disc}(r) + d]]$ for some quadratic form r of dimension $2n-2$ and some $b_1 \in F^*$. (Equivalently, for any nonsingular subform r_0 of dimension $2n-2$ in q_0, the subgroup $O(r_0) \times \mu_2 \subset SO(q_0)$ induces a surjection on H^1.) Assume that $n \geq 2$. We know the invariants for $O(r_0) \times \mu_2$ by Theorems 6.2 and 12.1. So every invariant u in $H^{m+1,m}$ for $SO(q_0)$ can be written, on a quadratic form $q = r + b_1\langle\langle \mathrm{disc}(r) + d]]$, as

$$u(q) = c + \mathrm{disc}(r)e + \mathrm{clif}(r)f + \{b_1\}g + \mathrm{disc}(r)\{b_1\}h + \mathrm{clif}(r)\{b_1\}\lambda$$

for some (unique) $c \in H^{m+1,m}(k)$, $e \in H^{m,m}(k)$, $f \in H^{m-1,m-1}(k)$, $g \in H^{m,m-1}(k)$, $h \in H^{m-1,m-1}(k)$, and $\lambda \in H^{m-2,m-2}(k)$.

We can apply this formula to $r = s + b_2\langle\langle a_2]]$, for any quadratic form s of dimension $2n-4$ over a field F/k and any $b_2 \in F^*$. This amounts to restricting the invariant u to a subgroup $O(2n-4) \times (\mathbf{Z}/2 \times \mu_2) \times \mu_2$. We compute that for a quadratic form $q = s + b_2\langle\langle a_2]] + b_1\langle\langle a_2 + \mathrm{disc}(s) + d]]$,

$$\begin{aligned} u(q) = c &+ \mathrm{disc}(s)e + [a_2]e + \mathrm{clif}(s)f + [a_2, b_2\}f + g\{b_1\} \\ &+ \mathrm{disc}(s)\{b_1\}h + [a_2, b_1\}h + \mathrm{clif}(s)\{b_1\}\lambda + [a_2, b_1, b_2\}\lambda. \end{aligned}$$

This must be unchanged when we switch b_1 and b_2 and simultaneously change a_2 to $a_2 + \mathrm{disc}(s) + d$. It follows that

$$\begin{aligned} 0 = [d]e &+ \mathrm{disc}(s)e + \{b_1\}([d]f + g) + \mathrm{disc}(s)\{b_1\}(f + h) \\ &+ [a_2, b_1\}(f + h) + \mathrm{clif}(s)\{b_1\}\lambda + \{b_2\}(g + [d]h) + \mathrm{clif}(s)\{b_2\}\lambda \\ &+ [a_2, b_2\}(f + h) + \{b_1, b_2\}[d]\lambda + \mathrm{disc}(s)\{b_1, b_2\}\lambda. \end{aligned}$$

Assume that $n \geq 3$, so that the invariants of $O(2n-4)$ are given by Theorem 12.1. Then our knowledge of the invariants of $O(2n-4)\times(\mathbf{Z}/2\times\mu_2)\times\mu_2$ from Theorems 6.2 and 8.3, in particular the uniqueness of the coefficients, implies from the formula above that $e=0$, $g=[d]f$, $h=f$, and $\lambda=0$.

So, on a quadratic form $q = r + b_1\langle\langle \mathrm{disc}(r)+d]]$, the invariant u is given by:

$$\begin{aligned} u(q) &= c + (\mathrm{clif}(r) + \mathrm{disc}(r)\{b_1\} + [d, b_1\})f \\ &= c + \mathrm{clif}(q)f, \end{aligned}$$

for some (unique) $c \in H^{m+1,m}(k)$ and $f \in H^{m-1,m-1}(k)$. Since the Clifford invariant is known to be an invariant of q, we have determined all the invariants of $SO(q_0)$ for $n \geq 3$.

We next consider the case $n=2$. In that case, the symmetry above (switching the two summands of a quadratic form $q = b_1\langle\langle a_1]] + b_2\langle\langle a_2]]$ with $a_1+a_2=d$ over a field F/k) gives only that $[d]e=0$, $g=[d]f$, $h=f$, and $[d]\lambda=0$. So the invariant has the form

$$u(q) = c + [a_1]e + \mathrm{clif}(q)f + [a_1, b_1, b_2\}\lambda$$

for some (unique) $c \in H^{m+1,m}(k)$, $e \in H^{m,m}(k)$, $f \in H^{m-1,m-1}(k)$, and $\lambda \in H^{m-2,m-2}(k)$ with $[d]e=0$ and $[d]\lambda=0$.

