Hodge theory of classifying stacks

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This paper creates a correspondence between the representation theory of algebraic groups and the topology of Lie groups. In more detail, we compute the Hodge and de Rham cohomology of the classifying space BG (defined as étale cohomology on the algebraic stack BG) for reductive groups G over many fields, including fields of small characteristic. These calculations have a direct relation with representation theory, yielding new results there. Eventually, p-adic Hodge theory should provide a more subtle relation between these calculations in positive characteristic and torsion in the cohomology of the classifying space $BG_{\mathbf{C}}$.

For the representation theorist, this paper's interpretation of certain Ext groups (notably for reductive groups in positive characteristic) as Hodge cohomology groups suggests spectral sequences that were not obvious in terms of Ext groups (Proposition 9.3). We apply these spectral sequences to compute Ext groups in new cases. The spectral sequences form a machine that can lead to further calculations.

One main result is an isomorphism between the Hodge cohomology of the classifying stack BG and the cohomology of G as an algebraic group with coefficients in the ring $O(\mathfrak{g}) = S(\mathfrak{g}^*)$ of polynomial functions on the Lie algebra $\mathfrak{g}$ (Theorem 3.1):

$$H^i(BG, \Omega^j) \cong H^{i-j}(G, S^j(\mathfrak{g}^*)).$$

This was shown by Bott over a field of characteristic 0 [8], but in fact the isomorphism holds in any characteristic, and even for group schemes over the integers. More generally, we give an analogous description of the equivariant Hodge cohomology of an affine scheme (Theorem 2.1). This was shown by Simpson and Teleman in characteristic 0 [29, Example 6.8(c)].

Using that isomorphism, we improve the known results on the cohomology of the representations $S^j(\mathfrak{g}^*)$. Namely, by Andersen, Jantzen, and Donkin, we have $H^{>0}(G, O(\mathfrak{g})) = 0$ for a reductive group G over a field of characteristic p if p is a "good prime" for G [13, Proposition and proof of Theorem 2.2], [21, II.4.22]. We strengthen that to an "if and only if" statement (Theorem 9.1):

Theorem 0.1. *Let G be a reductive group over a field k of characteristic $p \geq 0$. Then $H^{>0}(G, O(\mathfrak{g})) = 0$ if and only if p is not a torsion prime for G.*

For example, this cohomology vanishing holds for every symplectic group $Sp(2n)$ in characteristic 2 and for the exceptional group G_2 in characteristic 3; these are "bad primes" but not torsion primes.

Finally, we address the problem of computing the Hodge cohomology and de Rham cohomology of BG, especially at torsion primes. At non-torsion primes, we have a satisfying result, proved using ideas from topology (Theorem 9.2):

Theorem 0.2. *Let G be a split reductive group over $\mathbf{Z}$, and let p be a non-torsion prime for G. Then Hodge cohomology $H^*_{\mathrm{H}}(BG/\mathbf{Z})$ and de Rham cohomology $H^*_{\mathrm{dR}}(BG/\mathbf{Z})$, localized at p, are polynomial rings on generators of degrees equal to 2 times the fundamental degrees of G. These graded rings are isomorphic to the cohomology of the topological space $BG_{\mathbf{C}}$ with $\mathbf{Z}_{(p)}$ coefficients.*

At torsion primes p, it is an intriguing question how the de Rham cohomology of $BG_{\mathbf{F}_p}$ is related to the mod p cohomology of the topological space $BG_{\mathbf{C}}$. We show that these graded rings are isomorphic for $G = SO(n)$ with $p = 2$ (Theorem 11.1). On the other hand, we find that

$$\dim_{\mathbf{F}_2} H^{32}_{\mathrm{dR}}(B\,\mathrm{Spin}(11)/\mathbf{F}_2) > \dim_{\mathbf{F}_2} H^{32}(B\,\mathrm{Spin}(11)_{\mathbf{C}}, \mathbf{F}_2)$$

(Theorem 12.1). It seems that no existing results on integral p-adic Hodge theory address the relation between these two rings (because the stack BG is not proper over $\mathbf{Z}$), but the theory may soon reach that point. In particular, the results of Bhatt-Morrow-Scholze suggest that the de Rham cohomology $H^i_{\mathrm{dR}}(BG/\mathbf{F}_p)$ may always be an upper bound for the mod p cohomology of the topological space $BG_{\mathbf{C}}$ [6].

This work was supported by NSF grant DMS-1701237. Bhargav Bhatt convinced me to change some definitions in an earlier version of this paper: Hodge and de Rham cohomology of a smooth stack are now defined as étale cohomology. Thanks to Johan de Jong, Eric Primozic, Raphaël Rouquier, and the referees for their comments. Finally, I am grateful to Jungkai Chen for arranging my visit to National Taiwan University, where this work was completed.

1 Notation

The *fundamental degrees* of a reductive group G over a field k are the degrees of the generators of the polynomial ring $S(X^*(T) \otimes_{\mathbf{Z}} \mathbf{Q})^W$ of invariants under the Weyl group W, where $X^*(T)$ is the character group of a maximal torus T. For k of characteristic zero, the fundamental degrees of G can also be viewed as the degrees of the generators of the polynomial ring $O(\mathfrak{g})^G$ of invariant functions on the Lie algebra. Here are the fundamental degrees of the simple groups [17, section 3.7, Table 1]:

A_l	$2, 3, \ldots, l+1$
B_l	$2, 4, 6, \ldots, 2l$
C_l	$2, 4, 6, \ldots, 2l$
D_l	$2, 4, 6, \ldots, 2l-2; l$
G_2	$2, 6$
F_4	$2, 6, 8, 12$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$

For a commutative ring R and $j \geq 0$, write Ω^j for the sheaf of differential forms over R on any scheme over R. For an algebraic stack X over R, Ω^j is a sheaf of abelian groups on the big étale site of X. (In particular, for every scheme Y over X

of "size" less than a fixed limit ordinal α [32, Tag 06TN], we have an abelian group $\Omega^j(Y/R)$, and these groups form a sheaf in the étale topology.) We define Hodge cohomology $H^i(X,\Omega^j)$ to mean the étale cohomology of this sheaf [32, Tag 06XI]. In the same way, we define de Rham cohomology of a stack, $H^i_{\mathrm{dR}}(X/R)$, as étale cohomology with coefficients in the de Rham complex over R. (If X is an algebraic space, then the cohomology of a sheaf F on the big étale site of X coincides with the cohomology of the restriction of F to the small étale site, the latter being the usual definition of étale cohomology for algebraic spaces [32, Tag 0DGB].) For example, this gives a definition of equivariant Hodge or de Rham cohomology, $H^i_G(X,\Omega^j)$ or $H^i_{G,\mathrm{dR}}(X/R)$, as the Hodge or de Rham cohomology of the quotient stack $[X/G]$. Essentially the same definition was used for smooth stacks in characteristic zero by Teleman and Behrend [33, 3].

In particular, we have the Hodge spectral sequence for a stack X over R, meaning the "hypercohomology" spectral sequence [32, Tag 015J] associated to the de Rham complex of sheaves on X, $0 \to \Omega^0 \to \Omega^1 \to \cdots$:

$$E_1^{ij} = H^j(X,\Omega^i) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/R).$$

This definition of Hodge and de Rham cohomology is the "wrong" thing to consider for an algebraic stack which is not smooth over R. For non-smooth stacks, it would be better to define Hodge and de Rham cohomology using some version of Illusie and Bhatt's derived de Rham cohomology, or in other words using the cotangent complex [5, section 4]. Section 2 has further comments on possible definitions. In this paper, we will only consider Hodge and de Rham cohomology for smooth stacks over a commutative ring R. An important example for the paper is that the classifying stack BG is smooth over R even for non-smooth group schemes G [32, Tag 0DLS]:

Lemma 1.1. *Let G be a group scheme which is flat and locally of finite presentation over a commutative ring R. Then the algebraic stack BG is smooth over R. More generally, for a smooth algebraic space X over R on which G acts, the quotient stack $[X/G]$ is smooth over R.*

Let X be an algebraic stack over R, and let U be an algebraic space with a smooth surjective morphism to X. The *Čech nerve* $C(U/X)$ is the simplicial algebraic space:

$$\begin{tikzcd} U & U \times_X U \arrow[l, shift left] \arrow[l, shift right] & U \times_X U \times_X U \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[from=1-1, to=1-2, bend right] \arrow[from=1-2, to=1-3, bend right] \arrow[from=1-2, to=1-3, bend right=40] \end{tikzcd}$$

For any sheaf F of abelian groups on the big étale site of X, the étale cohomology of X with coefficients in F can be identified with the étale cohomology of the simplicial algebraic space $C(U/X)$ [32, Tags 06XJ, 0DGB]. In particular, there is a spectral sequence:

$$E_1^{ij} = H^j_{\mathrm{et}}(U_X^{i+1}, F) \Rightarrow H^{i+j}_{\mathrm{et}}(X,F).$$

Write $H^i_{\mathrm{H}}(X/R) = \oplus_j H^j(X,\Omega^{i-j})$ for the Hodge cohomology of an algebraic stack X over R, graded by total degree.

Let G be a group scheme which is flat and locally of finite presentation over a commutative ring R. Then the Hodge cohomology of the stack BG can be viewed, essentially by definition, as the ring of characteristic classes in Hodge cohomology for principal G-bundles (in the fppf topology). Concretely, for any scheme X over R, a principal G-bundle over X determines a morphism $X \to BG$ of stacks over R and hence a pullback homomorphism

$$H^i(BG, \Omega^j) \to H^i(X, \Omega^j).$$

Note that for a scheme X over R, $H^i(X, \Omega^j)$ can be computed either in the Zariski or in the étale topology, because the sheaf Ω^j (on the small étale site of X) is quasi-coherent [32, Tags 03OY, 0DGB].

For any scheme X over a commutative ring R, there is a simplicial scheme EX whose space $(EX)_n$ of n-simplices is $X^{\{0,\ldots,n\}} = X^{n+1}$ [11, 6.1.3]. For a group scheme G over R, the simplicial scheme $B_{\text{simp}}G$ over R is defined as the quotient of the simplicial scheme EG by the free left action of G:

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r, bend right] & G \arrow[l, shift left] \arrow[l, shift right] \arrow[r, bend right] \arrow[r, bend right=50] & G^2 \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \cdots \end{tikzcd}$$

If G is smooth over R, then Hodge cohomology $H^i(BG, \Omega^j)$ as defined above can be identified with the cohomology of $B_{\text{simp}}G$, because this simplicial scheme is the Čech nerve of the smooth surjective morphism $\mathrm{Spec}(R) \to BG$. For G not smooth, one has instead to use the Čech nerve of a smooth presentation of BG. See for example the calculation of the Hodge cohomology of $B\mu_p$ in characteristic p, Proposition 10.1.

It is useful that we can compute Hodge cohomology via any smooth presentation of a stack. For example, let H be a closed subgroup scheme of a smooth group scheme G over a commutative ring R, and assume that H is flat and locally of finite presentation over R. Then G/H is an algebraic space with a smooth surjective morphism $G/H \to BH$ over R, and so we can compute the Hodge cohomology of the stack BH using the associated Čech nerve. Explicitly, that is the simplicial algebraic space EG/H, and so we have:

Lemma 1.2.

$$H^i(BH, \Omega^j) \cong H^i(EG/H, \Omega^j).$$

Note that the cohomology theories we are considering are not A^1-homotopy invariant. Indeed, Hodge cohomology is usually not the same for a scheme X as for $X \times A^1$, even over a field of characteristic zero. For example, $H^0(\mathrm{Spec}(k), O) = k$, whereas $H^0(A^1_k, O)$ is the polynomial ring $k[x]$. In de Rham cohomology, $H^0_{\mathrm{dR}}(A^1/k)$ is just k if k has characteristic zero, but it is $k[x^p]$ if k has characteristic $p > 0$.

2 Equivariant Hodge cohomology and functions on the Lie algebra

In this section, we identify the Hodge cohomology of a quotient stack with the cohomology of an explicit complex of vector bundles (Theorem 2.1). As a special case, we relate the Hodge cohomology of a classifying stack BG to the cohomology of G as a group scheme (Corollary 2.2). In this section, we assume G is smooth. Undoubtedly, various generalizations of the statements here are possible. In particular, we will give an analogous description of the Hodge cohomology of BG for a non-smooth group G in Theorem 3.1.

The main novelty is that these results hold in any characteristic. In particular, Theorem 2.1 was proved in characteristic zero by Simpson and Teleman [29, Example 6.8(c)]. As discussed in section 1, equivariant Hodge cohomology $H^i_G(X,\Omega^j)$ is defined as cohomology of the quotient stack $[X/G]$.

Theorem 2.1. *Let G be a smooth affine group scheme over a commutative ring R. Let G act on a smooth affine scheme X over R. Then there is a canonical isomorphism*

$$H^i_G(X,\Omega^j)\cong H^i_G(X,\Lambda^j L_{[X/G]}),$$

where $\Lambda^j L_{[X/G]}$ is the complex of G-equivariant vector bundles on X, in degrees 0 to j:

$$0\to\Omega^j_X\to\Omega^{j-1}_X\otimes\mathfrak{g}^*_X\to\cdots\to S^j(\mathfrak{g}^*_X)\to 0,$$

associated to the map $\mathfrak{g}_X\to TX$.

Here the action of G on X gives an action of the Lie algebra $\mathfrak{g}$ by vector fields on X by differentiating the action $G\times_R X\to X$ at $1\in G(R)$. This can be viewed as a map from the trivial vector bundle $\mathfrak{g}_X$ over X to the tangent bundle TX. (The action of G on $\mathfrak{g}_X$ is nontrivial, coming from the adjoint representation of G on $\mathfrak{g}$.) Dualizing gives the map $\Omega^1_X\to\mathfrak{g}^*_X$ used in Theorem 2.1.

The isomorphism of Theorem 2.1 expresses the cohomology over $[X/G]$ of the "big sheaf" Ω^j, which is not a quasi-coherent sheaf on $[X/G]$, in terms of the cohomology of a complex of quasi-coherent sheaves on $[X/G]$. (Here differentials are over R unless otherwise stated. The sheaf Ω^j on the big étale site of $[X/G]$ is not quasi-coherent for $j>0$ because, for a morphism $f\colon Y\to Z$ of schemes over $[X/G]$, the pullback map $f^*\Omega^j_{Z/R}\to\Omega^j_{Y/R}$ need not be an isomorphism.)

One might prefer to take the right side of Theorem 2.1 as a definition of Hodge cohomology for algebraic stacks. This could be done without any smoothness assumption. Namely, Olsson defined the cotangent complex $L_{X/Y}$ as an inverse system for any quasi-compact and quasi-separated morphism $f\colon X\to Y$ of algebraic stacks, correcting the approach of Laumon and Moret-Bailly [27, section 8]. One could then define Hodge cohomology of X over Y as $R^if_*(L\Lambda^jL_{X/Y})$ (perhaps "Hodge-completed" in the sense of [5]). For X and G smooth over $Y=\mathrm{Spec}(R)$, this definition agrees with the right side of Theorem 2.1. using that

$$L_{[X/G]/R}\cong[\Omega^1_X\to\mathfrak{g}^*_X]$$

by the transitivity triangle [27, 8.1.5]. We have preferred to take the left side of Theorem 2.1 as the definition, using "big sheaves", because that definition is directly related to the cohomology of simplicial spaces as discussed in section 1. As a result, Theorem 2.1 makes a nontrivial connection between the two approaches.

Theorem 2.1 is useful already for $X=\mathrm{Spec}(R)$, where it gives the following result, proved over a field of characteristic zero by Bott [8].

Corollary 2.2. *Let G be a smooth affine group scheme over a commutative ring R. Then there is a canonical isomorphism*

$$H^i(BG,\Omega^j)\cong H^{i-j}(G,S^j(\mathfrak{g}^*)).$$

The group on the left is an étale cohomology group of the algebraic stack BG over R, as discussed in section 1. On the right is the cohomology of G as a group scheme, defined by $H^i(G,M)=\mathrm{Ext}^i_G(R,M)$ for a G-module M [21, section 4.2].

Proof. (Corollary 2.2) This follows from Theorem 2.1 applied to the stack $BG=[\mathrm{Spec}(R)/G]$. The deduction uses two facts. First, a quasi-coherent sheaf on BG is equivalent to a G-module [32, Tag 06WS]. Second, for a G-module M, the cohomology of the corresponding quasi-coherent sheaf on the big étale site of BG coincides with its cohomology as a G-module, $H^*(G,M)$, since both are computed by the same Čech complex (section 1 for the sheaf, [21, Proposition 4.16] for the module). □

Proof. (Theorem 2.1) The adjoint representation of G on $\mathfrak{g}$ determines a G-equivariant vector bundle $\mathfrak{g}_X$ on X. The action of G on X gives a morphism $\Omega^1_X\to\mathfrak{g}^*_X$ of G-equivariant quasi-coherent sheaves (in fact, vector bundles) on X. Consider these equivariant sheaves as quasi-coherent sheaves on $[X/G]$, according to [32, Tag 06WS].

We will define a map from the complex $\Omega^1_X\to\mathfrak{g}^*_X$ of quasi-coherent sheaves on $[X/G]$ (in degrees 0 and 1) to the sheaf Ω^1, in the derived category $D([X/G]_{\mathrm{et}},O_{[X/G]})$ of $O_{[X/G]}$-modules on the big étale site $[X/G]_{\mathrm{et}}$. To do this, define another sheaf S on the big étale site of $[X/G]$ by: for a scheme U over $[X/G]$, let $E=U\times_{[X/G]}X$ (so that $\pi\colon E\to U$ is a principal G-bundle), and define $S(U)=H^0(E,\Omega^1)^G$. (This space of invariants means the equalizer of the pullbacks via the two morphisms $G\times E\to E$, the projection and the group action.) Since G is smooth over R, there is a short exact sequence of quasi-coherent sheaves on E, $0\to\pi^*\Omega^1_U\to\Omega^1_E\to\Omega^1_{E/U}\to 0$. These are G-equivariant sheaves on E, and so this can be viewed as the pullback of a short exact sequence of sheaves on U, known as the *Atiyah sequence* [2, Theorem 1], [20, VII.2.4.2.13–14]:

$$0\to\Omega^1_U\to\pi_*(\Omega^1_E)^G\to{}^E\mathfrak{g}^*_U\to 0.$$

Here ${}^E\mathfrak{g}^*_U$ is the vector bundle on U associated to the G-bundle $E\to U$ and the action of G on $\mathfrak{g}^*$. Since the G-bundle $E\to U$ is arbitrary, we have produced an exact sequence

$$0\to\Omega^1\to S\to\mathfrak{g}^*_X\to 0$$

of sheaves on the big étale site of $[X/G]$. By definition of the Atiyah sequence, the map from $S(U)=H^0(E,\Omega^1_E)^G$ to $H^0(U,{}^E\mathfrak{g}^*_U)=H^0(E,\mathfrak{g}^*_E)^G$ arises from the map $\Omega^1_E\to\mathfrak{g}^*_E$ given by differentiating the action of G on E.

