The integral Hodge conjecture for 3-folds of Kodaira dimension zero

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The Hodge conjecture is true for all smooth complex projective 3-folds, by the Lefschetz $(1,1)$ theorem and the hard Lefschetz theorem [15, p. 164]. The integral Hodge conjecture is a stronger statement which fails for some 3-folds, in fact for some smooth hypersurfaces in $\mathbf{P}^4$, by Kollár [22]. Voisin made a dramatic advance by proving the integral Hodge conjecture for all uniruled 3-folds (or, equivalently, all 3-folds with Kodaira dimension $-\infty$) and all 3-folds X with trivial canonical bundle K_X and first Betti number zero [39]. Also, Grabowski proved the integral Hodge conjecture for abelian 3-folds [13, Corollary 3.1.9].

In this paper, we prove the integral Hodge conjecture for all smooth projective 3-folds X of Kodaira dimension zero with $h^0(X, K_X) > 0$ (hence equal to 1). This generalizes the results of Voisin and Grabowski in two directions. First, it includes all smooth projective 3-folds with trivial canonical bundle, not necessarily with first Betti number zero. For example, the integral Hodge conjecture holds for quotients of an abelian 3-fold by a free action of a finite group preserving a volume form, and for volume-preserving quotients of a K3 surface times an elliptic curve. Second, our result includes any smooth projective 3-fold whose minimal model is a possibly singular variety with trivial canonical bundle; this extends work of Höring-Voisin on singular 3-folds of this type [19, Proposition 3.18].

In contrast, Benoist and Ottem showed that the integral Hodge conjecture can fail for 3-folds of any Kodaira dimension ≥ 0. In particular, it can fail for an Enriques surface times an elliptic curve; in that case, X has Kodaira dimension zero, and in fact the canonical bundle is torsion of order 2 [4]. So our positive result is sharp in a strong sense.

The proof here covers all cases (including abelian 3-folds) in a unified way, building on the arguments of Voisin and Höring-Voisin. In order to show that a given homology class is represented by an algebraic 1-cycle on X, we consider a family of surfaces of high degree in a minimal model of X. The 1-cycle we want cannot be found on most surfaces in the family, but it will appear on some surface in the family. This uses an analysis of Noether-Lefschetz loci, which depends on the assumption that $h^0(X, K_X) > 0$.

As an application of what we know about the integral Hodge conjecture, we prove the integral Tate conjecture for all rationally connected 3-folds and all 3-folds of Kodaira dimension zero with $h^0(X, K_X) > 0$ in characteristic zero (Theorem 6.1). Finally, we prove the integral Tate conjecture for abelian 3-folds in any characteristic (Theorem 7.1).

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1 Notation

The *integral Hodge conjecture* for a smooth complex projective variety X asserts that every element of $H^{2i}(X,\mathbf{Z})$ whose image in $H^{2i}(X,\mathbf{C})$ is of type (i,i) is the class of an algebraic cycle of codimension i, that is, a $\mathbf{Z}$-linear combination of subvarieties of X. The *Hodge conjecture* is the analogous statement for rational cohomology and algebraic cycles with rational coefficients. The *integral Tate conjecture* for a smooth projective variety X over the separable closure F of a finitely generated field says: for k a finitely generated field of definition of X whose separable closure is F and l a prime number invertible in k, every element of $H^{2i}(X_F,\mathbf{Z}_l(i))$ fixed by some open subgroup of $\mathrm{Gal}(F/k)$ is the class of an algebraic cycle over F with $\mathbf{Z}_l$ coefficients. Although it does not hold for all varieties, this version of the integral Tate conjecture holds in more cases than the analogous statement over the finitely generated field k [37, section 1]. The *Tate conjecture* is the analogous statement with $\mathbf{Q}_l$ coefficients.

On a normal variety Y, we use a natural generalization of the vector bundle of differential forms on a smooth variety, the sheaf $\Omega_Y^{[j]}$ of *reflexive differentials*:

$$\Omega_Y^{[j]} := (\Omega_Y^j)^{**} = i_*\Omega_U^j,$$

where $i\colon U\to Y$ is the inclusion of the smooth locus.