If $b_2=b_1$, then $q = b_1(\langle\langle a_1]] + \langle\langle a_1+d]])$. A direct calculation shows that

$$\langle\langle a_1]] + \langle\langle a_1+d]] \cong \langle\langle d]] + H$$

[6, Example 7.23]. So, when $b_2=b_1$, q is independent of a_1, up to isomorphism. Also, when $b_2=b_1$, we have $\{b_1,b_2\}=0$ and $\mathrm{clif}(q)=[d,b_1\}$, so $u(q) = c+[a_1]e+[d,b_1\}f$. This must be independent of a_1. By the uniqueness in Theorem 8.4, it follows that $e=0$. Thus the invariant u has the form, for any a_1,a_2 in a field F over k with $a_1+a_2=d$ and $b_1,b_2\in F^*$:

$$u(q) = c + \mathrm{clif}(q)f + [a_1, b_1, b_2\}\lambda$$

for some (unique) $c \in H^{m+1,m}(k)$, $f \in H^{m-1,m-1}(k)$, and $\lambda \in H^{m-2,m-2}(k)$ with $[d]\lambda = 0$.

The calculation will be finished by showing that for any $\lambda \in H^{m-2,m-2}(k)$ with $[d]\lambda=0$, $b_\lambda(q) := [a_1,b_1,b_2\}\lambda$ is an invariant for $SO(q_0)$ in $H^{m+1,m}$. To show that $b_\lambda(q)$ is an invariant, we use Revoy's chain lemma for quadratic forms in characteristic 2 [29, Proposition 3]. Write $[a,b]$ for the 2-dimensional quadratic form $ax^2+xy+by^2$.

Theorem 14.2. *(Revoy) Let k be a field of characteristic 2. Then the quadratic form $\sum_{i=1}^n [a_i,b_i]$ over k is isomorphic to the form $\sum_{i=1}^n [a_i',b_i']$ if and only if these two elements of k^{2n} can be connected by a sequence of the following moves:*

$$A\colon [a_i,b_i] + [a_{i+1},b_{i+1}] \to [a_i+b_{i+1},b_i] + [a_{i+1}+b_i,b_{i+1}]$$

for some $1 \leq i \leq n-1$, or

$$\begin{aligned} B&\colon [a_i,b_i] \to [\beta^2 a_i, \beta^{-2} b_i] \\ C&\colon [a_i,b_i] \to [a_i+\beta^2 b_i+\beta, b_i] \\ D&\colon [a_i,b_i] \to [a_i, b_i+\beta^2 a_i+\beta] \end{aligned}$$

for some $1 \leq i \leq n$ and $\beta \in k^*$.

To relate this to the notation we have been using for quadratic forms: an easy calculation gives that the 2-dimensional form $[u, v]$ is isomorphic to $u\langle\langle uv]]$ if $u \neq 0$, and to the hyperbolic plane $H = 1\langle\langle 0]]$ if $u = 0$. So a 4-dimensional form $[u_1, v_1] + [u_2, v_2]$ is isomorphic to $u_1\langle\langle u_1v_1]] + u_2\langle\langle u_2v_2]]$ if u_1 and u_2 are nonzero, with the coefficient u_1 changed to 1 if $u_1 = 0$, and likewise for the coefficient u_2. So we want to show that for any $\lambda \in H^{m-2,m-2}(k)$ with $[d]\lambda = 0$,

$$b_\lambda(q) := [u_1v_1, u_1, u_2\}\lambda$$

is an invariant of 4-dimensional quadratic forms $q = [u_1, v_1] + [u_2, v_2]$ with discriminant $[d]$. (That is, we are assuming that $u_1v_1 + u_2v_2 = d \in H^{1,0}(k) = k/\mathcal{P}(k)$.) The formula for $b_\lambda(q)$ is understood to mean zero if $u_1 = 0$ or $u_2 = 0$.