Thus the sheaf Ω^1 on $[X/G]$ is isomorphic in the derived category to the complex $S\to\mathfrak{g}^*_X$ (in cohomological degrees 0 and 1) on $[X/G]$. Therefore, to produce the map in $D([X/G]_{\mathrm{et}},O_{[X/G]})$ promised above, it suffices to define a map α of complexes of sheaves on $[X/G]_{\mathrm{et}}$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Omega^1_X & \longrightarrow & \mathfrak{g}^*_X & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S & \longrightarrow & \mathfrak{g}^*_X & \longrightarrow & 0.
\end{array}$$

(As above, $\mathfrak{g}^*_X$ denotes the vector bundle on $[X/G]$ associated to the representation of G on $\mathfrak{g}^*$, and Ω^1_X denotes the vector bundle on $[X/G]$ corresponding to the G-equivariant vector bundle of the same name on X.) It is now easy to produce the map α of complexes: for any scheme U over $[X/G]$, with associated principal G-bundle $E \to U$ and G-equivariant morphism $h\colon E \to X$, the map from $\Omega^1_X(U) = H^0(E, h^*\Omega^1_X)^G$ to $S(U) = H^0(E, \Omega^1_E)^G$ is the pullback, and the map from ${}^E\mathfrak{g}^*_U$ to itself is the identity. In words, the difference between $\Omega^1_X(U)$ and $S(U)$ comes from the difference between (1) the pullback to E of the sheaf of differentials of X over R and (2) the sheaf of differentials on E over R.

The commutativity of the diagram above follows from the G-equivariance of the morphism $h\colon E \to X$, since the two horizontal maps arise by differentiating the actions of G on X and on E.

For any $j \geq 0$, taking the jth derived exterior power over $O_{[X/G]}$ of this map of complexes gives a map from the Koszul complex

$$0 \to \Omega^j_X \to \Omega^{j-1}_X \otimes \mathfrak{g}^*_X \to \cdots \to S^j(\mathfrak{g}^*_X) \to 0$$

(in degrees 0 to j) of vector bundles on $[X/G]$ to the big sheaf Ω^j, in $D([X/G]_{\text{et}}, O_{[X/G]})$. (The description of the derived exterior power of a 2-term complex of flat modules as a Koszul complex follows from Illusie [18, Proposition II.4.3.1.6], by the same argument used for derived divided powers in [20, Lemme VIII.2.1.2.1].) We want to show that this map of complexes induces an isomorphism on cohomology over $[X/G]$.

By the exact sequence above for the big sheaf Ω^1 on $[X/G]$, we can identify the big sheaf Ω^j in the derived category with a similar-looking Koszul complex:

$$0 \to \Lambda^j(S) \to \Lambda^{j-1}(S) \otimes \mathfrak{g}^*_X \to \cdots \to S^j(\mathfrak{g}^*_X) \to 0.$$

We want to show that the map $\Lambda^j(\alpha)$ from the Koszul complex of vector bundles (in the previous paragraph) to this complex of big sheaves induces an isomorphism on cohomology over $[X/G]$. For each of these complexes, we have a spectral sequence from the cohomology over $[X/G]$ of the individual sheaves to the "hypercohomology" over $[X/G]$ of the whole complex [32, Tag 015J]. We have a map of spectral sequences. Therefore, to show that the map on hypercohomology is an isomorphism, it suffices to show that the map on cohomology of the individual sheaves is an isomorphism. That is, it suffices to show that for each $0 \leq i \leq j$, the map

$$H^*_G(X, \Omega^i_X \otimes S^{j-i}(\mathfrak{g}^*_X)) \to H^*_G(X, \Lambda^i(S) \otimes S^{j-i}(\mathfrak{g}^*_X))$$

is an isomorphism. (Equivariant cohomology is defined as cohomology of the stack $[X/G]$, as discussed in section 1.)

By section 1, we can compute both of these cohomology groups on the Čech nerve of the smooth surjective morphism $X \to [X/G]$. This simplicial space can be written as $(X \times EG)/G$, where all products are over R:

$$\begin{tikzcd} X \arrow[r, bend right] & X \times G \arrow[l, shift left] \arrow[l, shift right] \arrow[r, bend right] \arrow[r, bend right=40] & X \times G^2 \cdots, \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

Since X is affine, all the spaces in this simplicial space are affine schemes. Therefore, for any $0 \leq i \leq j$, $H^*_G(X, \Omega^i_X \otimes S^{j-i}(\mathfrak{g}^*_X))$ is the cohomology of the complex of H^0 of the sheaves $\Omega^i_X \otimes O_{EG} \otimes S^{j-i}(\mathfrak{g}^*_X)$ over the spaces making up $(X \times EG)/G$. Likewise, $H^*_G(X, \Lambda^i(S) \otimes S^{j-i}(\mathfrak{g}^*_X))$ is the cohomology of the complex of H^0 of the sheaves $\Lambda^i(S) \otimes S^{j-i}(\mathfrak{g}^*_X)$ over the spaces making up $(X \times EG)/G$.

Let

$$\varphi\colon H^0(X \times EG, \Omega^i_X \otimes O_{EG} \otimes S^{j-i}(\mathfrak{g}^*)) \to H^0(X \times EG, \Lambda^i(S) \otimes S^{j-i}(\mathfrak{g}^*))$$

be the map of complexes of G-modules arising as H^0 of sheaves over the spaces making up $X \times EG$. The boundary maps in these complexes are alternating sums of pullbacks via the face maps in this simplicial space. By the previous paragraph, we want to show that the induced map φ^G on G-invariants is a quasi-isomorphism. Moreover, all of these G-modules arising as H^0 are induced from representations of the trivial group, because $X \times G^{r+1} \to (X \times G^{r+1})/G$ is a G-torsor with a section for each $r \geq 0$. Indeed, a choice of section of this G-torsor trivializes the torsor, and so the group of sections of a G-equivariant sheaf of $X \times G^{r+1}$ is the subspace of invariants tensored with $O(G)$, as a G-module. (Note that trivializations of these G-torsors cannot be made compatible with the face maps of the simplicial space, in general.) And every tensor product $O(G) \otimes_R M$ for a G-module M is injective as a G-module [21, Proposition 3.10]. It follows that $H^i(G, O(G) \otimes_R M) = 0$ for $i > 0$ [21, Lemma I.4.7].

Therefore, to show that the map φ^G of G-invariants is a quasi-isomorphism (as we want), it suffices to show that the map φ is a quasi-isomorphism. And for that, we can forget about the G-action. That is, we want to show that the map of complexes with rth term (for $r \geq 0$)

$$H^0(X \times G^{r+1}, \Omega^i_X \otimes O_{G^{r+1}} \otimes S^{j-i}(\mathfrak{g}^*)) \to H^0(X \times G^{r+1}, \Omega^i_{X \times G^{r+1}} \otimes S^{j-i}(\mathfrak{g}^*))$$

is a quasi-isomorphism.

We can write $\Omega^i_{X \times G^{r+1}}$ as the direct sum $\oplus_{l=0}^i \Omega^{i-l}_X \otimes \Omega^l_{G^{r+1}}$. Moreover, this splitting is compatible with pullback along the face maps of the simplicial scheme $X \times EG$. So the map of complexes above is the inclusion of a summand (corresponding to $l = 0$). It remains to show that for every $0 < l \leq i$, the lth summand is a complex A_l with cohomology zero. Its rth term is

$$\Omega^{i-l}(X) \otimes_R \Omega^l(G^{r+1}) \otimes_R S^{j-i}(\mathfrak{g}^*).$$

To analyze the cohomology of the complex A_l, we use the well-known "contractibility" of EG, in the following form:

Lemma 2.3. *Let Y be a scheme over a ring R with $Y(R)$ not empty. For any sheaf M of abelian groups on the big étale site of R, the cohomology of the simplicial scheme EY over R with coefficients in M coincides with the cohomology of* $\mathrm{Spec}(R)$*:*

$$H^i(EY, M) \cong H^i(R, M).$$

Proof. This is the standard result that a morphism $Y \to \mathrm{Spec}(R)$ with a section satisfies cohomological descent, via an explicit chain homotopy. More generally, it would suffice to have sections locally on $\mathrm{Spec}(R)$ [1, Proposition Vbis.3.3.1]. $\square$

Returning to the proof of Theorem 2.1: we want to show that for $l > 0$, the complex A_l has zero cohomology. Recall that the rth term of A_l is

$$\Omega^{i-l}(X) \otimes_R \Omega^l(G^{r+1}) \otimes_R S^{j-i}(\mathfrak{g}^*),$$

with boundary maps coming from the face maps of the simplicial scheme $X \times EG$. By Lemma 1.2 (applied to the sheaf Ω^l on the big étale site of R and the simplicial scheme EG), the complex A_l has cohomology equal to $\Omega^{i-l}(X) \otimes \Omega^l(\operatorname{Spec} R) \otimes S^{j-i}(\mathfrak{g}^*)$ in degree 0 and zero in other degrees. Since $l > 0$, the cohomology in degree 0 also vanishes. The proof is complete. □

The argument works verbatim to prove a twisted version of Corollary 2.2, where the sheaf Ω^j on BG is tensored with the vector bundle associated to any G-module. The generalization will not be needed in this paper, but we state it for possible later use.

Theorem 2.4. *Let G be a smooth affine group scheme over a commutative ring R. Let M be a G-module that is flat over R. Then there is a canonical isomorphism*

$$H^i(BG, \Omega^j \otimes M) \cong H^{i-j}(G, S^j(\mathfrak{g}^*) \otimes M).$$

3 Flat group schemes

We now describe the Hodge cohomology of the classifying stack of a group scheme G which need not be smooth, generalizing Corollary 2.2. The analog of the co-Lie algebra $\mathfrak{g}^*$ in this generality is the co-Lie complex l_G in the derived category of G-modules, defined by Illusie [20, section VII.3.1.2]. Namely, l_G is the pullback of the cotangent complex of $G \to \operatorname{Spec}(R)$ to $\operatorname{Spec}(R)$, via the section $1 \in G(R)$. (The cotangent complex $L_{X/Y}$ of a morphism $X \to Y$ of schemes is an object of the quasi-coherent derived category of X; if X is smooth over Y, then $L_{X/Y}$ is the sheaf $\Omega^1_{X/Y}$.)

The cohomology of l_G in degree 0 is the R-module ω^1_G, the restriction of Ω^1_G to the identity $1 \in G(R)$; thus ω^1_G is the co-Lie algebra $\mathfrak{g}^*$ if G is smooth over R. The complex l_G has zero cohomology except in cohomological degrees -1 and 0. If G is smooth, then l_G has cohomology concentrated in degree 0. More generally, a closed immersion of G into a smooth R-group scheme H yields an explicit formula for l_G in the derived category of G-modules: l_G is the complex

$$0 \to I/\mathfrak{m}I \to \mathfrak{m}/\mathfrak{m}^2 \to 0,$$

where I is the ideal defining G in H and $\mathfrak{m}$ is the ideal defining the point 1 in H, so that $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{h}^*$ [19, section 4.2].

Theorem 3.1. *Let G be a flat affine group scheme of finite presentation over a commutative ring R. Then there is a canonical isomorphism*

$$H^i(BG, \Omega^j) \cong H^{i-j}(G, S^j(l_G)).$$

This is an isomorphism of rings from $H^*_{\mathrm{H}}(BG/R)$ to $H^*(G, S^*(l_G))$.

Proof. As discussed in section 1, we can compute $H^*(BG, \Omega^j)$ as the étale cohomology with coefficients in Ω^j of the Čech nerve associated to any algebraic space U over R with a smooth surjective morphism from U to the stack BG. The assumption on G implies that BG is a quasi-compact algebraic stack over R, and so there is an affine scheme U with a smooth surjective morphism $U \to BG$ [32, Tags 06FI and 04YA]. By Lemma 1.1, BG is smooth over R, and so U is smooth over R. Let $E = U \times_{BG} \operatorname{Spec}(R)$; then E is a smooth R-space with a free G-action such that $U = E/G$. Also, E is affine because U and G are affine.

By section 1, $H^*(BG, \Omega^j)$ is the étale cohomology with coefficients in Ω^j of the simplicial algebraic space EE/G:

$$\begin{tikzcd} E/G \arrow[r, bend right] & E^2/G \arrow[l, shift left] \arrow[l, shift right] \arrow[r, bend right] \arrow[r, bend right=40] & E^3/G \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

By the properties of E and U above, E^{n+1}/G is an affine scheme for all $n \geq 0$. Since $H^*(BG, \Omega^j)$ is the cohomology with coefficients in Ω^j of the simplicial scheme EE/G, this is the cohomology of the cochain complex

$$0 \to \Omega^j(E/G) \to \Omega^j(E^2/G) \to \cdots .$$

As in the proof of Theorem 2.1, this complex is the G-invariants of the complex

$$0 \to H^0(E, \pi^*(\Omega^j_{E/G})) \to H^0(E^2, \pi^*(\Omega^j_{E^2/G})) \to \cdots ,$$

where we write π for the morphism $E^{n+1} \to E^{n+1}/G$ for any $n \geq 0$.

For any smooth R-scheme X with a free action of G, I claim that there is a canonical exact triangle in the quasi-coherent derived category of G-equivariant sheaves on X:

$$\pi^*(\Omega^1_{X/G}) \to \Omega^1_X \to l_G,$$

where we write l_G for the pullback of the co-Lie complex l_G from the stack BG over R to X. To deduce this from Illusie's results on the cotangent complex $L_{X/Y}$, let $Y = X/G$ and $S = \operatorname{Spec}(R)$, and use the transitivity exact triangle for $X \to Y \to S$ in the derived category of X [18, II.2.1.5.2]:

$$\pi^* L_{Y/S} \to L_{X/S} \to L_{X/Y}.$$

Since X is smooth over S, so is Y (even though G need not be); so $L_{Y/S} \cong \Omega^1_{Y/S}$ and $L_{X/S} \cong \Omega^1_{X/S}$. Also, since $X \to Y$ is a G-torsor in the fppf topology, $L_{X/Y}$ is the pullback of an object $l_{X/Y}$ on Y [20, VII.2.4.2.8]. Furthermore, $l_{X/Y}$ in the fppf topology is the pullback of l_G via the morphism from Y to the stack BG corresponding to the G-torsor $X \to Y$ [20, VII.3.1.2.6].

Applying this to E^{n+1}/G for any $n \geq 0$, we get an exact triangle

$$\pi^*(\Omega^1_{EE/G}) \to \Omega^1_{EE} \to l_G$$

in $D_G(EE)$, or equivalently

$$l_G[-1] \to \pi^*(\Omega^1_{EE/G}) \to \Omega^1_{EE}.$$

It follows that for any $j \geq 0$, $\pi^*(\Omega^j_{EE/G})$ has a filtration in the derived category with quotients $\pi^*(\Omega^{j-m}_{EE}) \otimes \Lambda^m(l_G[-1])$ for $m = 0, \ldots, j$.

If $E(R)$ is nonempty, then $H^i(EE, \Omega^j) \cong H^i(\operatorname{Spec}(R), \Omega^j)$, by Lemma 2.3. That group is zero unless $i = j = 0$, in which case it is R. By faithfully flat descent, the same conclusion holds under our weaker assumption that $E \to \operatorname{Spec}(R)$ is smooth and surjective. Therefore, in the filtration above, all objects but one have zero cohomology in all degrees over EE. We deduce that the homomorphism

$$H^i(EE, \Lambda^j(l_G[-1])) \to H^i(EE, \pi^*(\Omega^j_{EE/G}))$$

is an isomorphism of G-modules for all i. By Illusie's "décalage" isomorphism [18, Proposition I.4.3.2.1(i)], we can write $S^j(l_G)[-j]$ instead of $\Lambda^j(l_G[-1])$.

The cochain complex $O(EE)$ has cohomology R in degree 0 and 0 otherwise, by Lemma 2.3 again. So the complex of global sections of the trivial vector bundle $S^j(l_G)$ over EE is isomorphic, in the derived category of G-modules, to the complex of G-modules $S^j(l_G)$. We conclude that the complex of sections of $\pi^*(\Omega^j_{EE/G})$ over EE is isomorphic to $S^j(l_G)[-j]$ in the derived category of G-modules.

Finally, we observe that each G-module in this complex,

$$M := H^0(E^{n+1}, \pi^*(\Omega^j_{E^{n+1}/G}))$$

for $n \geq 0$, is acyclic (meaning that $H^{>0}(G, M) = 0$). More generally, for any affine R-scheme Y with a free G-action such that Y/G is affine, and any quasi-coherent sheaf F on Y/G, $M := H^0(Y, \pi^*F)$ is acyclic. Indeed, this holds if $Y \to Y/G$ is a trivial G-bundle, since then $M = O(G) \otimes F$ and so M is acyclic [21, Lemma 4.7]. We can prove acyclicity in general by pulling the G-bundle over Y/G back to a G-bundle over Y, which is trivial; then $H^{>0}(G, M) \otimes_{O(Y/G)} O(Y)$ is 0 by [21, Proposition 4.13], and so $H^{>0}(G, M) = 0$ by faithfully flat descent.

We conclude that the complex computing $H^*(BG, \Omega^j)$ is the same one that computes $H^*(G, S^j(l_G)[-j])$. ∎

4 Good filtrations

In this section, we explain how known results in representation theory imply calculations of the Hodge cohomology of classifying spaces in many cases, via Corollary 2.2. This is not logically necessary for the rest of the paper: Theorem 9.1 is a stronger calculation of Hodge cohomology, based on ideas from homotopy theory.