For a vector space V, $P(V)$ denotes the space of hyperplanes in V.

2 Examples

In this section, we discuss some examples of 3-folds satisfying our assumptions, and how our proof works in various cases. One interesting point is the following dichotomy among 3-folds satisfying our assumptions. This dichotomy will not be used in the rest of the paper, but the proof of Proposition 2.1 develops some basic properties of these 3-folds that will be used.

Proposition 2.1. *Let X be a smooth projective complex 3-fold of Kodaira dimension zero with $h^0(X,K_X)>0$ (hence equal to 1). Let Y be a minimal model of X. (Here Y is a terminal 3-fold with K_Y trivial.) Then either $H^1(X,O)=H^1(Y,O)$ is zero or Y is smooth (or both).*

Note that the integral Hodge conjecture for smooth projective 3-folds is a birationally invariant property [40, Lemma 15]. Therefore, to prove Theorem 4.1 (the integral Hodge conjecture for X as above), we could assume that $H^1(X,O)=0$ or else that K_X is trivial (although we will not in fact divide up the proof of Theorem 4.1 that way). The case with $H^1(X,O)=0$ follows from work of Höring and Voisin [19, Proposition 3.18] together with a relatively easy analysis of singularities below (Lemma 3.1). (To give examples of such 3-folds: there are many terminal hypersurface singularities in dimension 3, such as any isolated singularity of the form $xy+f(z,w)=0$ for some power series f [30, Definition 3.1, Corollary 3.12], and X could be any resolution of a terminal quintic 3-fold in $\mathbf{P}^4$.)

The case of smooth projective 3-folds X with K_X trivial (but $H^1(X,O)$ typically not zero) is harder, and requires a thorough reworking of Höring and Voisin's arguments. We discuss examples of such varieties after the following proof.

Proof. (Proposition 2.1) By Mori, there is a minimal model Y of X [24, 2.14]. That is, Y is a terminal projective 3-fold whose canonical divisor K_Y is nef, with a birational map from X to Y. Terminal varieties are smooth in codimension 2, and so Y is smooth outside finitely many points. Since X has Kodaira dimension zero, the Weil divisor class K_Y is torsion by the abundance theorem for 3-folds, proved by Kawamata and Miyaoka [21]. Since $h^0(Y,K_Y) = h^0(X,K_X) > 0$, K_Y is trivial (and hence $h^0(X,K_X) = h^0(Y,K_Y) = 1$).

Since K_Y is linearly equivalent to zero, K_Y is in particular a Cartier divisor. Also, since Y is terminal, it has rational singularities [30, 3.8]; therefore, Y is Cohen-Macaulay. So the line bundle K_Y is the dualizing sheaf of Y. Since Y is a terminal 3-fold with K_Y Cartier, it has only hypersurface (hence lci) singularities, by Reid [30, Theorem 3.2]. Let S be a smooth ample Cartier divisor in Y; then S is contained in the smooth locus of Y. Write $i\colon S \to X$ for the inclusion. By Goresky and MacPherson, $i_*\colon H_2(S,\mathbf{Z}) \to H_2(Y,\mathbf{Z})$ is surjective, using that Y has only lci singularities [11, p. 24].

For any scheme Y of finite type over the complex numbers, du Bois constructed a canonical object $\underline{\Omega}^*_Y$ in the filtered derived category of Y, isomorphic to the constant sheaf $\mathbf{C}_Y$ in the usual derived category $D(Y_{\text{an}})$ [9]. For Y smooth, this is simply the de Rham complex. Write $\underline{\Omega}^j_Y$ in $D(Y)$ for the jth graded piece of $\underline{\Omega}^*_Y$ with respect to the given filtration, shifted j steps to the left; for Y smooth, this is the sheaf Ω^j_Y in degree zero. For Y proper over $\mathbf{C}$, the resulting spectral sequence

$$E_1^{pq} = H^q(Y,\underline{\Omega}^p_Y) \Rightarrow H^{p+q}(Y,\mathbf{C})$$

degenerates at E_1 [9, Theorem 4.5]. The associated filtration on $H^*(Y,\mathbf{C})$ is the Hodge filtration defined by Deligne.