To show this, by Theorem 14.2, it suffices to show that $b_\lambda(q)$ is unchanged by moves A, B, C, or D. One helpful observation (*) is that $[u, u\} = 0$ in the Brauer group $H^{2,1}(k)$ for all $u \in k$, where the expression is defined to mean zero if $u = 0$. This follows from the description of $H^{2,1}(k)$ in terms of differential forms (section 2), using that $u(du/u) = du$ is exact. So we can rewrite $b_\lambda(q) = [u_1v_1, u_1, u_2\}\lambda$ as $[u_1v_1, v_1, u_2\}\lambda$. Also, we have $[u_1v_1]\lambda = [u_2v_2]\lambda$ because $[d]\lambda = 0$, and so we can also rewrite $b_\lambda(q)$ as $[u_2v_2, v_1, u_2\}\lambda$, and hence as $[u_2v_2, v_1, v_2\}\lambda$, for example.

We now check that $b_\lambda(q)$ is unchanged by move A. After move A, using the last formula for $b_\lambda(q)$ in the previous paragraph, $b_\lambda(q)$ becomes

$$[(u_2 + v_1)v_2, v_1, v_2\}\lambda = [u_2v_2, v_1, v_2\}\lambda + [v_1v_2, v_1, v_2\}\lambda.$$

By relation (*), the second term is equal to $[v_1v_2, v_2, v_2\}\lambda$, which is zero since $\{v_2, v_2\} = 0$. So the new $b_\lambda(q)$ is equal to the first term, which is the old $b_\lambda(q)$, as we want.

Applying move B with $i = 1$, the new $b_\lambda(q)$ is $[u_1v_1, \beta^2u_1, u_2\}\lambda = [u_1v_1, u_1, u_2\}\lambda$, which is the old $b_\lambda(q)$, as we want. The same argument works if $i = 2$.

Applying move C with $i = 1$, and using the last formula for $b_\lambda(q)$ above, the new $b_\lambda(q)$ is $[u_2v_2, v_1, v_2\}\lambda$, which is the old $b_\lambda(q)$. Applying move C with $i = 2$, the new $b_\lambda(q)$ is $[u_1v_1, v_1, v_2\}\lambda$, which is equal to the old $b_\lambda(q)$.

Applying move D with $i = 1$, the new $b_\lambda(q)$ is $[u_2v_2, u_1, u_2\}\lambda$, which is the old $b_\lambda(q)$. Applying move D with $i = 2$, the new $b_\lambda(q)$ is $[u_1v_1, u_1, u_2\}\lambda$, which is the old $b_\lambda(q)$. This completes the proof that $b_\lambda(q)$ is an invariant of 4-dimensional quadratic forms with discriminant $[d] \in H^{1,0}(k)$. Thus we have found all the invariants for $SO(q_0)$ for q_0 of dimension 4.

Finally, we turn to the case $n = 1$. That is, given an element $[d] \in H^{1,0}(k)$, we want to find the invariants u in $H^{m+1,m}$ for 2-dimensional quadratic forms with discriminant $[d]$ over fields F/k. Every such form can be written as $q = b_1\langle\langle d]]$ for some $b_1 \in F^*$. The form is determined up to isomorphism by $\{b_1\} \in H^{1,1}(F)$. So any invariant u determines an invariant for μ_2 over k with values in $H^{m+1,m}$. By Proposition 6.1, the invariant has the form

$$u(b_1\langle\langle d]]) = c + \{b_1\}e$$

for some (unique) $c \in H^{m+1,m}(k)$ and $e \in H^{m,m-1}(k)$. It remains to determine for which $e \in H^{m,m-1}(k)$ is $\{b_1\}e$ an invariant of q.

One invariant we know is the Clifford invariant of q, $\text{clif}(q) = [d, b_1\}$. It follows that any $e \in [d]H^{m-1,m-1}(k)$ gives an invariant of q. We show the converse. Let l

be the separable quadratic extension of k with discriminant d. Then, for any field F over l and any $b_1 \in F^*$, the form $q = b_1\langle\langle d]]$ is hyperbolic, and so $u(q)$ must be independent of b_1 on fields over l. By the uniqueness in Proposition 6.1, it follows that e maps to zero in $H^{m,m-1}(l)$. By Theorem 8.2,

$$\ker(H^{m,m-1}(k) \to H^{m,m-1}(l)) = [d]H^{m-1,m-1}(k).$$

So e is in $[d]H^{m-1,m-1}(k)$. This completes the determination of the invariants of $SO(q_0)$ for q_0 of dimension 2. Theorem 14.1 is proved. ☐

Remark 14.3. The invariant $b_\lambda(q)$ is easier to construct for 4-dimensional forms q with trivial discriminant, as in the case of characteristic not 2 [9, Example 20.3]. Namely, Theorem 14.1 says that $b_1(q) := [a_1, b_1, b_2\} \in H^{3,2}(F)$ is an invariant for quadratic forms $q = b_1\langle\langle a_1]] + b_2\langle\langle a_2]]$ over F with trivial discriminant (that is, $a_1 = a_2$ in $H^{1,0}(F)$).