Let G be a split reductive group over a field k. (A textbook reference on split reductive groups is [24, Chapter 21].) A *Schur module* for G is a module of the form $H^0(\lambda)$ for a dominant weight λ. By definition, $H^0(\lambda)$ means $H^0(G/B, L(\lambda))$, where B is a Borel subgroup and $L(\lambda)$ is the line bundle associated to λ. For k of characteristic zero, the Schur modules are exactly the irreducible representations of G. Kempf showed that the dimension of the Schur modules is independent of the characteristic of k [21, Chapter II.4]. They need not be irreducible in characteristic p, however.

A G-module M has a *good filtration* if there is a sequence of submodules $0 \subset M_0 \subset M_1 \subset \cdots$ such that $M = \cup M_j$ and each quotient M_i/M_{i-1} is a Schur module. One good feature of Schur modules is that their cohomology groups are known, by Cline-Parshall-Scott-van der Kallen [21, Proposition 4.13]. Namely,

$$H^i(G, H^0(\lambda)) \cong \begin{cases} k & \text{if } i = 0 \text{ and } \lambda = 0 \\ 0 & \text{otherwise.} \end{cases}$$

As a result, $H^i(G, M) = 0$ for all $i > 0$ when M has a good filtration.

The following result was proved by Andersen-Jantzen and Donkin [13, Proposition and proof of Theorem 2.2], [21, II.4.22]. The statement on the ring of invariants incorporates earlier work by Kac and Weisfeiler. Say that a prime number p is *bad* for a reductive group G if $p = 2$ and G has a simple factor not of type A_n, $p = 3$ and G has a simple factor of exceptional type, or $p = 5$ and G has an E_8 factor. Otherwise, p is *good* for G.

Theorem 4.1. *Let G be a split reductive group over a field k. Assume either that G is a simply connected semisimple group and* $\mathrm{char}(k)$ *is good for G, or that $G = GL(n)$. Then the polynomial ring $O(\mathfrak{g}) = S(\mathfrak{g}^*)$ has a good filtration as a G-module, and the ring of invariants $O(\mathfrak{g})^G$ is a polynomial ring over k, with generators in the fundamental degrees of G.*

It follows that, under these assumptions, $H^{>0}(G, S^j(\mathfrak{g}^*))$ is zero for all $j \geq 0$. Equivalently, $H^i(BG, \Omega^j) = 0$ for $i \neq j$, by Corollary 2.2. We prove this under the weaker assumption that p is not a torsion prime in Theorem 9.1.

5 Künneth formula

The Künneth formula holds for Hodge cohomology, in the following form. The hypotheses apply to the main case studied in this paper: classifying stacks BG with G an affine group scheme of finite type over a field.

Proposition 5.1. *Let X and Y be quasi-compact algebraic stacks with affine diagonal over a field k. Then*

$$H^*_{\mathrm{H}}((X \times_k Y)/k) \cong H^*_{\mathrm{H}}(X/k) \otimes_k H^*_{\mathrm{H}}(Y/k).$$

Proof. Since X and Y are quasi-compact, there are affine schemes A and B with smooth surjective morphisms $A \to X$ and $B \to Y$ [32, Tag 04YA]. Since X and Y have affine diagonal, the fiber products A^{n+1}_X and B^{n+1}_Y are affine over the products A^{n+1} and B^{n+1} over k, and so they are affine schemes, for all $n \geq 0$.

The morphism $A \times B \to X \times Y$ is smooth and surjective. Therefore, the Hodge cohomology of $X \times Y$ is the cohomology of the Čech nerve $C(A \times B/X \times Y)$ over k, with coefficients in Ω^* (with zero differential). This space is the product $C(A/X) \times C(B/Y)$ over k. By the previous paragraph, these are in fact simplicial affine schemes over k.

The quasi-coherent sheaf Ω^1 on the product of two affine schemes over k is the direct sum of the pullbacks of Ω^1 from the two factors. (No smoothness is needed for this calculation.) Therefore, the quasi-coherent sheaf Ω^* on the product affine scheme $A^{n+1}_X \times B^{n+1}_Y$ over k is the tensor product of the pullbacks on Ω^* on those two schemes. So $H^0(A^{n+1}_X \times B^{n+1}_Y, \Omega^*)$ is the tensor product of $H^0(A^{n+1}_X, \Omega^*)$ and $H^0(B^{n+1}_Y, \Omega^*)$ over k.

The spectral sequence of the simplicial scheme $C(A/X) \times C(B/Y)$ with coefficients in Ω^* reduces to one row, since all the schemes here are affine. Explicitly, by the previous paragraph, the cohomology of the product simplicial scheme is the cohomology of the tensor product over k of the two cosimplicial vector spaces $H^0(A^{n+1}_X, \Omega^*)$ and $H^0(B^{n+1}_Y, \Omega^*)$. By the Eilenberg-Zilber theorem, it follows that the cohomology of the product simplicial scheme is the tensor product over k of the cohomology of the two factors. [23, Theorem 29.3]. Equivalently,

$$H^*_{\mathrm{H}}((X \times_k Y)/k) \cong H^*_{\mathrm{H}}(X/k) \otimes_k H^*_{\mathrm{H}}(Y/k).$$

□

6 Parabolic subgroups

Theorem 6.1. *Let P be a parabolic subgroup of a reductive group G over a field k, and let L be the Levi quotient of P (the quotient of P by its unipotent radical). Then the restriction*

$$H^i(BP, \Omega^j) \to H^i(BL, \Omega^j)$$

is an isomorphism for all i and j. Equivalently,

$$H^a(P, S^j(\mathfrak{p}^*)) \to H^a(L, S^j(\mathfrak{l}^*))$$

is an isomorphism for all a and j.

Theorem 6.1 can be viewed as a type of homotopy invariance for Hodge cohomology of classifying spaces. This is not automatic, since Hodge cohomology is not A^1-homotopy invariant for smooth varieties. Homotopy invariance of Hodge cohomology also fails in general for classifying spaces. For example, let G_a be the additive group over a field k. Then the Hodge cohomology group $H^1(BG_a, O)$ is not zero for any k, and it is a k-vector space of infinite dimension for k of positive characteristic; this follows from Theorem 6.3, due to Cline, Parshall, Scott, and van der Kallen, together with Corollary 2.2.

Proof. (Theorem 6.1) Let U be the unipotent radical of P, so that $L = P/U$. It suffices to show that

$$H^a(P, S^j(\mathfrak{p}^*)) \to H^a(L, S^j(\mathfrak{l}^*))$$

is an isomorphism after extending the field k. So we can assume that G has a Borel subgroup B and that B is contained in P. Let R be the set of roots for G. We follow the convention that the weights of B acting on the Lie algebra of its unipotent radical are the *negative* roots R^-. There is a subset I of the set S of simple roots so that P is the associated subgroup P_I, in the notation of [21, II.1.8]. More explicitly, let $R_I = R \cap \mathbf{Z}I$; then $P = P_I$ is the semidirect product $U_I \rtimes L_I$, where L_I is the reductive group $G(R_I)$ and $U := U_I$ is the unipotent group $U(R^- \setminus R_I)$.

As a result, the weights of P on $\mathfrak{p}$ are all the roots $\sum_{\alpha \in S} n_\alpha \alpha$ such that $n_\alpha \leq 0$ for α not in I. The coefficients n_α for α not in I are all zero exactly for the weights of P on $\mathfrak{p}/\mathfrak{u}$. As a result, for any $j \geq 0$, the weights of P on $S^j(\mathfrak{p}^*)$ are all in the root lattice, with nonnegative coefficients for the simple roots not in I, and with those coefficients all zero only for the weights of P on the subspace $S^j((\mathfrak{p}/\mathfrak{u})^*) \subset S^j(\mathfrak{p}^*)$.

We now use the following information about the cohomology of P-modules [21, Proposition II.4.10]. For any element λ of the root lattice $\mathbf{Z}S$, $\lambda = \sum_{\alpha \in S} n_\alpha \alpha$, the *height* $\mathrm{ht}(\lambda)$ means the integer $\sum_{\alpha \in S} n_\alpha$.

Proposition 6.2. *Let P be a parabolic subgroup of a reductive group G over a field, and let M be a P-module. If $H^j(P, M) \neq 0$ for some $j \geq 0$, then there is a weight λ of M with $-\lambda \in \mathbf{N}R^+$ and* $\mathrm{ht}(-\lambda) \geq j$.

As mentioned above, for any $j \geq 0$, every weight of P on $M := \operatorname{coker}(S^j((\mathfrak{p}/\mathfrak{u})^*) \to S^j(\mathfrak{p}^*))$ has at least one positive coefficient in terms of the simple roots. By Proposition 6.2, it follows that $H^a(P, M) = 0$ for all a. Therefore, the homomorphism

$$H^a(P, S^j((\mathfrak{p}/\mathfrak{u})^*)) \to H^a(P, S^j(\mathfrak{p}^*))$$

is an isomorphism for all a and j. Here $\mathfrak{p}/\mathfrak{u} \cong \mathfrak{l}$ is a representation of the quotient group $L = P/U$. It remains to show that the pullback

$$H^a(L, S^j((\mathfrak{p}/\mathfrak{u})^*)) \to H^a(P, S^j((\mathfrak{p}/\mathfrak{u})^*))$$

is an isomorphism. This would not be true for an arbitrary representation of L; we will have to use what we know about the weights of L on $S^j((\mathfrak{p}/\mathfrak{u})^*)$.

We also use the following description of the cohomology of an additive group $V = (G_a)^n$ over a perfect field k [21, Proposition I.4.27]. (To prove Theorem 6.1, we can enlarge the field k, and so we can assume that k is perfect.) The following description is canonical, with respect to the action of $GL(V)$ on $H^*(V, k)$. Write $W^{(j)}$ for the jth Frobenius twist of a vector space W, as a representation of $GL(W)$.

Theorem 6.3. *(1) If k has characteristic zero, then $H^*(V, k) \cong \Lambda(V^*)$, with V^* in degree 1.*

(2) If k has characteristic 2, then

$$H^*(V, k) \cong S(\oplus_{j \geq 0} (V^*)^{(j)}),$$

with all the spaces $(V^*)^{(j)}$ in degree 1.

(3) If k has characteristic $p > 2$, then

$$H^*(V, k) \cong \Lambda(\oplus_{j \geq 0} (V^*)^{(j)}) \otimes S(\oplus_{j \geq 1} (V^*)^{(j)}),$$

with all the spaces $(V^*)^{(j)}$ in the first factor in degree 1, and all the spaces $(V^*)^{(j)}$ in the second factor in degree 2.

We also use the Hochschild-Serre spectral sequence for the cohomology of algebraic groups [21, I.6.5, Proposition I.6.6]:

Theorem 6.4. *Let G be an affine group scheme of finite type over a field k, and let N be a normal k-subgroup scheme of G. For every G-module (or complex of G-modules) V, there is a spectral sequence*

$$E_2^{ij} = H^i(G/N, H^j(N, V)) \Rightarrow H^{i+j}(G, V).$$

Theorems 6.3 and 6.4 give information about the weights of L on $H^*(U, k)$, that is, about the action of a maximal torus $T \subset L$ on $H^*(U, k)$. The method is to write U (canonically) as an extension of additive groups $V = (G_a)^n$ and use the Hochschild-Serre spectral sequence. We deduce that as a representation of L, all weights of $H^{>0}(U, k)$ are in the root lattice of G, with nonnegative coefficients for the simple roots not in I, and with at least one of those coefficients positive. (This is the same sign as we have for the action of L on $\mathfrak{u}^*$.)

Now apply the Hochschild-Serre spectral sequence to the normal subgroup U in P:

$$E_2^{ij}(L, H^j(U, k) \otimes S^l((\mathfrak{p}/\mathfrak{u})^*)) \Rightarrow H^{i+j}(P, S^l((\mathfrak{p}/\mathfrak{u})^*)).$$

By the analysis of $S^l(\mathfrak{p}^*)$ above, all the weights of L on the subspace $S^l((\mathfrak{p}/\mathfrak{u})^*)$ are in the root lattice of G, and the coefficients of all simple roots not in I are equal to zero. Combining this with the previous paragraph, we find: for $l \geq 0$ and $j > 0$, all weights of L on $H^j(U,k) \otimes S^l((\mathfrak{p}/\mathfrak{u})^*)$ have all coefficients of the simple roots not in I nonnegative, with at least one positive. By Proposition 6.2, it follows that

$$H^i(L, H^j(U,k) \otimes S^l((\mathfrak{p}/\mathfrak{u})^*)) = 0$$

for all i and l and all $j > 0$. So the spectral sequence above reduces to an isomorphism

$$H^i(P, S^l((\mathfrak{p}/\mathfrak{u})^*)) \cong H^i(L, S^l((\mathfrak{p}/\mathfrak{u})^*)),$$

as we wanted. Theorem 6.1 is proved. $\square$

7 Hodge cohomology of flag manifolds

We use the following result, proved by Srinivas [31, section 3]:

Proposition 7.1. *Let P be a parabolic subgroup of a split reductive group G over a field k. Then the cycle map*

$$CH^*(G/P) \otimes_{\mathbf{Z}} k \to H^*_{\mathrm{H}}((G/P)/k)$$

is an isomorphism of k-algebras. In particular, $H^i(G/P, \Omega^j) = 0$ for $i \neq j$.

There are many related results. In particular, Proposition 7.1 can also be deduced from the work of El Zein (who constructed the cycle map in Hodge cohomology over any field) and Gros (who constructed the pushforward homomorphism in Hodge cohomology over any perfect field) [15, Proposition 3.3.5], [16, sections II.2 and II.4]. That approach implies Proposition 7.1 more generally for any smooth proper variety with a cell decomposition. Also, Andersen gave the additive calculation of $H^i(G/P, \Omega^j)$ over any field [21, Proposition II.6.18].

Note that Chevalley and Demazure gave combinatorial descriptions of the Chow ring of G/P, which in particular show that this ring is independent of k, and isomorphic to the ordinary cohomology ring $H^*(G_{\mathbf{C}}/P_{\mathbf{C}}, \mathbf{Z})$ [10, Proposition 11], [12]. (That makes sense because the classification of split reductive groups and their parabolic subgroups is the same over all fields.)

8 Invariant functions on the Lie algebra

Theorem 8.1. *Let G be a simple group over a field k, T a maximal torus in G, $\mathfrak{g}$ and $\mathfrak{t}$ the Lie algebras. Assume that we are not in the case where* $\mathrm{char}(k) = 2$ *and $G_{\overline{k}}$ is a product of copies of $Sp(2n)$ for some positive integer n. Then the restriction $O(\mathfrak{g})^G \to O(\mathfrak{t})^W$ is an isomorphism.*

Theorem 8.1 was proved by Springer and Steinberg for any adjoint group G [30, II.3.17'], and generalized to any simple group by Chaput and Romagny [9, Theorem 1.1]. They assumed that G is split, but that implies Theorem 8.1 by passage to the algebraic closure $\overline{k}$.

The statement is optimal, in that the conclusion fails for the symplectic group $Sp(2n)$ in characteristic 2 for any positive integer n (for example, for $SL(2)$), as Chaput and Romagny pointed out. In their argument, the distinctive feature of the symplectic group is that it is the only simple group for which some roots are divisible by 2 in the weight lattice.

In particular, Theorem 8.1 applies to cases such as the spin group $\mathrm{Spin}(n)$ in characteristic 2 with $n \geq 6$, which we study further in Theorem 12.1.

Here is a related observation.

Lemma 8.2. *Let G be a smooth affine group over a field k whose identity component is reductive. Then there are canonical maps $H^a(BG, \Omega^a) \to H^{2a}_{\mathrm{dR}}(BG/k)$ and $H^{a+1}(BG, \Omega^a) \to H^{2a+1}_{\mathrm{dR}}(BG/k)$. These maps are compatible with products and with pullback under group homomorphisms.*

Proof. By Corollary 2.2, we have $H^a(BG, \Omega^b) = 0$ for all $a < b$. Therefore, the Hodge spectral sequence gives a canonical "edge map" $H^a(BG, \Omega^a) \to H^{2a}_{\mathrm{dR}}(BG/k)$.

In odd degrees, there is one possible differential on the group $H^{a+1}(BG, \Omega^a)$:

$$d_1 \colon H^{a+1}(BG, \Omega^a) \to H^{a+1}(BG, \Omega^{a+1}).$$

In fact, this differential is zero. To see this, let T be a maximal torus in G. Then the restriction $O(\mathfrak{g})^G \to O(\mathfrak{t})$ is injective, because the G-conjugates of elements of $\mathfrak{t}$ (over the algebraic closure of k) are the semisimple elements of $\mathfrak{g}$, which are Zariski dense in $\mathfrak{g}$. Equivalently, $H^b(BG, \Omega^b) \to H^b(BT, \Omega^b)$ is injective for all b. But $H^a(BT, \Omega^b) = 0$ for all $a \neq b$, and so $H^b(BT, \Omega^b)$ injects into $H^{2b}_{\mathrm{dR}}(BT/k)$. Therefore, $H^b(BG, \Omega^b)$ injects into $H^{2b}_{\mathrm{dR}}(BG/k)$. In particular, the d_1 differential into $H^{a+1}(BG, \Omega^{a+1})$ is zero, as we want. Therefore, we have a canonical "edge map" $H^{a+1}(BG, \Omega^a) \to H^{2a+1}_{\mathrm{dR}}(BG/k)$. ☐

9 Hodge cohomology of BG at non-torsion primes

Theorem 9.1. *Let G be a reductive group over a field k of characteristic $p \geq 0$. Then $H^{>0}(G, O(\mathfrak{g})) = 0$ if and only if p is not a torsion prime for G.*

Theorem 9.2. *Let G be a split reductive group over $\mathbf{Z}$, and let p be a non-torsion prime for G. Then $H^j(BG_{\mathbf{Z}}, \Omega^i)$ localized at p is zero for $i \neq j$. Moreover, the Hodge cohomology ring $H^*(BG_{\mathbf{Z}}, \Omega^*)$ and the de Rham cohomology $H^*_{\mathrm{dR}}(BG/\mathbf{Z})$, localized at p, are polynomial rings on generators of degrees equal to 2 times the fundamental degrees of G. These rings are isomorphic to the cohomology of the topological space $BG_{\mathbf{C}}$ with $\mathbf{Z}_{(p)}$ coefficients.*

We recall the definition of torsion primes for a reductive group G over a field k. Let B be a Borel subgroup of $G_{\overline{k}}$, and T a maximal torus in B. Then there is a natural homomorphism from the character group $X^*(T) = \mathrm{Hom}(T, G_m)$ (the weight lattice of G) to the Chow group $CH^1(G_{\overline{k}}/B)$. Therefore, for $N = \dim(G_{\overline{k}}/B)$, there is a homomorphism from the symmetric power $S^N(X^*(T))$ to $CH^N(G_{\overline{k}}/B)$; taking the degree of a zero-cycle on $G_{\overline{k}}/B$ gives a homomorphism (in fact, an isomorphism) $CH^N(G_{\overline{k}}/B) \to \mathbf{Z}$. A prime number p is said to be a torsion prime for G if the image of $S^N(X^*(T)) \to \mathbf{Z}$ is zero modulo p. Borel showed that p is a torsion prime for G if and only if the cohomology $H^*(BG_{\mathbf{C}}, \mathbf{Z})$ has p-torsion, where $G_{\mathbf{C}}$ is the corresponding complex reductive group [7, Proposition 4.2]. It is also equivalent to say that $G(\mathbf{C})$ contains an elementary abelian p-subgroup that is not contained in a torus [7, Théorème 4.5].