The objects $\underline{\Omega}^j_Y$ need not be sheaves, even in our very special situation, where Y has terminal 3-fold hypersurface singularities. In particular, Steenbrink showed that $\underline{\Omega}^1_Y$ has nonzero cohomology in degree 1 (not just degree 0) for any isolated rational complete intersection 3-fold singularity other than a node or a smooth point [33, p. 1374].

In our case, because $H_2(S,\mathbf{Z}) \to H_2(Y,\mathbf{Z})$ is surjective, the pullback $H^2(Y,\mathbf{Q}) \to H^2(S,\mathbf{Q})$ is injective. By strict compatibility of pullback maps with the weight filtration, it follows that the mixed Hodge structure on $H^2(Y,\mathbf{Q})$ is pure of weight 2 [7]. By the discussion above, the graded pieces of the Hodge filtration on $H^2(Y,\mathbf{C})$ are $H^2(Y,\underline{\Omega}^0_Y)$, $H^1(Y,\underline{\Omega}^1_Y)$, and $H^0(Y,\underline{\Omega}^2_Y)$. Since Y is terminal (log canonical would be enough), it is du Bois, which means that $\underline{\Omega}^0_Y \cong O_Y$ [23].

Let $TY = (\Omega^1_Y)^*$, which is a reflexive sheaf on Y. The Lie algebra of the automorphism group of Y is $H^0(Y,TY)$. We have $TY \cong \Omega^{[2]}_Y \otimes K^*_Y \cong \Omega^{[2]}_Y$, the sheaf of reflexive 2-forms, since K_Y is trivial. So $H^0(Y,TY) \cong H^0(Y,\Omega^{[2]})$. Du Bois's object $\underline{\Omega}^2_Y$ in the derived category of Y has $\mathcal{H}^0(\underline{\Omega}^2_Y) \cong \Omega^{[2]}$ since Y is klt [20, Theorems 5.4 and 7.12]. Since the cohomology sheaves of $\underline{\Omega}^2_Y$ are concentrated in degrees ≥ 0, it follows that $H^0(Y,\Omega^{[2]}_Y) \cong H^0(Y,\underline{\Omega}^2_Y)$.

The polarization of $H^2(S_{t_0}, \mathbf{Q})$ by the intersection form gives a canonical direct-sum decomposition of Hodge structures [38, Lemma 7.36]:

$$H^2(S_{t_0}, \mathbf{Q}) = H^2(Y, \mathbf{Q}) \oplus H^2(Y, \mathbf{Q})^{\perp}.$$

The restriction of this polarization gives a polarization of the Hodge structure $H^2(Y, \mathbf{Q})$; this can be described as the polarization of $H^2(Y, \mathbf{Q})$ given by the ample line bundle $H = O(S)$ on Y.

In particular, the polarization of $H^2(Y, \mathbf{Q})$ gives an isomorphism $H^0(Y, \underline{\Omega}^2_Y) \cong H^2(Y, \underline{\Omega}^0_Y)^* \cong H^2(Y, O)^*$. By Serre duality and the triviality of K_Y, we have $H^2(Y, O)^* \cong H^1(Y, K_Y) \cong H^1(Y, O)$. Putting this all together, we have $H^0(Y, TY) \cong H^1(Y, O)$.

Thus, if $H^1(X, O)$ is not zero, then the identity component $\text{Aut}^0(Y)$ of $\text{Aut}(Y)$ has positive dimension. By the Barsotti-Chevalley theorem, $\text{Aut}^0(Y)$ is an extension of an abelian variety by a connected linear algebraic group [25, Theorem 8.27]. Any connected linear algebraic group over $\mathbf{C}$ is unirational [25, Theorem 17.93]. Since Y has Kodaira dimension 0, it is not uniruled, and so it has no nontrivial action of a connected linear algebraic group. We conclude that $A := \text{Aut}^0(Y)$ is an abelian variety of positive dimension.