To prove this directly, note that q is a scalar multiple of a quadratic Pfister form, namely $q = b_1\langle\langle b_1b_2, a_1]]$. (Following the notation of [6, section 9.B], a bilinear Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle$ means $\langle\langle a_1\rangle\rangle \otimes \cdots \otimes \langle\langle a_n\rangle\rangle$, where $\langle\langle a\rangle\rangle$ is the 2-dimensional bilinear form $\langle 1, -a\rangle_{\mathrm{b}}$. A quadratic Pfister form $\langle\langle a_1, \ldots, a_n]]$ means $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \otimes \langle\langle a_n]]$, where $\langle\langle a]]$ is the 2-dimensional quadratic form $[1, a] = x^2 + xy + ay^2$.)

It follows that q is a difference of two quadratic Pfister forms, $q = \langle\langle b_1, b_1b_2, a_1]] - \langle\langle b_1b_2, a_1]] = \varphi_3 - \varphi_2$, in the quadratic Witt group $I_q(F)$. So the class of q in $I_q^2/I_q^3 \cong H^{2,1}(F)$ (also known as the Clifford invariant $\mathrm{clif}(q)$) is equal to the class of φ_2, and that class determines the Pfister form φ_2 up to isomorphism, by the Arason-Pfister Hauptsatz [6, Theorem 23.7]. So q also determines φ_3 up to isomorphism, as $\varphi_3 = q + \varphi_2$ in $I_q(F)$. The class of φ_3 in $H^{3,2}(F)$ is $[a_1, b_1, b_1b_2\} = [a_1, b_1, b_2\}$, and so we have shown that the latter expression is an invariant of q.

15 Cohomological invariants in degree 1

In this section, we compute the mod p cohomological invariants in degree 1 (that is, in $H^{1,0}$ or $H^{1,1}$) for any affine group scheme in characteristic p. The analogous mod l result is easier, using $\mathbf{A}^1$-invariance of mod l étale cohomology: for an affine group scheme G over a field k and a prime number l invertible in k, the group of degree-1 invariants for G over k with coefficients in $\mathbf{Z}/l$ is

$$H^1(k, \mathbf{Z}/l) \oplus \mathrm{Hom}_k(G, \mathbf{Z}/l).$$

A reference for this mod l isomorphism is Guillot [12, Corollary 5.1.5]. (Guillot assumes k algebraically closed, but his proof gives this statement for any field k with l invertible in k.)

Theorem 15.1. *Let G be an affine group scheme of finite type over a field k of characteristic $p > 0$. Then*

$$\mathrm{Inv}_k^{1,0}(G) \cong H^{1,0}(k) \oplus \mathrm{Hom}_k(G, \mathbf{Z}/p).$$

Here $H^{1,0}(k)$ can also be written as $H^1_{\text{ét}}(k, \mathbf{Z}/p)$.

Proof. We have $\mathrm{Inv}_k^{1,0}(G) \cong H^{1,0}(k) \oplus \mathrm{NormInv}_k^{1,0}(G)$, as for invariants in any degree. So it suffices to identify the group of normalized invariants with $\mathrm{Hom}_k(G, \mathbf{Z}/p)$. A homomorphism $G \to \mathbf{Z}/p$ over k clearly gives a normalized invariant for G-torsors with values in $H^{1,0}(F) = H^1_{\text{ét}}(F, \mathbf{Z}/p)$, for fields F over k.