In most cases, Theorem 9.1 follows from Theorem 4.1. Explicitly, a prime number p is torsion for a simply connected simple group G if $p = 2$ and G has a simple factor not of type A_n or C_n, $p = 3$ and G has a simple factor of type F_4, E_6, E_7, or E_8, pr $p = 5$ and G has an E_8 factor. So the main new cases in Theorem 9.1 are the symplectic groups $Sp(2n)$ in characteristic 2 and G_2 in characteristic 3. (These are non-torsion primes, but not good primes in the sense of Theorem 4.1.) In these cases, the representation-theoretic result that $H^{>0}(G, O(\mathfrak{g})) = 0$ seems to be new. Does $O(\mathfrak{g})$ have a good filtration in these cases?

The following spectral sequence, modeled on the Leray-Serre spectral sequence in topology, will be important for the rest of the paper.

Proposition 9.3. *Let P be a parabolic subgroup of a split reductive group G over a field k. Let L be the quotient of P by its unipotent radical. Then there is a spectral sequence of algebras*

$$E_2^{ij} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k).$$

Proof. Consider $\Omega^* = \oplus \Omega^i$ as a presheaf of commutative dgas on smooth k-schemes, with zero differential.

For a smooth morphism $f\colon X \to Y$ of smooth k-schemes, consider the object $Rf_*(\Omega^*_X)$ in the derived category $D(Y)$ of étale sheaves on Y. Here the sheaf Ω^*_X on X has an increasing filtration, compatible with its ring structure, with 0th step the subsheaf $f^*(\Omega^*_Y)$ and jth graded piece $f^*(\Omega^*_Y) \otimes \Omega^j_{X/Y}$. So $Rf_*(\Omega^*_X)$ has a corresponding filtration in $D(Y)$, with jth graded piece $Rf_*(f^*(\Omega^*_Y) \otimes \Omega^j_{X/Y}) \cong \Omega^*_Y \otimes Rf_* \Omega^j_{X/Y}$. This gives a spectral sequence

$$E_2^{ij} = H^{i+j}(Y, \Omega^*_Y \otimes Rf_* \Omega^j_{X/Y}) \Rightarrow H^{i+j}(X, \Omega^*_X).$$

Now specialize to the case where $f\colon X \to Y$ is the G/P-bundle associated to a principal G-bundle over Y. The Hodge cohomology of G/P is essentially independent of the base field, by the isomorphism $H^*_{\mathrm{H}}((G/P)/k) \cong CH^*(G/P) \otimes_{\mathbf{Z}} k$ (Proposition 7.1). Here $CH^*(G/P)$ is a free abelian group with a fixed basis (independent of k), as discussed in section 7, and G acts trivially on $CH^*(G/P)$. Therefore, each object $Rf_*(\Omega^j_{X/Y})$ is a trivial vector bundle on Y, with fiber $H^j(G/P, \Omega^j)$, viewed as a complex in degree j. So we can rewrite the spectral sequence as

$$E_2^{ij} = H^i(Y, \Omega^*) \otimes H^j(G/P, \Omega^j) \Rightarrow H^{i+j}(X, \Omega^*).$$

All differentials in the spectral sequence above preserve the degree in the grading of Ω^*. Therefore, we can renumber the spectral sequence so that it is graded by total degree:

$$E_2^{ij} = H^i_{\mathrm{H}}(Y/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(X/k).$$

Finally, we consider the analogous spectral sequence for the morphism $f\colon EG/P \to B_{\mathrm{simp}}G$ of simplicial schemes:

$$E_2^{ij} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}((EG/P)/k).$$

By Lemma 1.2, the output of the spectral sequence is isomorphic to $H^*_{\mathrm{H}}(BP/k)$, or equivalently (by Theorem 6.1) to $H^*_{\mathrm{H}}(BL/k)$. This is a spectral sequence of algebras. All differentials preserve the degree in the grading of Ω^*. ☐

Proof. (Theorem 9.1) First, suppose that $H^{>0}(G, O(\mathfrak{g})) = 0$; then we want to show that $\mathrm{char}(k)$ is not a torsion prime for G. By Corollary 2.2, the assumption implies that $H^j(BG, \Omega^i) = 0$ for all $i \neq j$. Apply Proposition 9.3 when P is a Borel subgroup B in G; this gives a spectral sequence

$$E_2^{ij} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/B)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BT/k),$$

where T is a maximal torus in B. Under our assumption, this spectral sequence degenerates at E_2, because the differential d_r (for $r \geq 2$) takes $H^i(BG, \Omega^i) \otimes H^j(G/B, \Omega^j)$ into $H^{i+r}(BG, \Omega^{i+r-1}) \otimes H^{j-r+1}(G/B, \Omega^{j-r+1})$, which is zero. It follows that $H^*_{\mathrm{H}}(BT/k) \to H^*_{\mathrm{H}}((G/B)/k)$ is surjective. Here $H^*_{\mathrm{H}}(BT/k)$ is the polynomial ring $S(X^*(T) \otimes k)$ by Theorem 4.1, and $H^*_{\mathrm{H}}((G/B)/k) = CH^*(G/B) \otimes k$ by Proposition 7.1. It follows that the ring $CH^*(G/B) \otimes k$ is generated as a k-algebra by the image of $X^*(T) \to CH^1(G/B)$. Equivalently, p is not a torsion prime for G.

Conversely, suppose that p is not a torsion prime for G. That is, the homomorphism $S(X^*(T) \otimes k) \to CH^*(G/B) \otimes k$ is surjective. Equivalently, $H^*_{\mathrm{H}}(BT/k) \to H^*_{\mathrm{H}}((G/B)/k)$ is surjective. By the product structure on the spectral sequence above, it follows that the spectral sequence degenerates at E_2. Since $H^j(BT, \Omega^i) = 0$ for $i \neq j$, it follows that $H^j(BG, \Omega^i) = 0$ for $i \neq j$. Equivalently, $H^{>0}(G, O(\mathfrak{g})) = 0$. ☐

Proof. (Theorem 9.2) Let G be a split reductive group over $\mathbf{Z}$, and let p be a non-torsion prime for G. We have a short exact sequence

$$0 \to H^j(BG_{\mathbf{Z}}, \Omega^i)/p \to H^j(BG_{\mathbf{F}_p}, \Omega^i) \to H^{j+1}(BG_{\mathbf{Z}}, \Omega^i)[p] \to 0.$$

By Theorem 9.1 and Corollary 2.2, the Hodge cohomology ring $H^*(BG_{\mathbf{Z}}, \Omega^*)$ localized at p is concentrated in bidegrees $H^{i,i}$ and is torsion-free. This ring tensored with $\mathbf{Q}$ is the ring of invariants $O(\mathfrak{g}_{\mathbf{Q}})^G$, which is a polynomial ring on generators of degrees equal to the fundamental degrees of G.

To show that the Hodge cohomology ring over $\mathbf{Z}_{(p)}$ is a polynomial ring on generators in $H^{i.i}$ for i running through the fundamental degrees of G, it suffices to show that the Hodge cohomology ring $H^*_{\mathrm{H}}(BG/\mathbf{F}_p)$ is a polynomial ring in the same degrees. Given that, the other statements of the theorem will follow. Indeed, the statement on Hodge cohomology implies that the de Rham cohomology ring $H^*_{\mathrm{dR}}(BG/\mathbf{Z})$ localized at p is also a polynomial ring, on generators in 2 times the fundamental degrees of G. The cohomology of the topological space $BG_{\mathbf{C}}$ localized at p is known to be a polynomial ring on generators in the same degrees, by Borel [7, Proposition 4.2, Théorème 4.5], [25, Theorem VII.2.12].

From here on, let $k = \mathbf{F}_p$, and write G for G_k. By definition of the Weyl group W as $W = N_G(T)/T$, the image of $H^*_{\mathrm{H}}(BG/k)$ in $H^*_{\mathrm{H}}(BT/k) = S(X(T) \otimes k)$ is contained in the subring of W-invariants. We now use that p is not a torsion prime for G. By Demazure, except in the case where $p = 2$ and G has an $Sp(2n)$ factor, the ring of W-invariants in $S(X(T) \otimes k)$ is a polynomial algebra over k, with the degrees of generators equal to the fundamental degrees of G [12, Théorème].

By Theorem 9.1, $H^*_{\mathrm{H}}(BG/k)$ is equal to the ring of invariants $O(\mathfrak{g})^G$. By Theorem 8.1 (due to Chaput and Romagny), for any simple group G over a field k of characteristic p with p not a torsion prime, except for $G = Sp(2n)$ with $p = 2$, the restriction $O(\mathfrak{g})^G \to O(\mathfrak{t})^W$ is an isomorphism, and hence $O(\mathfrak{g})^G$ is the polynomial ring we want.

The case of $Sp(2n)$ in characteristic 2 (including $SL(2) = Sp(2)$) is a genuine exception: here $O(\mathfrak{g})^G$ is a subring of $O(\mathfrak{t})^W$, not equal to it. However, it is still true in this case that $O(\mathfrak{g})^G$ is a polynomial ring with generators in the fundamental degrees of G, that is, $2, 4, \ldots, 2n$, by Chaput and Romagny [9, Theorem 6.6]. □

10 μ_p

Proposition 10.1. *Let k be a field of characteristic $p > 0$. Let G be the group scheme μ_p of pth roots of unity over k. Then*

$$H^*_{\mathrm{H}}(B\mu_p/k) \cong k[c_1]\langle v_1\rangle,$$

where c_1 is in $H^1(B\mu_p, \Omega^1)$ and v_1 is in $H^0(B\mu_p, \Omega^1)$. Likewise, $H^*_{\mathrm{dR}}(B\mu_p/k) \cong k[c_1]\langle v_1\rangle$ with $|v_1| = 1$ and $|c_1| = 2$.

Here $R\langle v\rangle$ denotes the exterior algebra over a graded-commutative ring R with generator v; that is, $R\langle v\rangle = R \oplus R \cdot v$, with product $v^2 = 0$. See section 1 for the definitions of Hodge and de Rham cohomology we are using for a non-smooth group scheme such as μ_p. Proposition 10.1 can help to compute Hodge cohomology of BG for smooth group schemes G, as we will see in the proof of Theorem 11.1 for $G = SO(n)$.

Proposition 10.1 is roughly what the topological analogy would suggest. Indeed, the group scheme μ_p of pth roots of unity is defined over $\mathbf{Z}$, with $(\mu_p)_{\mathbf{C}}$ isomorphic to the group $\mathbf{Z}/p$. For k of characteristic p, the ring $H^*((B\mu_p)_{\mathbf{C}}, k)$ is a polynomial ring $k[x]$ with $|x| = 1$ if $p = 2$, or a free graded-commutative algebra $k\langle x, y\rangle$ with $|x| = 1$ and $|y| = 2$ if p is odd. So $H^*_{\mathrm{dR}}(B\mu_p/k)$ is isomorphic to $H^*((B\mu_p)_{\mathbf{C}}, k)$ additively for any prime p, and as a graded ring if $p > 2$.

Proof. Let $G = \mu_p$ over k. The co-Lie complex l_G in the derived category of G-modules, discussed in section 3, has $H^0(l_G) \cong \mathfrak{g}^* \cong k$ and also $H^{-1}(l_G) \cong k$, with other cohomology groups being zero. (In short, this is because G is a complete intersection in the affine line, defined by the one equation $x^p = 1$.)

Since representations of G are completely reducible, we have $\mathrm{Ext}^{>0}_G(M, N) = 0$ for all G-modules M and N [21, Lemma I.4.3]. The isomorphism class of l_G is described by an element of $\mathrm{Ext}^2_G(k, k)$, which is zero. So $l_G \cong k \oplus k[1]$ in the derived category of G-modules.

By Theorem 3.1, we have

$$H^i(BG, \Omega^j) \cong H^{i-j}(G, S^j(l_G)).$$

Here

$$\begin{aligned} S^j(l_G) &\cong \oplus_{m=0}^{j} S^m(k) \otimes S^{j-m}(k[1]) \\ &\cong \oplus_{m=0}^{j} S^m(k) \otimes \Lambda^{j-m}(k)[j-m], \end{aligned}$$

which is isomorphic to $k \oplus k[1]$ if $j \geq 1$ and to k if $j = 0$. Therefore, $H^i(BG, \Omega^j)$ is isomorphic to k if $0 \leq i = j$ or if $0 \leq i = j - 1$, and is otherwise zero.

Write c_1 for the generator of $H^1(BG, \Omega^1)$, which is pulled back from the Chern class c_1 in $H^1(BG_m, \Omega^1)$ via the inclusion $G \hookrightarrow G_m$. Write v_1 for the generator of $H^0(BG, \Omega^1)$. We have $v_1^2 = 0$ because $H^0(BG, \Omega^2) = 0$. Theorem 3.1 also describes the ring structure on the Hodge cohomology of BG. In particular, $\oplus_i H^i(BG, \Omega^i)$ is the ring of invariants of G acting on $O(\mathfrak{g})$, which is the polynomial ring $k[c_1]$. Finally, the description of $S^j(l_G)$ also shows that $\oplus_i H^i(BG, \Omega^{i+1})$ is the free module over $k[c_1]$ on the generator v_1. This completes the proof that

$$H^*_{\mathrm{H}}(BG/k) \cong k[c_1]\langle v_1 \rangle.$$

Finally, consider the Hodge spectral sequence for BG from section 1. The element v_1 is a permanent cycle because $H^0(BG, \Omega^2) = 0$, and c_1 is a permanent cycle because it is pulled back from a permanent cycle on BG_m. Therefore, the Hodge spectral sequence degenerates at E_1. We have $v_1^2 = 0$ in de Rham cohomology as in Hodge cohomology, because $\oplus_i H^0(BG, \Omega^i)$ is a subring of de Rham cohomology, using degeneration of the Hodge spectral sequence. Therefore, the de Rham cohomology of BG is isomorphic to $k[c_1]\langle v_1 \rangle$ as a graded ring. □

Lemma 10.2. *Let G be a discrete group, considered as a group scheme over a field k. Then the Hodge cohomology of the algebraic stack BG is the group cohomology of G:*

$$H^i(BG, \Omega^j) \cong \begin{cases} H^i(G, k) & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $H^*_{\mathrm{dR}}(BG/k) \cong H^*(G, k)$.

Proof. Since G is smooth over k, we can compute the Hodge cohomology of the stack BG as the étale cohomology of the simplicial scheme $B_{\mathrm{simp}}G$ with coefficients in Ω^j. Since G is discrete, the sheaf Ω^j is zero for $j > 0$. For $j = 0$, the spectral sequence

$$E_1^{ab} = H^b(G^a, O) \Rightarrow H^{a+b}(BG, O)$$

reduces to a single row, since $H^b(G^a, O) = 0$ for $b > 0$. That is, $H^*(BG, O)$ is the cohomology of the standard complex that computes the cohomology of the group G with coefficients in k. □

Although Lemma 10.2 applies to any discrete group G, it is probably most meaningful in the case where G is finite (so that the associated group scheme over a field k is affine). A generalization of that case is the following "Hochschild-Serre" spectral sequence for the Hodge cohomology of a non-connected group scheme:

Lemma 10.3. *Let G be an affine group scheme of finite type over a field k. Let G^0 be the identity component of G, and suppose that the finite group scheme G/G^0 is the k-group scheme associated to a finite group Q. Then there is a spectral sequence*

$$E_2^{ij} = H^i(Q, H^j(BG^0, \Omega^a)) \Rightarrow H^{i+j}(BG, \Omega^a).$$

for any $a \geq 0$.

Proof. By Theorem 3.1, $H^r(BG, \Omega^a)$ is isomorphic to $H^{r-a}(G, S^a(l_G))$. The lemma then becomes a special case of the Hochschild-Serre spectral sequence for the cohomology of G as an algebraic group, Theorem 6.4:

$$E_2^{ij} = H^i(Q, H^j(G^0, S^a(l_G))) \Rightarrow H^{i+j}(G, S^a(l_G)).$$

(Strictly speaking, move the Hochschild-Serre spectral sequence up by a rows to obtain the spectral sequence of the lemma.) □

11 The orthogonal groups

Theorem 11.1. *Let G be the split group $SO(n)$ (also called $O^+(n)$) over a field k of characteristic 2. Then the Hodge cohomology ring of BG is a polynomial ring $k[u_2, u_3, \ldots, u_n]$, where u_{2a} is in $H^a(BG, \Omega^a)$ and u_{2a+1} is in $H^{a+1}(BG, \Omega^a)$. Also, the Hodge spectral sequence degenerates at E_1, and so $H^*_{\mathrm{dR}}(BG/k)$ is also isomorphic to $k[u_2, u_3, \ldots, u_n]$.*

Likewise, the Hodge and de Rham cohomology rings of $BO(2r)$ are isomorphic to the polynomial ring $k[u_1, u_2, \ldots, u_{2r}]$. Finally, the Hodge and de Rham cohomology rings of $BO(2r+1)$ are isomorphic to $k[v_1, c_1, u_2, \ldots, u_{2r+1}]/(v_1^2)$, where v_1 is in $H^0(BO(2r+1), \Omega^1)$ and c_1 is in $H^1(BO(2r+1), \Omega^1)$.