By Brion, extending work of Nishi and Matsumura, any faithful action of an abelian variety on a normal quasi-projective variety has finite stabilizer groups [6, Theorem 2]. In our case, A preserves the singular locus of Y, which has dimension at most 0 because Y is a terminal 3-fold. Since A has positive dimension, the singular locus of Y must be empty. QED

We conclude the section by giving examples of smooth projective 3-folds X with K_X trivial and $H^1(X, O_X) \neq 0$, beyond the obvious examples: a K3 surface times an elliptic curve, or an abelian 3-fold. The Beauville-Bogomolov structure theorem implies that X is a quotient of a variety Z of one of those special types by a free action of a finite group [3]. Knowing the integral Hodge conjecture for Z does not obviously imply it for X, which helps to motivate this paper.

Example 2.2. An action of a finite group G on a complex K3 surface S is said to be *symplectic* if G acts as the identity on $H^0(S, K_S) \cong \mathbf{C}$. Mukai (completing earlier work of Nikulin) classified the finite groups that can act faithfully and symplectically on some K3 surface. In particular, the abelian groups that can occur are: $\mathbf{Z}/a$ for $1 \leq a \leq 8$, $(\mathbf{Z}/2)^2$, $(\mathbf{Z}/2)^3$, $(\mathbf{Z}/2)^4$, $(\mathbf{Z}/3)^2$, $(\mathbf{Z}/4)^2$, $\mathbf{Z}/2 \times \mathbf{Z}/4$, and $\mathbf{Z}/2 \times \mathbf{Z}/6$ [29, Theorem 4.5(b) and note added in proof], [27, Theorem 0.6].

Let G be a nontrivial group on this list other than $(\mathbf{Z}/2)^3$ or $(\mathbf{Z}/2)^4$, and let G act symplectically on a K3 surface S. Let E be any complex elliptic curve. We can choose an embedding of G as a subgroup of E. Let $X = (S \times E)/G$, where G acts in the given way on S and by translations on E. Then X is a smooth projective 3-fold with K_X trivial. Moreover, $H^1(X, O)$ is not zero, because X maps onto the elliptic curve E/G. Finally, X is not the product of a K3 surface with an elliptic curve. So it is a new case for which Theorem 4.1 proves the integral Hodge conjecture.

Example 2.3. Theorem 4.1 also applies to some quotients of abelian 3-folds. For example, let S be a complex abelian surface, and let G be a finite abelian group with at most 2 generators which acts faithfully and symplectically on S as an abelian surface. Let E be any elliptic curve. Choose an embedding of G as a subgroup of E. Let $X = (S \times E)/G$, where G acts in the given way on S and by translations on E. Then X is a smooth projective 3-fold, K_X is trivial, and $H^1(X, O)$ is not zero, because X maps onto the elliptic curve E/G. Here X is not an abelian 3-fold, and so it is a new case for which this paper proves the integral Hodge conjecture. The simplest case is $G = \mathbf{Z}/2$, acting on any abelian surface S by ± 1.

3 Terminal 3-folds

We here analyze the homology of the exceptional divisor of a resolution of an isolated rational 3-fold singularity (Lemma 3.1). This will be used in proving the integral Hodge conjecture for certain 3-folds whose minimal model is singular (Theorem 4.1). Benoist and Wittenberg used a similar argument in their work on the integral Hodge conjecture for real varieties, while studying a 3-fold fibered over a curve [5, proof of Proposition 8.6].

Lemma 3.1. *Let Y be a complex 3-fold with isolated rational singularities. Let $\pi\colon X \to Y$ be a projective birational morphism with X smooth such that π is an isomorphism over the smooth locus of Y and the inverse image of the singular locus of Y is a divisor D in X with simple normal crossings. Then $H_2(D, \mathbf{Z})$ is generated by classes of algebraic 1-cycles on D.*

Here a complex projective curve C (possibly singular) has a fundamental class in $H_2(C, \mathbf{Z})$, which pushes forward to a class in $H_2(D, \mathbf{Z})$ when C is contained in a complex scheme D.