Conversely, let α be a normalized invariant for G with values in $H^{1,0}$. Let V be a representation of G over k with a nonempty open subset U such that G acts freely on U with a quotient scheme U/G over k. Applying α to the obvious G-torsor ξ over the function field $k(U/G)$ yields a $\mathbf{Z}/p$-torsor Y_1 over $k(U/G)$. By Theorem 2.1, the invariant α is determined by the $\mathbf{Z}/p$-torsor Y_1. Let G^0 be the identity component of G, and let Y_2 be the pullback of Y_1 over $k(U/G^0)$. Let Y_3 be the pullback of Y_1 over $k(U)$:

$$\begin{tikzcd}
Y_3 \arrow[r] \arrow[d] & \mathrm{Spec}\ k(U) \arrow[r] \arrow[d] & U \arrow[d] \\
Y_2 \arrow[r] \arrow[d] & \mathrm{Spec}\ k(U/G^0) \arrow[r] \arrow[d] & U/G^0 \arrow[d] \\
Y_1 \arrow[r] & \mathrm{Spec}\ k(U/G) \arrow[r] & U/G.
\end{tikzcd}$$

Since ξ pulls back to a trivial G-torsor over U and α is normalized, the pullback torsor Y_3 over $k(U)$ must be trivial; that is, $Y_3 \cong \mathbf{Z}/p \times \mathrm{Spec}\ k(U)$. Write $\pi_{13}\colon U \to U/G$ and $\pi_{12}\colon U/G^0 \to U/G$ for the obvious quotient morphisms.

Here Y_1 extends to a $\mathbf{Z}/p$-torsor Z over some nonempty open subset W of U/G. So Z pulls back to a $\mathbf{Z}/p$-torsor over $\pi_{13}^{-1}(W) \subset U$ which is trivial over the generic point $\mathrm{Spec}\ k(U)$ of $\pi_{13}^{-1}(W)$, and which therefore has p irreducible components. Since $\pi_{13}^{-1}(W)$ is smooth over k and connected, it follows that this pullback $\mathbf{Z}/p$-torsor over $\pi_{13}^{-1}(W)$ is trivial. Since G^0 is connected, the pullback of Z to $\pi_{12}^{-1}(W) \subset U/G^0$ also has p irreducible components, and so it is a trivial $\mathbf{Z}/p$-torsor over $\pi_{12}^{-1}(W)$. Restricting to the generic point, we deduce that the $\mathbf{Z}/p$-torsor $Y_2 \to \mathrm{Spec}\ k(U/G^0)$ is trivial.

The group scheme G/G^0 is finite and étale over k. Therefore, $k(U/G^0)$ is a finite separable extension field of $k(U/G)$. First consider the case where G/G^0 is the k-group scheme associated to a finite group, which we also call G/G^0. Then $k(U/G^0)$ is a finite Galois extension of $k(U/G)$ with Galois group G/G^0. Let H be the absolute Galois group of $k(U/G)$. By Galois theory, the $\mathbf{Z}/p$-torsor Y_1 over $k(U/G)$ corresponds to a homomorphism $\alpha\colon H \to \mathbf{Z}/p$, and the Galois extension $k(U/G^0)$ of $k(U/G)$ corresponds to a surjective homomorphism $\beta\colon H \to G/G^0$. Because Y_1 pulls back to a trivial torsor over $k(U/G^0)$, α factors uniquely through a homomorphism $G/G^0 \to \mathbf{Z}/p$ of finite groups. Equivalently, $\alpha \in \mathrm{NormInv}_k^{1,0}(G)$ is the invariant associated to a homomorphism $G \to \mathbf{Z}/p$ of k-group schemes, as we want.

Now consider the general case, where the finite étale k-group scheme G/G^0 need not be "split" (meaning the k-group scheme associated to a finite group). Let K be the subgroup of the Galois group H corresponding to the extension $k_s(U/G)$ of $k(U/G)$, so that $H/K \cong \mathrm{Gal}(k_s/k)$. Then $H^{1,0}(k(U/G)) = \mathrm{Hom}(H, \mathbf{Z}/p)$ (the group of continuous homomorphisms), which fits into an exact sequence

$$\begin{array}{ccccc} \mathrm{Hom}(H/K, \mathbf{Z}/p) & \longrightarrow & \mathrm{Hom}(H, \mathbf{Z}/p) & \longrightarrow & \mathrm{Hom}(K, \mathbf{Z}/p)^{H/K} \\ \| & & \| & & \| \\ H^{1,0}(k) & \longrightarrow & H^{1,0}(k(U/G)) & \longrightarrow & H^{1,0}(k_s(U/G))^{\mathrm{Gal}(k_s/k)}. \end{array}$$