Thus the de Rham cohomology ring of $BSO(n)_{\mathbf{F}_2}$ is isomorphic to the mod 2 cohomology ring of the topological space $BSO(n)_{\mathbf{C}}$ as a graded ring:

$$H^*(BSO(n)_{\mathbf{C}}, \mathbf{F}_2) \cong \mathbf{F}_2[w_2, w_3, \ldots, w_n],$$

where the classes w_i are the Stiefel-Whitney classes. Theorem 11.1 gives a new analog of the Stiefel-Whitney classes for quadratic bundles in characteristic 2. (Note that the k-group scheme $O(2r+1)$ is not smooth in characteristic 2. Indeed, it is isomorphic to $SO(2r+1) \times \mu_2$. By contrast, $O(2r)$ is smooth but not connected, and we write $SO(2r)$ for the kernel of the *Dickson determinant* $O(2r) \to \mathbf{Z}/2$, which describes the action of $O(2r)$ on the center $k \times k$ of the even Clifford algebra.)

The proof is inspired by topology. In particular, it involves some hard work with spectral sequences, related to Borel's transgression theorem and Zeeman's comparison theorem. The method should be useful for other reductive groups.

The formula for the classes u_i of a direct sum of two quadratic bundles is not the same as for the Stiefel-Whitney classes in topology. To state this, define a quadratic form (q, V) over a field k to be *nondegenerate* if the radical $V^\perp$ of the associated bilinear form is zero, and *nonsingular* if $V^\perp$ has dimension at most 1 and q is nonzero on any nonzero element of $V^\perp$. (In characteristic not 2, nonsingular and nondegenerate are the same.) The orthogonal group is defined as the automorphism group scheme of a nonsingular quadratic form; the precise group over k depends on the choice of form [22, section VI.23]. For example, over a field k of characteristic 2, the quadratic form

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r}$$

is nonsingular of even dimension $2r$, while the form

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r} + x_{2r+1}^2$$

is nonsingular of odd dimension $2r+1$, with $V^\perp$ of dimension 1. (These particular forms define the *split* orthogonal groups over k.) Let $u_0 = 1$.

Proposition 11.2. *Let X be a scheme of finite type over a field k of characteristic 2. Let E and F be vector bundles with nondegenerate quadratic forms over X (hence of even rank). Write u_l for the characteristic classes from Theorem 11.1. Then, for any $a \geq 0$, in either Hodge cohomology or de Rham cohomology,*

$$u_{2a}(E \oplus F) = \sum_{j=0}^{a} u_{2j}(E)u_{2a-2j}(F)$$

and

$$u_{2a+1}(E \oplus F) = \sum_{l=0}^{2a+1} u_l(E)u_{2a+1-l}(F).$$

Thus the *even* u-classes of $E \oplus F$ depend only on the even u-classes of E and F. By contrast, Stiefel-Whitney classes in topology satisfy

$$w_m(E \oplus F) = \sum_{l=0}^{m} w_l(E)w_{m-l}(F)$$

for all m [25, Theorem III.5.11].

Theorem 12.1 gives an example of a reductive group G for which the de Rham cohomology of $BG_{\mathbf{F}_p}$ and the mod p cohomology of $BG_{\mathbf{C}}$ are not isomorphic. It is a challenge to find out how close these rings are, in other examples.

Via Corollary 2.2, Theorem 11.1 can be viewed as a calculation in the representation theory of the algebraic group $G = SO(n)$ for any n, over a field k of characteristic 2. For example, when $G = SO(3) = PGL(2)$ over k of characteristic 2, we find (what seems to be new):

$$H^i(G, S^j(\mathfrak{g}^*)) \cong \begin{cases} k & \text{if } 0 \leq i \leq j \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (Theorem 11.1) We will assume that $k = \mathbf{F}_2$. This implies the theorem for any field of characteristic 2.

We begin by computing the ring $\oplus_i H^i(BG, \Omega^i)$ for $G = SO(n)$. By Corollary 2.2, this is equal to the ring of G-invariant polynomial functions on the Lie algebra $\mathfrak{g}$ over k. By Theorem 8.1, since G is not a symplectic group, the restriction $O(\mathfrak{g})^G \to O(\mathfrak{t})^W$ is an isomorphism.

Let $r = \lfloor n/2 \rfloor$. For $n = 2r+1$, the Weyl group W is the semidirect product $S_r \ltimes (\mathbf{Z}/2)^r$. There is a basis $e_1, \ldots, e_r$ for $\mathfrak{t}$ on which $(\mathbf{Z}/2)^r$ acts by changing the signs, and so that action is trivial since k has characteristic 2. The group S_r has its standard permutation action on $e_1, \ldots, e_r$. Therefore, the ring of invariants $O(\mathfrak{t})^W$ is the ring of symmetric functions in r variables. Let $u_2, u_4, \ldots, u_{2r}$ denote the elementary symmetric functions. By the isomorphisms mentioned, we can view u_{2a} as an element of $H^a(BSO(2r+1), \Omega^a)$ for $1 \leq a \leq r$, and $\oplus_i H^i(BSO(2r+1), \Omega^i)$ is the polynomial ring $k[u_2, u_4, \ldots, u_{2r}]$.

For $n = 2r$, the Weyl group W of $SO(2r)$ is the semidirect product $S_r \ltimes (\mathbf{Z}/2)^{r-1}$. Again, the subgroup $(\mathbf{Z}/2)^{r-1}$ acts trivially on $\mathfrak{t}$, and S_r acts by permutations as usual. So $\oplus_i H^i(BSO(2r), \Omega^i)$ is also the polynomial ring $k[u_2, u_4, \ldots, u_{2r}]$, with u_{2a} in $H^a(BSO(2r), \Omega^a)$ for $1 \leq a \leq r$.

For the smooth k-group $G = O(2r)$, we can also compute the ring $\oplus_i H^i(BG, \Omega^i)$. By Corollary 2.2, this is the ring of G-invariant polynomial functions on the Lie algebra $\mathfrak{g} = \mathfrak{so}(2r)$. This is contained in the ring of $SO(2r)$-invariant functions on $\mathfrak{g}$, and I claim that the two rings are equal. It suffices to show that an $SO(2r)$-invariant function on $\mathfrak{g}$ is also invariant under the normalizer N in $O(2r)$ of a maximal torus T in $SO(2r)$, since that normalizer meets both connected components of $O(2r)$. Here $N = S_r \ltimes (\mathbf{Z}/2)^r$, which acts on $\mathfrak{t}$ in the obvious way; in particular, $(\mathbf{Z}/2)^r$ acts trivially on $\mathfrak{t}$. Therefore, an $SO(2r)$-invariant function on $\mathfrak{g}$ (corresponding to an S_r-invariant function on $\mathfrak{t}$) is also $O(2r)$-invariant. Thus we have $\oplus_i H^i(BO(2r), \Omega^i) = k[u_2, u_4, \ldots, u_{2r}]$.

For a smooth group scheme G over $R = \mathbf{Z}/4$, define the *Bockstein*

$$\beta \colon H^i(BG_k, \Omega^j) \to H^{i+1}(BG_k, \Omega^j)$$

on the Hodge cohomology of BG_k (where $k = \mathbf{Z}/2$) to be the boundary homomorphism associated to the short exact sequence of sheaves

$$0 \to \Omega^j_k \to \Omega^j_R \to \Omega^j_k \to 0$$

on BG_R. (The Bockstein on Hodge cohomology is also defined for group schemes G such as μ_2 which are flat but not smooth over $R = \mathbf{Z}/4$, because the Hodge cohomology of BG can be described using smooth schemes (Lemma 1.2).)

Next, define elements $u_1, u_3, \ldots, u_{2r-1}$ of $H^*_{\mathrm{H}}(BO(2r)/k)$ as follows. First, let $u_1 \in H^1(BO(2r), \Omega^0)$ be the pullback of the generator of $H^1(\mathbf{Z}/2, k) = k$ via the surjection $O(2r) \to \mathbf{Z}/2$ (Lemma 10.2). Next, use that the split group $O(2r)$ over $k = \mathbf{F}_2$ lifts to a smooth group $O(2r)$ over $\mathbf{Z}$. As a result, we have a Bockstein homomorphism on the Hodge cohomology of $BO(2r)$. For $0 \leq a \leq r-1$, let $u_{2a+1} = \beta u_{2a} + u_1 u_{2a} \in H^{a+1}(BO(2r), \Omega^a)$. This agrees with the previous formula for u_1, if we make the convention that $u_0 = 1$. (The definition of u_{2a+1} is suggested by the formula for odd Stiefel-Whitney classes in topology: $w_{2a+1} = \beta w_{2a} + w_1 w_{2a}$ [25, Theorem III.5.12].)

I claim that the homomorphism

$$k[u_1, u_2] \to H^*_{\mathrm{H}}(BO(2)/k)$$

is an isomorphism. To see this, consider the Hochschild-Serre spectral sequence of Lemma 10.3,

$$E_2^{ij} = H^i(\mathbf{Z}/2, H^j(BSO(2)_k, \Omega^*)) \Rightarrow H^{i+j}(BO(2)_k, \Omega^*).$$

Here $SO(2)$ is isomorphic to G_m, and so we know the Hodge cohomology of $BSO(2)$ by Theorem 4.1: $H^*_{\mathrm{H}}(BSO(2)/k) \cong k[c_1]$ with c_1 in $H^1(BSO(2), \Omega^1)$. We read off that the E_2 page of the spectral sequence is the polynomial ring $k[u_1, u_2]$, with u_1 in $H^1(\mathbf{Z}/2, H^0(BSO(2), \Omega^0))$ and u_2 in $H^0(\mathbf{Z}/2, H^1(BSO(2), \Omega^1))$. Here u_1 is a permanent cycle, because all differentials send u_1 to zero groups. Also, because the surjection $O(2) \to \mathbf{Z}/2$ of k-groups is split, there are no differentials into the bottom row of the spectral sequence; so u_2 is also a permanent cycle. It follows that the spectral sequence degenerates at E_2, and hence that $H^*_{\mathrm{H}}(BO(2)/k) \cong k[u_1, u_2]$.

We also need to compute the Bockstein on the Hodge cohomology of $BO(2)$, which is defined because $O(2)$ lifts to a smooth group scheme over $R := \mathbf{Z}/4$. The Bockstein is related to the Hodge cohomology of $BO(2)_R$ by the exact sequence

$$H^i(BO(2)_R, \Omega^j) \to H^i(BO(2)_k, \Omega^j) \underset{\beta}{\to} H^{i+1}(BO(2)_k, \Omega^j).$$

Consider the Hochschild-Serre spectral sequence of Lemma 10.3 for $BO(2)_R$:

$$E_2^{ij} = H^i(\mathbf{Z}/2, H^j(BSO(2)_R, \Omega^*)) \Rightarrow H^{i+j}(BO(2)_R, \Omega^*).$$

Here $H^1(BO(2)_R, \Omega^1)$ is isomorphic to $H^0(\mathbf{Z}/2, H^1(BSO(2)_R, \Omega^1))$, where $\mathbf{Z}/2$ acts by -1 on $H^1(BSO(2)_R, \Omega^1) \cong \mathbf{Z}/4$. So the generator of $H^1(BO(2)_R, \Omega^1) \cong \mathbf{Z}/2$ maps to zero in $H^1(BO(2)_k, \Omega^1) = k \cdot u_2$. Therefore, $\beta(u_2) \neq 0$. Since $k = \mathbf{F}_2$, the element $\beta(u_2)$ in $H^2(BO(2)_k, \Omega^1) = k \cdot u_1 u_2$ must be equal to $u_1 u_2$. A similar analysis shows that $\beta(u_1) = u_1^2$.

Finally, think of $O(2)$ as the isometry group of the quadratic form $q(x, y) = xy$ on $V = A_k^2$. There is an inclusion $H = \mathbf{Z}/2 \times \mu_2 \subset O(2)$, where $\mathbf{Z}/2$ switches x and y and μ_2 acts by scalars on V. For later use, it is convenient to say something about the restriction from $BO(2)$ to BH on Hodge cohomology. By Lemma 10.2, the Hodge cohomology of $B(\mathbf{Z}/2)$ over k is the cohomology of $\mathbf{Z}/2$ as a group, namely the polynomial ring $k[s]$ with $s \in H^1(B(\mathbf{Z}/2), O)$. Also, by Proposition 10.1, the Hodge cohomology of $B\mu_2$ is $k[t, v]/(v^2)$ with $t \in H^1(B\mu_2, \Omega^1)$ and $v \in H^0(B\mu_2, \Omega^1)$. Thus we have a homomorphism from $H^*_{\mathrm{H}}(BO(2)/k) = k[u_1, u_2]$ to $H^*_{\mathrm{H}}(BH/k) \cong k[s, t, v]/(v^2)$ (by the Künneth theorem, Proposition 5.1). Here u_1 restricts to s, since both elements are pulled back from the generator of $H^1(B\mathbf{Z}/2, O)$. Also, u_2 restricts to either t or $t+sv$, because u_2 restricts to the generator c_1 of $H^1(BG_m, \Omega^1)$ and hence to t in $H^1(B\mu_2, \Omega^1)$. Thus the homomorphism from $H^*_{\mathrm{H}}(BO(2)/k)$ to $H^*_{\mathrm{H}}(BH/k)/\mathrm{rad} = k[s, t]$ is an isomorphism. (Here the *radical* of a commutative ring means the ideal of nilpotent elements.) A direct cocycle computation shows that u_2 restricts to $t+sv$ in $H^1(BH, \Omega^1)$, but we do not need that fact in this paper.

We now return to the group $O(2r)$ over $k = \mathbf{F}_2$ for any r. To formulate the following lemma, let $s_1, \ldots, s_r \in H^1(BO(2)^r, \Omega^0)$ be the pullbacks of u_1 from the r $BO(2)$ factors, and let $t_1, \ldots, t_r$ be the pullbacks of u_2 from those r factors. By the Künneth theorem (Proposition 5.1), the Hodge cohomology of $BO(2)^r$ is the polynomial ring $k[s_1, \ldots, s_r, t_1, \ldots, t_r]$.

Lemma 11.3. *The homomorphism*

$$\psi\colon k[u_1, u_2, \ldots, u_{2r}] \to H^*_{\mathrm{H}}(BO(2r)/k)$$

is injective. Also, the composition of ψ with the restriction ρ to the Hodge cohomology of $BO(2)^r$ is given by

$$u_{2a} \mapsto e_a(t_1, \ldots, t_r) = \sum_{1 \leq i_1 < \cdots < i_a \leq r} t_{i_1} \cdots t_{i_a}$$

and

$$u_{2a+1} \mapsto \sum_{m=1}^{r} s_m \sum_{\substack{1 \leq i_1 < \cdots < i_a \leq r \\ \text{none equal to } m}} t_{i_1} \cdots t_{i_a}.$$

Finally, the same formulas hold in de Rham cohomology as well as Hodge cohomology.

Proof. The formula for the restriction $\rho\psi(u_{2a})$ on Hodge cohomology follows from the definition of u_{2a}. Likewise, it is immediate that

$$u_1 \mapsto s_1 + \cdots + s_r.$$

The inclusion $O(2)^2 \subset O(2r)$ lifts to an inclusion of smooth groups over $\mathbf{Z}$, and so the restriction homomorphism commutes with the Bockstein. Therefore, for $0 \leq a \leq r-1$,

$$\begin{aligned}
u_{2a+1} &= \beta u_{2a} + u_1 u_{2a} \\
&\mapsto \beta\Bigg(\sum_{1\leq i_1<\cdots<i_a\leq r} t_{i_1}\cdots t_{i_a}\Bigg) + (s_1+\cdots+s_r)\Bigg(\sum_{1\leq i_1<\cdots<i_a\leq r} t_{i_1}\cdots t_{i_a}\Bigg) \\
&= \sum_{1\leq i_1<\cdots<i_a\leq r}\Bigg(\sum_{j=1}^{a} s_{i_j} + \sum_{m=1}^{r} s_m\Bigg) t_{i_1}\cdots t_{i_a} \\
&= \sum_{m=1}^{r} s_m \sum_{\substack{1\leq i_1<\cdots<i_a\leq r\\ \text{none equal to } m}} t_{i_1}\cdots t_{i_a},
\end{aligned}$$

as we want.

These formulas remain true in de Rham cohomology as well as in Hodge cohomology, using Lemma 8.2: for a smooth affine k-group G whose identity component is reductive, there are *canonical* maps $H^a(BG,\Omega^a) \to H^{2a}_{\mathrm{dR}}(BG/k)$ and $H^{a+1}(BG,\Omega^a) \to H^{2a+1}_{\mathrm{dR}}(BG/k)$. These maps are compatible with products and with pullback under a homomorphism of smooth affine k-groups.

To show that the homomorphism $\psi\colon k[u_1,\ldots,u_{2r}] \to H^*_{\mathrm{H}}(BO(2r)/k)$ is injective, it suffices to show that the composition $\rho\psi\colon k[u_1,\ldots,u_{2r}] \to k[s_1,\ldots,s_r,t_1,\ldots,t_r]$ is injective. We can factor this homomorphism through $k[u_1,u_3,\ldots,u_{2r-1},t_1,\ldots,t_r]$, by the homomorphism μ sending $u_2,u_4,\ldots,u_{2r}$ to the elementary symmetric polynomials in $t_1,\ldots,t_r$. Since μ is injective, it remains to show that

$$\sigma\colon k[u_1,u_3,\ldots,u_{2r-1},t_1,\ldots,t_r] \to k[s_1,\ldots,s_r,t_1,\ldots,t_r]$$

is injective.