Proof. We start with the following result by Steenbrink [32, Lemma 2.14].

Lemma 3.2. *Let $\pi\colon X \to Y$ be a log resolution of an isolated rational singularity, with exceptional divisor D. Then $H^i(D, O) = 0$ for all $i > 0$.*

We continue the proof of Lemma 3.1. Let $D_1, \ldots, D_r$ be the irreducible components of D, which are smooth projective surfaces. Write $D_{i_0 \cdots i_l}$ for an intersection $D_{i_0} \cap \cdots \cap D_{i_l}$. We have an exact sequence of coherent sheaves on D:

$$0 \to O_D \to \oplus_i O_{D_i} \to \oplus_{i<j} O_{D_{ij}} \to \oplus_{i<j<k} O_{D_{ijk}} \to 0.$$

Taking cohomology gives a Mayer-Vietoris spectral sequence

$$E_1^{p,q} = \oplus_{i_0 < \cdots < i_p} H^q(D_{i_0 \cdots i_p}, O) \Rightarrow H^{p+q}(D, O).$$

$$\begin{array}{ccccccc}
\oplus H^2(D_i, O) & \longrightarrow & 0 & & 0 & & 0 \\
\oplus H^1(D_i, O) & \longrightarrow & \oplus H^1(D_{ij}, O) & \longrightarrow & 0 & & 0 \\
& & & \searrow & & & \\
\oplus H^0(D_i, O) & \longrightarrow & \oplus H^0(D_{ij}, O) & \longrightarrow & \oplus H^0(D_{ijk}, O) & \longrightarrow & 0
\end{array}$$

We have $H^2(D, O) = 0$ by Lemma 3.2. It follows from the spectral sequence that each irreducible component D_i of D has $H^2(D_i, O) = 0$.

There is also a Mayer-Vietoris spectral sequence for the integral homology of D:

$$E^1_{p,q} = \oplus_{i_0<\cdots<i_p} H_q(D_{i_0\cdots i_p}, \mathbf{Z}) \Rightarrow H_{p+q}(D, \mathbf{Z}).$$

$$\begin{array}{ccccccc}
\oplus H_2(D_i,\mathbf{Z}) & \longleftarrow & \oplus H_2(D_{ij},\mathbf{Z}) & \longleftarrow & 0 & & 0 \\
 & & & & & & \\
\oplus H_1(D_i,\mathbf{Z}) & \longleftarrow & \oplus H_1(D_{ij},\mathbf{Z}) & \longleftarrow & 0 & & 0 \\
 & \nwarrow & & & & & \\
\oplus H_0(D_i,\mathbf{Z}) & \longleftarrow & \oplus H_0(D_{ij},\mathbf{Z}) & \longleftarrow & \oplus H_0(D_{ijk},\mathbf{Z}) & \longleftarrow & 0
\end{array}$$

Finally, we have a Mayer-Vietoris spectral sequence converging to $H^*(D, \mathbf{C})$, which can be obtained from the integral homology spectral sequence by applying $\mathrm{Hom}(\cdot, \mathbf{C})$. We have a map of spectral sequences from the one converging to $H^*(D, \mathbf{C})$ to the one converging to $H^*(D, O)$. Since $H^2(D, O) = 0$, we know that the groups $E^{1,1}_\infty$ and $E^{2,0}_\infty$ are zero in the spectral sequence converging to $H^*(D, O)$. That is, the d_1 and d_2 differentials together map onto $\oplus H^1(D_{ij}, O)$ and $\oplus H^0(D_{ijk}, O)$. We will deduce that the d_1 and d_2 differentials together map onto $\oplus H^1(D_{ij}, \mathbf{C})$ and $\oplus H^0(D_{ijk}, \mathbf{C})$. In the first case, we are given that $d_1\colon \oplus H^1(D_i, O) \to \oplus H^1(D_{ij}, O)$ is surjective, and we want to deduce that $d_1\colon \oplus H^1(D_i, \mathbf{C}) \to \oplus H^1(D_{ij}, \mathbf{C})$ is surjective. That follows from $d_1\colon \oplus H^1(D_i, \mathbf{C}) \to \oplus H^1(D_{ij}, \mathbf{C})$ being a morphism of Hodge structures of weight 1, so that $H^1(D_i, \mathbf{C}) = H^1(D_i, O) \oplus \overline{H^1(D_i, O)}$ and this grading is compatible with the differential.