The group scheme G/G^0 becomes split over the separable closure k_s, and so the previous paragraph implies that the image of $\alpha \in H^{1,0}(k(U/G))$ in $H^{1,0}(k_s(U/G))$ is the one associated to a homomorphism $(G/G^0)_{k_s} \to \mathbf{Z}/p$. Since this image is also fixed by $\mathrm{Gal}(k_s/k)$, it corresponds to a homomorphism $G \to \mathbf{Z}/p$ of k-group schemes. Thus, letting α' be α minus the invariant of G associated to this homomorphism $G \to \mathbf{Z}/p$, the exact sequence above shows that α' is the image of an element of $H^{1,0}(k)$. Since α' is a normalized invariant, it follows that $\alpha' = 0$. Thus we have shown that α is the invariant associated to a homomorphism $G \to \mathbf{Z}/p$ of k-group schemes. □

The following result overlaps with independent work of Alexander Wertheim. He determines all invariants of a group scheme of multiplicative type with values in $H^1_{\text{ét},\mathbf{Z}/m(1)}$, for any positive integer m [37, Theorem A].

Theorem 15.2. *Let G be an affine group scheme of finite type over a field k of characteristic $p > 0$. Then*

$$\mathrm{Inv}_k^{1,1}(G) \cong H^{1,1}(k) \oplus \mathrm{Hom}_k(G, \mu_p).$$

Proof. Let V be a representation of G over k such that G acts freely on an open subset U with a quotient scheme U/G over k. We can assume that $V - U$ has codimension at least 2 in V. Let α be an invariant for G over k with values in $H^{1,1}$. By Theorem 2.1, α is determined by its class in $H^{1,1}(k(U/G))$, and this class is unramified over U/G; that is, it lies in $H^0_{\mathrm{Zar}}(U/G, H^{1,1})$. The restriction map $H^{1,1}(U/G) \to H^0_{\mathrm{Zar}}(U/G, H^{1,1})$ is an isomorphism, since both groups can be identified with the group $H^0(U/G, \Omega^1_{\log})$ of differential forms. So we can view α as a μ_p-torsor over U/G.

Equivalently, α is a G-equivariant μ_p-torsor over U. We can also view this as a G-equivariant line bundle L on U with a G-equivariant trivialization of $L^{\otimes p}$. Since $V - U$ has codimension at least 2 in V and V is smooth over k, the direct image of L from U to V is a line bundle. The G-action on L and the trivialization of $L^{\otimes p}$ clearly extend to V. So α extends uniquely to a G-equivariant μ_p-torsor over V.

The G-equivariant Picard group of V can be viewed as the Picard group of the stack $[V/G]$ over k. By the homotopy invariance of equivariant K-theory proved by Thomason, $\mathrm{Pic}_G(V)$ is isomorphic to $\mathrm{Pic}_G(\mathrm{Spec}\, k) = \mathrm{Hom}_k(G, \mathbf{G}_m)$ [32, Theorem 4.1]. By the exact sequence $1 \to \mu_p \to \mathbf{G}_m \to \mathbf{G}_m \to 1$ of sheaves in the flat topology, we have an exact sequence of flat cohomology groups over $[V/G]$:

$$(O(V)^*)^G \xrightarrow[p]{} (O(V)^*)^G \to H^1_G(V, \mu_p) \to \mathrm{Pic}_G(V) \xrightarrow[p]{} \mathrm{Pic}_G(V).$$

Here the group of units $O(V)^*$ is equal to k^*, on which G acts trivially. Note that $(k^*)/(k^*)^p$ is isomorphic to $H^1(k, \mu_p) = H^{1,1}(k)$. So this exact sequence can be rewritten as

$$0 \to H^{1,1}(k) \to H^1_G(V, \mu_p) \to \mathrm{Hom}_k(G, \mu_p) \to 0.$$

Every homomorphism $G \to \mu_p$ determines an element of $H^1_G(V, \mu_p)$, and so we can write

$$H^1_G(V, \mu_p) = H^{1,1}(k) \oplus \mathrm{Hom}_k(G, \mu_p).$$

We have an obvious homomorphism from $H^{1,1}(k) \oplus \mathrm{Hom}_k(G, \mu_p)$ to the group of invariants $\mathrm{Inv}^{1,1}_k(G)$, and this homomorphism is an isomorphism by the description of $H^1_G(V, \mu_p)$ above. □

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