More strongly, we will show that σ is generically étale; that is, its Jacobian determinant is not identically zero. Because σ is the identity on the t_i coordinates, it suffices to show that the determinant of the matrix of derivatives of $u_1,u_3,\ldots,u_{2r-1}$ with respect to $s_1,\ldots,s_r$ is nonzero for $s_1,\ldots,s_r,t_1,\ldots,t_r$ generic. This matrix of derivatives in fact only involves $t_1,\ldots,t_r$, because $u_1,u_3,\ldots,u_{2r-1}$ have degree 1 in $s_1,\ldots,s_r$. For example, for $r=3$, this matrix of derivatives is

$$\begin{pmatrix} 1 & t_2+t_3 & t_2t_3 \\ 1 & t_1+t_3 & t_1t_3 \\ 1 & t_1+t_2 & t_1t_2 \end{pmatrix},$$

where the ath column gives the derivatives of u_{2a-1} with respect to $s_1,\ldots,s_r$. For any r, column 1 consists of 1s, while entry (j,a) for $a \geq 2$ is

$$\sum_{\substack{1\leq i_1<\cdots<i_{a-1}\leq r\\ \text{none equal to } j}} t_{i_1}\cdots t_{i_{a-1}}.$$

This determinant is equal to the Vandermonde determinant $\delta := \prod_{i<j}(t_i - t_j)$, and in particular it is not identically zero [14, Theorem 1]. (The reference works over $\mathbf{C}$, but it amounts to an identity of polynomials over $\mathbf{Z}$, which therefore holds over any field.)

Thus we have shown that the composition $\psi \colon k[u_1, \ldots, u_{2r}] \to H^*_{\mathrm{H}}(BO(2r)/k)$ is injective, because the composition $\rho\psi$ to $H^*_{\mathrm{H}}(BO(2)^r/k)$ is injective. $\square$

To avoid an excess of notation, let us also write ψ for the homomorphism $k[u_2, u_3, \ldots, u_n] \to H^*_{\mathrm{H}}(BSO(n)/k)$.

Lemma 11.4. *The homomorphism*

$$\psi \colon k[u_2, u_3, \ldots, u_n] \to H^*_{\mathrm{H}}(BSO(n)/k)$$

is injective. Also, in the case $n = 2r + 1$, the composition of ψ with the restriction ρ to the Hodge cohomology of $BO(2)^r$ is given by

$$u_{2a} \mapsto e_a(t_1, \ldots, t_r) = \sum_{1 \leq i_1 < \cdots < i_a \leq r} t_{i_1} \cdots t_{i_a},$$

$$u_{2a+1} \mapsto \sum_{m=1}^{r} s_m \sum_{\substack{1 \leq i_1 < \cdots < i_a \leq r \\ \text{one equal to } m}} t_{i_1} \cdots t_{i_a}$$

for $1 \leq a \leq r$. Finally, the same formulas hold in de Rham cohomology as well as Hodge cohomology.

Proof. For $n = 2r + 1$, this is an easy consequence of Lemma 11.3, using the inclusions $O(2)^r \subset O(2r) \subset SO(2r + 1)$. Write $u_2, u_3, \ldots, u_{2r+1}$ for the elements of the Hodge cohomology of $BSO(2r + 1)$ defined by the same formulas as used above for $BO(2r)$ (which simplify to $u_{2a+1} = \beta u_{2a}$, since there is no element u_1 for $BSO(2r + 1)$). Also, let $v_1, \ldots, v_{2r}$ be the elements of the Hodge cohomology of $BO(2r)$ that were called $u_1, \ldots, u_{2r}$ above. Then restricting from $BSO(2r + 1)$ to $BO(2r)$ sends $u_{2a} \mapsto v_{2a}$ and $u_{2a+1} = \beta u_{2a} \mapsto \beta v_{2a} = v_{2a+1} + v_1 v_{2a}$ for $1 \leq a \leq r-1$. It is not immediate how to compute the restriction of the remaining element u_{2r+1} to $BO(2r)$, but we can compute its restriction to $BO(2)^r$:

$$\begin{aligned} u_{2r+1} &= \beta u_{2r} \\ &\mapsto \beta v_{2r} \\ &= \beta(t_1 \cdots t_r) \\ &= (s_1 + \cdots + s_r)(t_1 \cdots t_r). \end{aligned}$$

Thus we have proved the desired formulas for the restriction on Hodge cohomology from $BSO(2r + 1)$ to $BO(2)^r$. Since the generators are in $H^i(BSO(2r + 1), \Omega^i)$ or $H^{i+1}(BSO(2r + 1), \Omega^i)$, the same formulas hold in de Rham cohomology.

Thus, the restriction from $BSO(2r + 1)$ to $BO(2)^r$ sends $k[u_2, \ldots, u_{2r+1}]$ into the subring

$$k[v_1, \ldots, v_{2r}] \subset k[s_1, \ldots, s_r, t_1, \ldots, t_r],$$

by $u_{2a} \mapsto v_{2a}$ for $1 \leq a \leq r$, $u_{2a+1} \mapsto v_{2a+1} + v_1 v_{2a}$ for $1 \leq a \leq r - 1$, and $u_{2r+1} \mapsto v_1 v_{2r}$. This homomorphism is injective, because the corresponding morphism $A^{2r} \to A^{2r}$ is birational (for $u_{2r} \neq 0$, one can solve for $v_1, \ldots, v_{2r}$ in terms of

$u_2, \ldots, u_{2r+1})$. So the homomorphism $\psi\colon k[u_2, \ldots, u_{2r+1}] \to H^*_{\mathrm{H}}(BSO(2r+1)/k)$ is injective (because its composition to $H^*_{\mathrm{H}}(BO(2)^r/k)$ is injective).

For $SO(2r)$, we argue a bit differently. As discussed above, there is a subgroup $\mathbf{Z}/2 \times \mu_2 \subset O(2)$. Therefore, we have a k-subgroup scheme $(\mathbf{Z}/2 \times \mu_2)^r \subset O(2)^r \subset O(2r)$. Since $SO(2r)$ is the kernel of a homomorphism from $O(2r)$ onto $\mathbf{Z}/2$, $SO(2r)$ contains a k-subgroup scheme $H \cong (\mathbf{Z}/2)^{r-1} \times (\mu_2)^r$. By Lemma 10.2, the Hodge cohomology of $B(\mathbf{Z}/2)$ over k is the cohomology of $\mathbf{Z}/2$ as a group, namely the polynomial ring $k[x]$ with $x \in H^1(B(\mathbf{Z}/2), \Omega^0)$. Also, by Proposition 10.1, the Hodge cohomology of $B\mu_2$ is $k[t,v]/(v^2)$ with $t \in H^1(B\mu_2, \Omega^1)$ and $v \in H^0(B\mu_2, \Omega^1)$. Thus we have a homomorphism ψ from $k[u_2, u_3, \ldots, u_{2r}]$ to $H^*_{\mathrm{H}}(BSO(2r)/k)$, and a homomorphism from there to $H^*_{\mathrm{H}}(BH/k) \cong k[x_1, \ldots, x_{r-1}, t_1, \ldots, t_r, v_1, \ldots, v_r]/(v_i^2)$ (by the Künneth theorem, Proposition 5.1). We want to show that this composition is injective. For convenience, we will prove the stronger statement that the composition $\rho\psi$ from $k[u_2, u_3, \ldots, u_{2r}]$ to

$$H^*_{\mathrm{H}}(BH/k)/\mathrm{rad} = k[x_1, \ldots, x_{r-1}, t_1, \ldots, t_r]$$

is injective.

We compare the restriction from $O(2r)$ to $(\mathbf{Z}/2)^r \times (\mu_2)^r$ with that from $SO(2r)$ to H:

$$\begin{tikzcd}
k[u_1, \ldots, u_{2r}] \arrow[r] \arrow[d] & k[u_2, u_3, \ldots, u_{2r}] \arrow[d] \\
H^*_{\mathrm{H}}(BO(2r)/k) \arrow[r] \arrow[d] & H^*_{\mathrm{H}}(BSO(2r)/k) \arrow[d] \\
k[s_1, \ldots, s_r, t_1, \ldots, t_r] \arrow[r] & k[x_1, \ldots, x_{r-1}, t_1, \ldots, t_r]
\end{tikzcd}$$

The bottom homomorphism is given (for a suitable choice of generators $x_1, \ldots, x_{r-1}$) by $s_i \mapsto x_i$ for $1 \leq i \leq r-1$ and $s_r \mapsto x_1 + \cdots + x_{r-1}$ (agreeing with the fact that $u_1 \mapsto s_1 + \cdots + s_r \mapsto 0$ in the Hodge cohomology of BH). By the formulas for $O(2r)$, we know how the elements $u_2, \ldots, u_{2r}$ restrict to $k[s_1, \ldots, s_r, t_1, \ldots, t_r]$, and hence to $k[x_1, \ldots, x_{r-1}, t_1, \ldots, t_r]$. Namely,

$$u_{2a} \mapsto e_a(t_1, \ldots, t_r) = \sum_{1 \leq i_1 < \cdots < i_a \leq r} t_{i_1} \cdots t_{i_a},$$

and, for $1 \leq a \leq r-1$,

$$\begin{aligned}
u_{2a+1} &\mapsto \sum_{1 \leq i_1 < \cdots < i_a \leq r} \left(\sum_{j=1}^{a} s_{i_j} + \sum_{m=1}^{r} s_m \right) t_{i_1} \cdots t_{i_a} \\
&\mapsto \sum_{1 \leq i_1 < \cdots < i_a \leq r-1} \left(\sum_{j=1}^{a} x_{i_j} \right) t_{i_1} \cdots t_{i_a} \\
&\qquad + \sum_{1 \leq i_1 < \cdots < i_{a-1} \leq r-1} \left(x_1 + \cdots + x_{r-1} + \sum_{j=1}^{a-1} x_{i_j} \right) t_{i_1} \cdots t_{i_{a-1}} t_r \\
&= \sum_{j=1}^{r-1} x_j (t_j + t_r) \sum_{\substack{1 \leq i_1 < \cdots < i_{a-1} \leq r-1 \\ \text{none equal to } j}} t_{i_1} \cdots t_{i_{a-1}}.
\end{aligned}$$

We want to show that this homomorphism $\rho\psi\colon k[u_2, u_3, \ldots, u_{2r}] \to k[x_1, \ldots, x_{r-1}, t_1, \ldots, t_r]$ is injective. It can be factored through $k[u_3, u_5, \ldots, u_{2r-1}, t_1, \ldots, t_r]$, by the homomorphism μ sending $u_2, u_4, \ldots, u_{2r}$ to the elementary symmetric polynomials in $t_1, \ldots, t_r$. Since μ is injective, it remains to show that $\sigma\colon k[u_3, u_5, \ldots, u_{2r-1}, t_1, \ldots, t_r] \to k[x_1, \ldots, x_{r-1}, t_1, \ldots, t_r]$ is injective.

As in the argument for $O(2r)$, we will show (more strongly) that σ is generically étale; that is, its Jacobian determinant is not identically zero. Because σ is the identity on the t_i coordinates, it suffices to show that the determinant of the matrix of derivatives of $u_3, u_5, \ldots, u_{2r-1}$ with respect to $x_1, \ldots, x_{r-1}$ is nonzero for $x_1, \ldots, x_{r-1}, t_1, \ldots, t_r$ generic. This matrix of derivatives in fact only involves $t_1, \ldots, t_r$, because $u_3, u_5, \ldots, u_{2r-1}$ have degree 1 as polynomials in $x_1, \ldots, x_{r-1}$. For example, for $r = 3$, this $(r-1) \times (r-1)$ matrix of derivatives is

$$\begin{pmatrix} t_1 + t_3 & (t_1 + t_3)(t_2) \\ t_2 + t_3 & (t_2 + t_3)(t_1) \end{pmatrix},$$

where the ath column gives the derivatives of u_{2a+1} with respect to $x_1, \ldots, x_{r-1}$. For any r, the entry (j, a) of the matrix (with $j, a \in \{1, \ldots, r-1\}$) is $(t_j + t_r)e_{ja}$, where

$$e_{ja} = \sum_{\substack{1 \leq i_1 < \cdots < i_{a-1} \leq r-1 \\ \text{none equal to } j}} t_{i_1} \cdots t_{i_{a-1}}.$$

Since row j is a multiple of $(t_j + t_r)$ for each r, the determinant is $(t_1 + t_r)(t_2 + t_r) \cdots (t_{r-1} + t_r)$ times the determinant of the $(r-1) \times (r-1)$ matrix $E = (e_{ja})$. So it suffices to show that the determinant of E is not identically zero. Indeed, the determinant of E is the same determinant shown to be nonzero in the calculation above for $O(2r)$, but with r replaced by $r - 1$.

Thus we have shown that $\psi\colon k[u_2, \ldots, u_n] \to H^*_{\mathrm{H}}(BSO(n)/k)$ is injective for n even as well as for n odd. $\square$

Having shown that $\psi\colon k[u_2, \ldots, u_n] \to H^*_{\mathrm{H}}(BSO(n)/k)$ is injective, we now show that it is an isomorphism.

Let $r = \lfloor n/2 \rfloor$ and $s = \lfloor (n-1)/2 \rfloor$. Let P be the parabolic subgroup of $G = SO(n)$ that stabilizes a maximal isotropic subspace (that is, an isotropic subspace of dimension r). Then the quotient of P by its unipotent radical is isomorphic to $GL(r)$. By Proposition 9.3, we have a spectral sequence

$$E_2^{ij} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BGL(r)/k).$$

The Chow ring of G/P is isomorphic to

$$\mathbf{Z}[e_1, \ldots, e_s]/(e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} - \cdots + (-1)^i e_{2i}),$$

where $e_i \in CH^i(G/P)$ is understood to mean zero if $i > s$ [25, III.6.11]. (This uses the theorem, discussed in section 7, that the Chow ring of G/P for a split group G is independent of the characteristic of k, and is isomorphic to the integral cohomology ring of $G_{\mathbf{C}}/P_{\mathbf{C}}$. The reference assumes that n is even, but that is enough, because the obvious map $SO(2r+1)/P \to SO(2r+2)/P$ is an isomorphism.) By Proposition 7.1, it follows that the Hodge cohomology ring of G/P is isomorphic to

$$k[e_1, \ldots, e_s]/(e_i^2 = e_{2i}),$$

where e_i is in $H^i(G/P, \Omega^i)$. For any list of variables $x_1, \ldots, x_m$, write $\Delta(x_1, \ldots, x_m)$ for the k-vector space with basis consisting of all products $x_{i_1} \ldots x_{i_j}$ with $1 \leq i_1 < \cdots < i_j \leq m$ and $0 \leq j \leq m$. Then we can say that

$$H^*_{\mathrm{H}}((G/P)/k) = \Delta(e_1, \ldots, e_s).$$

The spectral sequence converges to $H^*_{\mathrm{H}}(BGL(r)/k) = k[c_1, \ldots, c_r]$, by Theorem 9.2. Write $\Phi\colon H^*_{\mathrm{H}}(BG/k) \to H^*_{\mathrm{H}}(BGL(r)/k)$ for the restriction homomorphism, which is the "edge map" associated to the 0th row in the spectral sequence. The restriction Φ takes the elements $u_2, u_4, \ldots, u_{2r}$ (where u_{2i} is in $H^i(BG, \Omega^i)$) to $c_1, c_2, \ldots, c_r$. So the E_∞ term of the spectral sequence is concentrated on the 0th row and consists of the polynomial ring $k[u_2, u_4, \ldots, u_{2r}]$.

To analyze the structure of the spectral sequence further, we use Zeeman's comparison theorem, which he used to simplify the proof of the Borel transgression theorem [25, Theorem VII.2.9]. The key point is to show that the elements e_i (possibly after adding decomposable elements) are transgressive. (By definition, an element u of $E_2^{0,q}$ in a first-quadrant spectral sequence is *transgressive* if $d_2 = \cdots = d_q = 0$ on u; then u determines an element $\tau(u) := d_{q+1}(u)$ of $E_{q+1}^{q+1,0}$, called the transgression of u.)

In order to apply Zeeman's comparison theorem, we define a model spectral sequence that maps to the spectral sequence we want to analyze. (To be precise, we consider spectral sequences of k-vector spaces, not of k-algebras.) As above, let $k = \mathbf{F}_2$. For a positive integer q, define a spectral sequence G_* with E_2 page given by $G_2 = \Delta(y) \otimes k[u]$, y in bidegree $(0, q)$, u in bidegree $(q+1, 0)$, and $d_{q+1}(yu^j) = u^{j+1}$.

$$\begin{tikzcd}
k \cdot y \arrow[dr, "\cong"] & k \cdot yu \arrow[dr, "\cong"] & k \cdot yu^2 & \cdots \\
k \cdot 1 & k \cdot u & k \cdot u^2 & \cdots
\end{tikzcd}$$

Suppose that, for some positive integer a, we have found elements y_i of $H^{2i}_{\mathrm{H}}((G/P)/k)$ for $1 \leq i \leq a$ which are transgressive in the spectral sequence E_* above. Because y_i is transgressive, there is a map of spectral sequences $G_* \to E_*$ that takes the element y (in degree $q = 2i$) to y_i. Since E_* is a spectral sequence of algebras, tensoring these maps gives a map of spectral sequences

$$\alpha\colon F_* := G_*(y_1) \otimes \cdots \otimes G_*(y_a) \otimes k[u_2, u_4, \ldots, u_{2r}] \to E_*.$$

(Here we are using that the elements $u_2, u_4, \ldots, u_{2r}$ are in $H^*_{\mathrm{H}}(BG/k)$, which is row 0 of the E_2 page on the right, and so they are permanent cycles.) Although we do not view the domain as a spectral sequence of algebras, its E_2 page is the tensor product of row 0 and column 0, and the map $\alpha\colon F_2 \to E_2$ of E_2 pages is the tensor product of the maps on row 0 and column 0.

Using these properties, we have the following version of Zeeman's comparison theorem, as sharpened by Hilton and Roitberg [25, Theorem VII.2.4]:

Theorem 11.5. *Let N be a natural number. Suppose that the homomorphism $\alpha\colon F_* \to E_*$ of spectral sequences is bijective on $E_\infty^{i,j}$ for $i + j \leq N$ and injective for $i + j = N + 1$, and that α is bijective on row 0 of the E_2 page in degrees $\leq N + 1$ and injective in degree $N + 2$. Then α is bijective on column 0 of the E_2 page in degree $\leq N$ and injective in degree $N + 1$.*

The inductive step for computing the Hodge cohomology of $BSO(n)$ is as follows.