A similar argument applies to H^0. First, the differential $d_1\colon \oplus H^0(D_{ij}, \mathbf{C}) \to \oplus H^0(D_{ijk}, \mathbf{C})$ maps isomorphically to $d_1\colon \oplus H^0(D_{ij}, O) \to \oplus H^0(D_{ijk}, O)$. Also, by the comment about Hodge structures of weight 1, $E^{0,1}_2(\mathbf{C}) = \ker(\oplus H^1(D_i, \mathbf{C}) \to \oplus H^1(D_{ij}, \mathbf{C}))$ is the direct sum of $E^{0,1}_2(O) = \ker(\oplus H^1(D_i, O) \to \oplus H^1(D_{ij}, O))$ and its conjugate, and so $E^{0,1}_2(\mathbf{C}) \to E^{0,1}_2(O)$ is surjective. Since $d_2\colon E^{0,1}_2(O) \to E^{2,0}_2(O)$ is onto and $E^{0,1}_2(\mathbf{C}) \to E^{0,1}_2(O)$ is onto, it follows that $d_2\colon E^{0,1}_2(\mathbf{C}) \to E^{2,0}_2(\mathbf{C})$ $(= E^{2,0}_2(O))$ is onto. That is, $E^{2,0}_\infty(\mathbf{C})$ as well as $E^{1,1}_\infty(\mathbf{C})$ are zero. Therefore $H^2(D, \mathbf{C}) \to \oplus H^2(D_i, \mathbf{C})$ is injective. (In particular, the mixed Hodge structure on $H^2(D, \mathbf{Q})$ is pure of weight 2 [7].) By the universal coefficient theorem, it follows that $\oplus H_2(D_i, \mathbf{Q}) \to H_2(D, \mathbf{Q})$ is surjective.

The groups $H_0(D_{ijk}, \mathbf{Z})$ and $H_1(D_{ij}, \mathbf{Z})$ are torsion-free, since D_{ijk} is a point or empty and D_{ij} is a smooth projective curve or empty. It follows that the subgroups $E^\infty_{2,0}(\mathbf{Z})$ and $E^\infty_{1,1}(\mathbf{Z})$ of these groups are also torsion-free. Since $\oplus H_2(D_i, \mathbf{Q}) \to H_2(D, \mathbf{Q})$ is surjective, those two E^∞ groups are zero after tensoring with the rationals, and so they are zero. Therefore, $\oplus H_2(D_i, \mathbf{Z}) \to H_2(D, \mathbf{Z})$ is surjective. Since $H^2(D_i, O) = 0$, the Lefschetz $(1, 1)$ theorem gives that the smooth projective surface D_i has $H_2(D_i, \mathbf{Z})$ spanned by algebraic cycles. We deduce that $H_2(D, \mathbf{Z})$ is spanned by algebraic cycles. QED

4 3-folds of Kodaira dimension zero

In this section, we begin the proof of our main result on the integral Hodge conjecture, Theorem 4.1. We reduce the problem to a statement on the variation of Hodge structure associated to a family of surfaces of high degree in a minimal model of the 3-fold, to be proved in the next section (Proposition 5.3).

Theorem 4.1. *Let X be a smooth projective complex 3-fold of Kodaira dimension zero such that $h^0(X, K_X) > 0$. Then X satisfies the integral Hodge conjecture.*

Proof. Let Y be a minimal model of X. Then Y is terminal and hence has singular set of dimension at most zero. As in the proof of Proposition 2.1, K_Y is trivial (and hence $h^0(X, K_X) = h^0(Y, K_Y) = 1$).