Lemma 11.6. *Let G be $SO(n)$ over $k = \mathbf{F}_2$, P the parabolic subgroup above, $r = \lfloor n/2 \rfloor$, $s = \lfloor (n-1)/2 \rfloor$. Let N be a natural number, and let $a = \min(s, \lfloor N/2 \rfloor)$. Then, for each $1 \le i \le a$, there is an element y_i in $H^i(G/P, \Omega^i)$ with the following properties. First, y_i is equal to e_i modulo polynomials in $e_1, \ldots, e_{i-1}$ with exponents ≤ 1. Also, each element y_i is transgressive, and any lift v_{2i+1} to $H^{i+1}(BG, \Omega^i)$ of the element $\tau(y_i)$ has the property that*

$$k[u_2, u_4, \ldots, u_{2r}; v_3, v_5, \ldots, v_{2a+1}] \to H^*_{\mathrm{H}}(BG/k)$$

is bijective in degree $\le N+1$ and injective in degree $N+2$. Finally, each element v_{2i+1} is equal to u_{2i+1} modulo polynomials in $u_2, u_3, \ldots, u_{2i}$.

More precisely, if this statement holds for $N-1$, then it holds for N with the same elements y_i, possibly with one added.

We will apply Lemma 11.6 with $N = \infty$, but the formulation with N arbitrary is convenient for the proof.

Proof. As discussed earlier, the E_∞ page of the spectral sequence

$$E_2^{ij} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BGL(r)/k)$$

is isomorphic to $k[u_2, u_4, \ldots, u_{2r}]$, concentrated on row 0.

We prove the lemma by induction on N. For $N = 0$, it is true, using that $H^0_{\mathrm{H}}(BG/k) = k$ and $H^1_{\mathrm{H}}(BG/k) = 0$, as one checks using our knowledge of the E_∞ term.

We now assume the result for $N-1$, and prove it for N. By the inductive assumption, for $b := \min(s, \lfloor (N-1)/2 \rfloor)$, we can choose $y_1, \ldots, y_b$ such that $y_i \in H^i(G/P, \Omega^i)$ is equal to e_i modulo polynomials in $e_1, \ldots, e_{i-1}$ with exponents ≤ 1, y_i is transgressive for the spectral sequence, and, if we define $v_{2i+1} \in H^{i+1}(BG, \Omega^i)$ to be any lift (from the E_{2i+1} page to the E_2 page) of the transgression $\tau(y_i)$ for $1 \le i \le b$, the homomorphism

$$k[u_2, u_4, \ldots, u_{2r}; v_3, v_5, \ldots, v_{2b+1}] \to H^*_{\mathrm{H}}(BG/k)$$

is bijective in degree $\le N$ and injective in degree $N+1$. Finally, the element v_{2i+1} for $1 \le i \le b$ is equal to u_{2i+1} modulo polynomials in $u_2, u_3, \ldots, u_{2i}$.

Also, by the injectivity in degree $N+1$ (above), it follows that there is a k-linearly independent set (possibly empty) of elements z_i in $H^{N+1}_{\mathrm{H}}(BG/k)$ such that

$$\varphi\colon k[u_2, u_4, \ldots, u_{2r}; v_3, v_5, \ldots, v_{2b+1}; z_i] \to H^*_{\mathrm{H}}(BG/k)$$

is bijective in degrees at most $N+1$. (Recall that $b = \min(s, \lfloor (N-1)/2 \rfloor)$.) The elements z_i do not affect the domain of φ in degree $N+2$ (because that ring is zero in degree 1). Therefore, φ is injective in degree $N+2$, because

$$k[u_2, u_4, \ldots, u_{2r}; v_3, v_5, \ldots, v_{2b+1}] \to H^*_{\mathrm{H}}(BG/k)$$

is injective. (This uses that v_{2i+1} is equal to u_{2i+1} modulo polynomials in $u_2, u_3, \ldots, u_{2i}$, together with the injectivity of $k[u_2, u_3, \ldots, u_n] \to H^*_{\mathrm{H}}(BG/k)$, shown in Lemma 11.4.)

The elements z_i can be chosen to become zero in the E_∞ page, because the E_∞ page is just $k[u_2, u_4, \ldots, u_{2r}]$ on row 0. Therefore, there are transgressive elements $w_i \in H^N_{\mathrm{H}}((G/P)/k)$ with $z_i = \tau(w_i)$ in the E_{N+1} page. (If z_i is killed before E_{N+1}, we can simply take $w_i = 0$.)

Consider the map of spectral sequences

$$\alpha\colon F_* := \Delta(y_1, \ldots, y_b; w_i) \otimes k[u_2, u_4, \ldots, u_{2r},\ v_3, v_3, \ldots, v_{2b+1}; z_i] \to E_*.$$

The map on E_∞ terms is an isomorphism (to $k[u_2, u_4, \ldots, u_{2r}]$), and we showed two paragraphs back that the map on column 0 of the E_2 terms is bijective in degrees at most $N+1$ and injective in degree $N+2$. Therefore, Zeeman's comparison theorem (Theorem 11.5) gives that the homomorphism

$$\psi\colon \Delta(y_1, \ldots, y_b; w_i) \to H^*_{\mathrm{H}}((G/P)/k)$$

is bijective in degrees $\leq N$ and injective in degree $N + 1$.

Let $a = \min(s, \lfloor N/2 \rfloor)$. We know that $\Delta(e_1, \ldots, e_a) \to H^*_{\mathrm{H}}((G/P)/k)$ is bijective in degrees $\leq N$. Since the elements w_i are in degree N, while $b = \min(s, \lfloor (N-1)/2 \rfloor)$, we deduce that there is no element w_i if N is odd or $N > 2s$, and there is exactly one w_i if N is even and $N \leq 2s$. In the latter case, we have $a = N/2$; in that case, let y_a denote the single element w_i. Since we know that $H^*_{\mathrm{H}}((G/P)/k) = \Delta(e_1, \ldots, e_s)$, y_a must be equal to e_a modulo polynomials in $e_1, \ldots, e_{a-1}$ with exponents ≤ 1. By construction, y_a is transgressive. Also, in the case where N is even and $N \leq 2s$, let v_{2a+1} in $H^{a+1}(BG, \Omega^a)$ be a lift to the E_2 page of the element $\tau(y_a)$ (formerly called z_i). Then we know that

$$\varphi\colon k[u_2, u_4, \ldots, u_{2r}; v_3, v_5, \ldots, v_{2a+1}] \to H^*_{\mathrm{H}}(BG/k)$$

is bijective in degree $\leq N + 1$. In the case where N is even and $N \leq 2s$ (where we have added one element v_{2a+1} to those constructed before), this bijectivity in degree $N+1 = 2a+1$ together with the injectivity of $k[u_2, u_3, \ldots, u_n] \to H^*_{\mathrm{H}}(BG/k)$ in all degrees implies that v_{2a+1} must be equal to u_{2a+1} modulo polynomials in $u_2, u_3, \ldots, u_{2a}$. By the same injectivity, it follows that φ is injective in degree $N + 2$. $\square$

We can take $N = \infty$ in Lemma 11.6, because the elements $y_1, \ldots, y_s$ do not change as we increase N. This gives that $k[u_2, u_3, \ldots, u_n] \to H^*_{\mathrm{H}}(BSO(n)/k)$ is an isomorphism. (The element v_{2i+1} produced by Lemma 11.6 need not be the element u_{2i+1} defined earlier, but v_{2i+1} is equal to u_{2i+1} modulo decomposable elements, which gives this conclusion.)

Using the Hodge cohomology of $BSO(2r)$, we can compute the Hodge cohomology of $BO(2r)$ over k using the Hochschild-Serre spectral sequence of Lemma 10.3:

$$E_2^{ij} = H^i(\mathbf{Z}/2, H^j(BSO(2r), \Omega^*)) \Rightarrow H^{i+j}(BO(2r), \Omega^*).$$

We have a homomorphism $k[u_1, u_2, \ldots, u_{2r}] \to BO(2r)$ whose composition to $BSO(2r)$ is surjective. Therefore, $\mathbf{Z}/2$ acts trivially on the Hodge cohomology of $BSO(2r)$, and all differentials are zero on column 0 of this spectral sequence. It follows that the spectral sequence degenerates at E_2, and hence

$$\begin{aligned} H^*_{\mathrm{H}}(BO(2r)/k) &\cong H^*(\mathbf{Z}/2, k) \otimes H^*_{\mathrm{H}}(BSO(2r)/k) \\ &\cong k[u_1, u_2, \ldots, u_{2r}]. \end{aligned}$$

Finally, we show that the Hodge spectral sequence

$$E_1^{ij} = H^j(BG, \Omega^i) \Rightarrow H^{i+j}_{\mathrm{dR}}(BG/k)$$

degenerates for $G = SO(n)$ over k. Indeed, by restricting to a maximal torus $T = (G_m)^r$ of G, the elements $u_2, u_4, \ldots, u_{2r}$ restrict to the elementary symmetric polynomials in the generators of $H^*_{\mathrm{dR}}(BT/k) = k[t_1, \ldots, t_r]$. Therefore, the ring $k[u_2, u_4, \ldots, u_{2r}]$ injects into $H^*_{\mathrm{dR}}(BG/k)$. So all differentials into the main diagonal $\oplus_i H^{i,i}$ of the Hodge spectral sequence for BG are zero.

$$\begin{tikzcd}
H^2(BG,\Omega^0) \arrow[r, "d_1"] \arrow[drr, dashed, "d_2"] & H^2(BG,\Omega^1) \arrow[r, "d_1"] & H^2(BG,\Omega^2) \\
H^1(BG,\Omega^0) \arrow[r, "d_1"] \arrow[drr, dashed, "d_2"] & H^1(BG,\Omega^1) \arrow[r] & 0 \\
H^0(BG,\Omega^0) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

It follows that all differentials are zero on the elements $u_{2i+1} \in H^{i+1}(BG, \Omega^i)$: only d_1 maps u_{2i+1} into a nonzero group, and that is on the main diagonal. Also, all differentials are zero on the elements u_{2i} in the main diagonal (since they map into zero groups). This proves the degeneration of the Hodge spectral sequence. Therefore, $H^*_{\mathrm{dR}}(BSO(n)/k)$ is isomorphic to $k[u_2, u_3, \ldots, u_n]$.

The same argument proves the degeneration of the Hodge spectral sequence for $BO(2r)$. Therefore, $H^*_{\mathrm{dR}}(BO(2r)/k)$ is isomorphic to $k[u_1, u_2, \ldots, u_{2r}]$.

Finally, $O(2r+1)$ is isomorphic to $SO(2r+1) \times \mu_2$, and so the calculation for $BO(2r+1)$ follows from those for $BSO(2r+1)$ (above) and $B\mu_2$ (Proposition 10.1), by the Künneth theorem (Proposition 5.1). Theorem 11.1 is proved. ☐

Proof. (Proposition 11.2) Let $2r$ and $2s$ be the ranks of the quadratic bundles E and F. The problem amounts to computing the restriction from $BO(2r+2s)$ to $BO(2r) \times BO(2s)$ on Hodge cohomology or de Rham cohomology. We first compute $u(E \oplus F)$ in Hodge cohomology. The formula for $u_{2a}(E \oplus F)$ follows from the definition of u_{2a} in $H^a(BO(2r+2s), \Omega^a)$. (Since u_{2a} is in $H^a(BO(2r+2s), \Omega^a)$, its restriction to the Hodge cohomology of $BO(2r) \times BO(2s)$ must be in $H^a(BO(2r) \times BO(2s), \Omega^a)$, which explains why only the even u-classes of E and F appear in the formula.) The formula for $u_{2a+1}(E \oplus F)$ follows from the formula for $u_{2a}(E \oplus F)$, using that $u_{2a+1} = \beta u_{2a} + u_1 u_{2a}$.

In de Rham cohomology, the same formulas hold for $u(E \oplus F)$. This uses that the subring $\oplus_i H^i(BG, \Omega^i)$ of Hodge cohomology canonically maps into de Rham cohomology (Lemma 8.2). ☐

12 The spin groups

In contrast to the other calculations in this paper, we now exhibit a reductive group G such that the mod 2 cohomology of the topological space $BG_{\mathbf{C}}$ is not isomorphic to the de Rham cohomology of the algebraic stack $BG_{\mathbf{F}_2}$, even additively. The example was suggested by the observation of Feshbach, Benson, and Wood that the restriction $H^*(BG_{\mathbf{C}}, \mathbf{Z}) \to H^*(BT_{\mathbf{C}}, \mathbf{Z})^W$ fails to be surjective for $G = \mathrm{Spin}(n)$ if $n \geq 11$ and $n \equiv 3, 4, 5 \pmod 8$ [4]. For simplicity, we work out the case of $\mathrm{Spin}(11)$.

It would be interesting to make a full computation of the de Rham cohomology of $B\,\mathrm{Spin}(n)$ in characteristic 2.

Theorem 12.1.

$$\dim_{\mathbf{F}_2} H^{32}_{\mathrm{dR}}(B\,\mathrm{Spin}(11)/\mathbf{F}_2) > \dim_{\mathbf{F}_2} H^{32}(B\,\mathrm{Spin}(11)_{\mathbf{C}}, \mathbf{F}_2).$$

Proof. Let $k = \mathbf{F}_2$. Let n be an integer at least 6; eventually, we will restrict to the case $n = 11$. Let G be the split group $\mathrm{Spin}(n)$ over k, and let T be a maximal torus in G. Let $r = \lfloor n/2 \rfloor$. The Weyl group W of G is $S_r \ltimes (\mathbf{Z}/2)^r$ for $n = 2r+1$, and the subgroup $S_r \ltimes (\mathbf{Z}/2)^{r-1}$ for $n = 2r$. We start by computing the ring $O(\mathfrak{t})^W$ of W-invariant functions on the Lie algebra $\mathfrak{t}$ of T.

First consider the easier case where n is odd, $n = 2r+1$. The element -1 in $(\mathbf{Z}/2)^r \subset W$ acts as the identity on $\mathfrak{t}$, since we are in characteristic 2. The ring $O(\mathfrak{t})^W$ can also be viewed as $S(X^*(T) \otimes k)^W$. Computing this ring is similar to, but simpler than, Benson and Wood's calculation of $S(X^*(T))^W = H^*(BT_{\mathbf{C}}, \mathbf{Z})^W$ [4]. We follow their notation.

We have

$$S(X^*(T)) \cong \mathbf{Z}[x_1, \ldots, x_r, A]/(2A = x_1 + \cdots + x_r),$$

by thinking of T as the double cover of a maximal torus in $SO(2r+1)$. The symmetric group S_r in W permutes $x_1, \ldots, x_r$ and fixes A. The elementary abelian group $E_r = (\mathbf{Z}/2)^r$ in W, with generators $\epsilon_1, \ldots, \epsilon_r$, acts by: ϵ_i changes the sign of x_i and fixes x_j for $j \neq i$, and $\epsilon_i(A) = A - x_i$. So

$$S(X^*(T) \otimes k) \cong k[x_1, \ldots, x_r, A]/(x_1 + \cdots + x_r).$$

Note that $-1 := \epsilon_1 \cdots \epsilon_r$ in W acts as the identity on $S^*(X^*(T) \otimes k)$.

We first compute the invariants of the subgroup E_r on $S(X^*(T) \otimes k)$, using the following lemma.

Lemma 12.2. *Let R be an $\mathbf{F}_2$-algebra which is a domain, S the polynomial ring $R[x]$, and a a nonzero element of R. Let $G = \mathbf{Z}/2$ act on S by fixing R and sending x to $x + a$. Then the ring of invariants is*

$$S^G = R[u],$$

where $u = x(x+a)$.

Proof. Clearly $u = x(x+a)$ in S is G-invariant. Since u is a monic polynomial of degree 2 in x, we have $S = R[u] \oplus x \cdot R[u]$. Let σ be the generator of $G = \mathbf{Z}/2$. Any element of S can be written as $f + xg$ for some (unique) elements $f, g \in R[u]$. If $f + xg$ is G-invariant, then $0 = \sigma(f + xg) - (f + xg) = (x+a)g - xg = ag$. Since a is a non-zero-divisor in R, it is a non-zero-divisor in $R[u]$; so $g = 0$. Thus $S^G = R[u]$. ☐

Let $E_j \cong (\mathbf{Z}/2)^j$ be the subgroup of W generated by $\epsilon_1, \ldots, \epsilon_j$. Let

$$\eta_j = \prod_{I \subset \{1, \ldots, j\}} \Big(A - \sum_{i \in I} x_i \Big),$$

which is E_j-invariant. Here η_j has degree 2^j in $S^*(X^*(T) \otimes k)$. By Lemma 12.2 (with $R = k[x_1, \ldots, x_r]/(x_1 + \cdots + x_r)$) and induction on j, we have

$$S^*(X^*(T) \otimes k)^{E_j} = k[x_1, \ldots, x_r, \eta_j]/(x_1 + \cdots + x_r = 0)$$

for $1 \leq j \leq r-1$. Since $-1 = \epsilon_1 \cdots \epsilon_r$ acts as the identity on these rings, we also have

$$S^*(X^*(T) \otimes k)^{E_r} = k[x_1, \ldots, x_r, \eta_{r-1}]/(x_1 + \cdots + x_r = 0).$$

The symmetric group S_r permutes $x_1, \ldots, x_r$, and it fixes η_{r-1}. Therefore, computing the invariants of the Weyl group on $S^*(X^*(T) \otimes k)$ reduces to computing the invariants of the symmetric group S_r on $k[x_1, \ldots, x_r]/(x_1 + \cdots + x_r)$. That is known, by the following result [26, Proposition 4.1]. Write $c_1, \ldots, c_r$ for the elementary symmetric polynomials in $k[x_1, \ldots, x_r]$.

Lemma 12.3. *Let k be a field of characteristic 2. If $r \geq 3$, then the ring of invariants of S_r on $R = k[x_1, \ldots, x_r]/(x_1 + \cdots + x_r)$ is equal to $k[c_1, \ldots, c_n]/(c_1) = k[c_2, \ldots, c_r]$. If $r = 2$, on the other hand, then S_2 acts trivially on $R = k[x_1, x_2]/(x_1 + x_2)$, and so $R^{S_2} = R = k[x_1]$.*

Combining Lemma 12.3 with the calculations above, we have found the invariants for the Weyl group W of $G = \mathrm{Spin}(2r+1)$: for $r \geq 1$,

$$S^*(X^*(T) \otimes k)^W = \begin{cases} k[c_2, \ldots, c_r, \eta_{r-1}] & \text{if } r \neq 2, \\ k[x_1, \eta_1] & \text{if } r = 2. \end{cases}$$

Here $|c_i| = i$ for $2 \leq i \leq r$, $|x_1| = 2$, and $|\eta_{r-1}| = 2^{r-1}$.

We now compute $S^*(X^*(T) \otimes k)^W$ for $G = \mathrm{Spin}(2r)$. Note that a maximal torus in $\mathrm{Spin}(2r)$ is also a maximal torus in $\mathrm{Spin}(2r+1)$. So we have again

$$S^*(X^*(T) \otimes k) \cong k[x_1, \ldots, x_r, A]/(x_1 + \cdots + x_r).$$

The Weyl group $W = S_r \ltimes (\mathbf{Z}/2)^{r-1}$ acts on this ring by: S_r permutes $x_1, \ldots, x_r$, and fixed A, and $(\mathbf{Z}/2)^{r-1}$ is the subgroup $\langle \epsilon_1\epsilon_2, \ldots, \epsilon_1\epsilon_r \rangle$ in the notation above. Thus $\epsilon_1\epsilon_j$ fixes each x_j (since we are working modulo 2) and sends A to $A - x_1 - x_j$.

For $1 \leq j \leq r$, let F_j be the subgroup $\langle \epsilon_1\epsilon_2, \ldots, \epsilon_1\epsilon_j \rangle \cong (\mathbf{Z}/2)^{j-1} \subset W$. Let

$$\mu_j = \prod_{\substack{I \subset \{1, \ldots, j\} \\ |I| \text{ even}}} \left(A - \sum_{i \in I} x_i \right).$$

Then $|\mu_j| = 2^{j-1}$ and $\mu_1 = A$. Clearly μ_j is F_j-invariant. Benson and Wood observed (or one can check directly) that if r is even and $r \geq 4$, then μ_{r-1} is in fact W-invariant, while if r is odd and $r \geq 3$, then μ_r is W-invariant [4, Proposition 4.1].

For $1 \leq j \leq r-1$, an induction on j using Lemma 12.2 gives that

$$S^*(X^*(T) \otimes k)^{F_j} = k[x_1, \ldots, x_r, \mu_j]/(x_1 + \cdots + x_r).$$

If r is even, then $-1 := \epsilon_1 \cdots \epsilon_r$ is in $F_r \subset W$, and it acts trivially on $S^*(X^*(T) \otimes k)$. Therefore, for r even, we have

$$S^*(X^*(T) \otimes k)^{F_r} = k[x_1, \ldots, x_r, \mu_{r-1}]/(x_1 + \cdots + x_r).$$

If r is odd, then we can apply Lemma 12.2 one more time, yielding that

$$S^*(X^*(T)\otimes k)^{F_r} = k[x_1,\ldots,x_r,\mu_r]/(x_1+\cdots+x_r).$$

The subgroup $S_r \subset W$ permutes $x_1,\ldots,x_r$, and fixes μ_{r-1}, resp. μ_r. We showed above that

$$k[x_1,\ldots,x_r]/(x_1+\cdots+x_r)^{S_r} = k[c_2,\ldots,c_r].$$

Therefore, for $G = \mathrm{Spin}(2r)$, we have

$$S^*(X^*(T)\otimes k)^W = \begin{cases} k[c_2,\ldots,c_r,\mu_{r-1}] & \text{if } r \text{ is even} \\ k[c_2,\ldots,c_r,\mu_r] & \text{if } r \text{ is odd.}\end{cases}$$

Here $|c_i| = i$ for $2 \le i \le r$ and $|\mu_{r-1}| = 2^{r-2}$, resp. $|\mu_r| = 2^{r-1}$.

Thus we have determined $S^*(X^*(T)\otimes k)^W$ for $G = \mathrm{Spin}(n)$ for all n, even or odd. Now think of $G = \mathrm{Spin}(n)$ as a split reductive group over k. By Theorem 8.1 (due to Chaput and Romagny), the ring $S^*(X^*(T)\otimes k)^W = O(\mathfrak{t})^W$ can be identified with $O(\mathfrak{g})^G$ for all $n \ge 6$. (The exceptional cases $\mathrm{Spin}(3), \mathrm{Spin}(4), \mathrm{Spin}(5)$ are the spin groups that have a factor isomorphic to a symplectic group: $\mathrm{Spin}(3) \cong Sp(2)$, $\mathrm{Spin}(4) \cong Sp(2)\times Sp(2)$, and $\mathrm{Spin}(5) \cong Sp(4)$.) We deduce that for $n \ge 6$,

$$O(\mathfrak{g})^G = \begin{cases} k[c_2,\ldots,c_r,\eta_{r-1}] & \text{if } n = 2r+1 \\ k[c_2,\ldots,c_r,\mu_{r-1}] & \text{if } n = 2r \text{ and } r \text{ is even} \\ k[c_2,\ldots,c_r,\mu_r] & \text{if } n = 2r \text{ and } r \text{ is odd.}\end{cases}$$

For $G = \mathrm{Spin}(n)$ and any $n \ge 6$, we have homomorphisms

$$O(\mathfrak{g})^G \to H^*_{\mathrm{dR}}(BG/k) \to H^*_{\mathrm{dR}}(BT/k)^W = O(\mathfrak{t})^W,$$

whose composition is the obvious inclusion. (The first homomorphism comes from the isomorphism of $O(\mathfrak{g})^G$ with $\oplus_i H^i(BG_k,\Omega^i)$, using that $H^i(BG_k,\Omega^j) = 0$ for $i < j$.) In this case, the restriction $O(\mathfrak{g})^G \to O(\mathfrak{t})^W$ is a bijection. So $H^*_{\mathrm{dR}}(BG/k)$ contains the ring computed above (with degrees multiplied by 2), and retracts onto it. It follows that for all $n \ge 6$, $H^*_{\mathrm{dR}}(BG/k)$ has an indecomposable generator in degree 2^r if $n = 2r+1$, in degree 2^{r-1} if $n = 2r$ and r is even, and in degree 2^r if $n = 2r$ and r is odd. (For this argument, we do not need to find all the indecomposable generators of $H^*_{\mathrm{dR}}(BG/k)$.) For our application to $\mathrm{Spin}(11)$, we note the following information:

Lemma 12.4. *The image of $H^*_{\mathrm{dR}}(B\,\mathrm{Spin}(2r+1)/k) \to H^*_{\mathrm{dR}}(BT/k)$ is the polynomial ring $k[c_2,\ldots,c_r,\eta_{r-1}]$, where $|c_i| = 2i$ and $|\eta_{r-1}| = 2^r$.*

Compare this with Quillen's calculation of the cohomology of the classifying space of the complex reductive group $\mathrm{Spin}(n)_{\mathbf{C}}$, or equivalently of the compact Lie group $\mathrm{Spin}(n)$ [28, Theorem 6.5]:

$$H^*(B\,\mathrm{Spin}(n)_{\mathbf{C}},k) \cong H^*(BSO(n)_{\mathbf{C}},k)/J \otimes k[w_{2^h}(\Delta_\theta)].$$

Here Δ_θ is a faithful orthogonal representation of $\mathrm{Spin}(n)_{\mathbf{C}}$ of minimal dimension, and J is the ideal generated by the regular sequence

$$w_2, Sq^1 w_2, \ldots, Sq^{2^{h-2}}\cdots Sq^2 Sq^1 w_2$$

in the polynomial ring $H^*(BSO(n)_{\mathbf{C}}, k) = k[w_2, w_3, \ldots, w_n]$, where $|w_i| = i$. Finally, the number h is given by the following table:

n	h
$8l+1$	$4l+0$
$8l+2$	$4l+1$
$8l+3$	$4l+2$
$8l+4$	$4l+2$
$8l+5$	$4l+3$
$8l+6$	$4l+3$
$8l+7$	$4l+3$
$8l+8$	$4l+3$

The Steenrod operations on the mod 2 cohomology of $BSO(n)_{\mathbf{C}}$, as used in the formula above, are known, by Wu's formula [25, Theorem III.5.12]:

$$Sq^i w_j = \sum_{l=0}^{i} \binom{j-l-1}{i-l} w_l w_{i+j-l}$$

for $0 \leq i \leq j$, where by convention $\binom{-1}{0} = 1$.

Write $r = \lfloor n/2 \rfloor$. If $n = 2r+1$, then the generator $w_{2^h}(\Delta_\theta)$ is in degree 2^r if $r \equiv 0, 3 \pmod 4$ and in degree 2^{r+1} if $r \equiv 1, 2 \pmod 4$. If $n = 2r$, then the generator $w_{2^h}(\Delta_\theta)$ is in degree 2^{r-1} if $r \equiv 0 \pmod 4$ and in degree 2^r if $r \equiv 1, 2, 3 \pmod 4$. Therefore, for $n \geq 11$, $H^*(B\,\mathrm{Spin}(n)_{\mathbf{C}}, k)$ has no indecomposable generator in degree 2^r if $n \equiv 3, 5 \pmod 8$, and no indecomposable generator in degree 2^{r-1} if $n \equiv 4 \pmod 8$. But $H^*_{\mathrm{dR}}(BG/k)$ does have an indecomposable generator in the indicated degree 2^a, as shown above. Thus, for $G = \mathrm{Spin}(n)$, $H^*(BG_{\mathbf{C}}, k)$ is not isomorphic to $H^*_{\mathrm{dR}}(BG/k)$ as a graded ring when $n \geq 11$ and $n \equiv 3, 4, 5 \pmod 8$.

We want to show, more precisely, that for $n = 11$, $H^{32}_{\mathrm{dR}}(BG/k)$ is bigger than $H^{32}(BG_{\mathbf{C}}, k)$. We know the cohomology of $BG_{\mathbf{C}}$ by Quillen (above), and so it remains to give a lower bound for the de Rham cohomology of BG over k.

We do this by restricting to a suitable abelian k-subgroup scheme of $G = \mathrm{Spin}(n)$. Assume that $n \not\equiv 2 \pmod 4$; this includes the case Spin(11) that we are aiming for. Then the Weyl group W of $\mathrm{Spin}(n)$ contains -1. So $\mathrm{Spin}(n)$ contains an extension of $\mathbf{Z}/2$ by a split maximal torus $T \cong (G_m)^r$, where $\mathbf{Z}/2$ acts by inversion on T. Let L be the subgroup of the form $1 \to T[2] \to L \to \mathbf{Z}/2 \to 1$; then L is abelian (because inversion is the identity on $T[2] \cong (\mu_2)^r$). Since the field $k = \mathbf{F}_2$ is perfect, the reduced subscheme of L is a k-subgroup scheme (isomorphic to $\mathbf{Z}/2$) [24, Corollary 1.39], and so the extension splits. That is, $L \cong (\mu_2)^r \times \mathbf{Z}/2$.

Let us compute the pullbacks of the generators u_i of $H^*_{\mathrm{dR}}(BSO(n)/k)$ (Theorem 11.1) to the subgroup L of $G = \mathrm{Spin}(n)$. It suffices to compute the restrictions of the classes u_i to the image K of L in $SO(n)$; clearly $K \cong (\mu_2)^{r-1} \times \mathbf{Z}/2$. In notation similar to that used earlier in this proof, the ring of polynomial functions on the Lie algebra of the subgroup $(\mu_2)^{r-1}$ here is

$$k[t_1, \ldots, t_r]/(t_1 + \cdots + t_r).$$

This ring can be viewed as the Hodge cohomology ring of $B(\mu_2)^{r-1}$ modulo its radical, with the generators t_i in $H^1(B(\mu_2)^{r-1}, \Omega^1)$ (by Propositions 10.1 and 5.1).

Using Lemma 10.2, we conclude that

$$H^*_{\mathrm{H}}(BK/k)/\mathrm{rad} \cong k[s, t_1, \ldots, t_r]/(t_1 + \cdots + t_r),$$

where s is pulled back from the generator of $H^1(B(\mathbf{Z}/2), O)$. The Hodge spectral sequence for BK degenerates at E_1, since we know this degeneration for $B\mathbf{Z}/2$ and $B(\mu_2)^{r-1}$. Therefore,

$$H^*_{\mathrm{dR}}(BK/k)/\mathrm{rad} \cong k[s, t_1, \ldots, t_r]/(t_1 + \cdots + t_r),$$

Note that the surjection $L \to K$ is split. So if we compute that an element of $H^*_{\mathrm{dR}}(BSO(n)/k)$ has nonzero restriction to K, then it has nonzero restriction to L, hence a fortiori to $G = \mathrm{Spin}(n)$.

Now strengthen the assumption $n \not\equiv 2 \pmod 4$ to assume that n is odd and $n \geq 7$. In Lemma 11.4, we computed the restriction of $u_2, u_3, \ldots, u_{2r+1}$ in de Rham cohomology from $SO(2r+1)$ to its subgroup $O(2)^r$, and hence to its subgroup $(\mu_2)^r \times (\mathbf{Z}/2)^r$. We now want to restrict to the smaller subgroup $K = (\mu_2)^{r-1} \times \mathbf{Z}/2$. This last step sends $H^*_{\mathrm{dR}}(B((\mu_2)^r \times (\mathbf{Z}/2)^r)/k)/\mathrm{rad} = k[s_1, \ldots, s_r, t_1, \ldots, t_r]$ to $H^*_{\mathrm{dR}}(BK/k)/\mathrm{rad} = k[s, t_1, \ldots, t_r]/(t_1 + \cdots + t_r)$ by $s_i \mapsto s$ for all i and $t_i \mapsto t_i$. By Lemma 11.4, the element u_{2a} (for $1 \leq a \leq r$) restricts to the elementary symmetric polynomial

$$c_a = \sum_{1 \leq i_1 < \cdots < i_a \leq r} t_{i_1} \cdots t_{i_a}.$$

Thus u_2 restricts to 0 on K, but $u_4, u_6, \ldots, u_{2r}$ restrict to generators of the polynomial ring

$$(k[t_1, \ldots, t_r]/(t_1 + \cdots + t_r))^{S_r} \subset H^*_{\mathrm{dR}}(BK/k)/\mathrm{rad},$$

using that $r \geq 3$, by Lemma 12.3.

Next, by Lemma 11.4, for $1 \leq a \leq r$, the restriction of u_{2a+1} to $H^*_{\mathrm{dR}}(BK/k)/\mathrm{rad}$ is (first restricting from $SO(2r+1)$ to its subgroup $(\mu_2)^r \times (\mathbf{Z}/2)^r$, and then to $K = (\mu_2)^{r-1} \times \mathbf{Z}/2$):

$$\begin{aligned} u_{2a+1} &\mapsto \sum_{m=1}^{r} s_m \sum_{\substack{1 \leq i_1 < \cdots < i_a \leq r \\ \text{one equal to } m}} t_{i_1} \cdots t_{i_a} \\ &\mapsto as \sum_{1 \leq i_1 < \cdots < i_a \leq r} t_{i_1} \cdots t_{i_a} \\ &= as u_{2a}. \end{aligned}$$

Thus, for all $1 \leq a \leq r$, u_{2a+1} restricts in $H^*_{\mathrm{dR}}(BK/k)/\mathrm{rad}$ to su_{2a} if a is odd, and otherwise to zero. (But u_2 restricts to 0, and so this also means that u_3 restricts to 0.)

This gives a lower bound for the image of $H^*_{\mathrm{dR}}(BSO(n)/k) \to H^*_{\mathrm{dR}}(B\,\mathrm{Spin}(n)/k)$ for n odd. In particular, for $n = 11$, this image has Hilbert series at least that of the ring

$$k[u_4, u_6, u_7, u_8, u_{10}, u_{11}]/(u_{11}u_6 + u_{10}u_7),$$

since the latter ring is isomorphic to the image of restriction from $SO(11)$ to $H^*_{\mathrm{dR}}(BL/k)/\mathrm{rad}$, where $L \subset \mathrm{Spin}(11)$.

We now compare this to Quillen's computation (above) in the case of Spin(11):

$$H^*(B\,\mathrm{Spin}(11)_{\mathbf{C}}, k) = k[w_4, w_6, w_7, w_8, w_{10}, w_{11}, w_{64}(\Delta_\theta)]/(w_{11}w_6 + w_{10}w_7, \\ w_{11}^3 + w_{11}^2 w_7 w_4 + w_{11}w_8w_7^2).$$

Since the last generator $w_{64}(\Delta_\theta)$ is in degree 64 and the last relation is in degree 33, the degree-32 component of this ring has the same dimension as the degree-32 component of the lower bound above for $H^*_{\mathrm{dR}}(B\,\mathrm{Spin}(11)/k)$. However, Lemma 12.4 shows that $H^*_{\mathrm{dR}}(B\,\mathrm{Spin}(11)/k)$ has an extra generator η_4 in degree 32. This is linearly independent of the image of restriction from $SO(11)$, as we see by restricting to a maximal torus T in Spin(11). Indeed, by Lemma 12.4, the image of $H^*_{\mathrm{dR}}(B\,\mathrm{Spin}(11)/k) \to H^*_{\mathrm{dR}}(BT/k)$ is the polynomial ring $k[c_2, \ldots, c_5, \eta_4]$, whereas the image of the pullback from $SO(11)$ to $T \subset \mathrm{Spin}(11)$ is just $k[c_2, \ldots, c_5]$ $(= k[w_4, w_6, w_8, w_{10}])$. Thus we have shown that

$$\dim_k H^{32}_{\mathrm{dR}}(B\,\mathrm{Spin}(11)/k) > \dim_k H^{32}(B\,\mathrm{Spin}(11)_{\mathbf{C}}, k).$$

□

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