For codimension-1 cycles, the integral Hodge conjecture always holds, by the Lefschetz $(1, 1)$ theorem. It remains to prove the integral Hodge conjecture for codimension-2 cycles on X. This is a birationally invariant property for smooth projective varieties X [40, Lemma 15]. Therefore, we can assume that the birational map $X \dashrightarrow Y$ is a morphism, and that X is whatever resolution of Y we like. Explicitly, we can assume that $X \to Y$ is an isomorphism over the smooth locus and that the fiber over each of the (finitely many) singular points of Y is a divisor with simple normal crossings. (We do this in order to apply Lemma 3.1.)

Let H be a very ample line bundle on Y, and S a smooth surface in the linear system $|H|$. As shown in the proof of Proposition 2.1, the pushforward homomorphism $H_2(S, \mathbf{Z}) \to H_2(Y, \mathbf{Z})$ is surjective.

We assume that the Hilbert scheme $\mathcal{H}$ of smooth surfaces in Y in the homology class of S is smooth, which holds if H is sufficiently ample. We are free to replace H by a large multiple in the course of the argument.

The following lemma was suggested by Schoen's argument on the integral Tate conjecture [31, Theorem 0.5], combined with Voisin's paper [39].

Lemma 4.2. *Let Y be a terminal projective complex 3-fold. Write S_{t_0} for the surface in Y corresponding to a point t_0 in $\mathcal{H}$, with inclusion $i\colon S_{t_0} \to Y$. Write $H_2(S_{t_0}, \mathbf{Z})_{\text{van}} = \ker(i_*\colon H_2(S_{t_0}, \mathbf{Z}) \to H_2(Y, \mathbf{Z}))$. By Poincaré duality, identify $H^2(S_{t_0}, \mathbf{Z})$ with $H_2(S_{t_0}, \mathbf{Z})$. Let C be a nonempty open cone in $H^2(S_{t_0}, \mathbf{R})_{\text{van}}$. Suppose that there is a contractible open neighborhood U of t_0 in $\mathcal{H}$ such that every element of $H^2(S_{t_0}, \mathbf{Z})_{\text{van}} \cap C$ becomes a Hodge class on S_t for some t in U. Then every element of $H_2(Y, \mathbf{Z})$ whose image in $H_2(Y, \mathbf{C})$ is in $H_{1,1}(Y)$ is algebraic.*

Proof. By the proof of Proposition 2.1, the pushforward $H_2(S_{t_0}, \mathbf{Z}) \to H_2(Y, \mathbf{Z})$ is surjective, and so the pullback $H^2(Y, \mathbf{Q}) \to H^2(S_{t_0}, \mathbf{Q})$ is injective. Therefore, the Hodge structure on $H^2(S_{t_0}, \mathbf{Q})$ is pure of weight 2. Still following the proof of Proposition 2.1, the polarization of $H^2(S_{t_0}, \mathbf{Q})$ by the intersection form gives a canonical direct-sum decomposition of Hodge structures

$$H^2(S_{t_0}, \mathbf{Q}) = H^2(Y, \mathbf{Q}) \oplus H^2(Y, \mathbf{Q})^{\perp}.$$

In fact, this argument shows that the surjection $i_*\colon H_2(S_t, \mathbf{Q}) \to H_2(Y, \mathbf{Q})$ is split as a map of variations of $\mathbf{Q}$-Hodge structures over the space $\mathcal{H}$ of smooth surfaces S. In particular, any element of $H_2(Y, \mathbf{Q}) \cap H_{1,1}(Y) \subset H_2(Y, \mathbf{C})$ is the image of some element in $H_2(S_{t_0}, \mathbf{Q})$ whose translate to every surface S_t is in $H_{1,1}(S_t)$. Therefore, for any element α of $H_2(Y, \mathbf{Z})$ that maps into $H_{1,1}(Y) \subset H_2(Y, \mathbf{C})$, there is a positive integer N and an element β of $H_2(S_{t_0}, \mathbf{Z})$ that lies in $H_{1,1}(S_t) \subset H_2(S_t, \mathbf{C})$ for every surface S_t such that $i_*\beta = N\alpha$.

Also, because $i_*\colon H_2(S_{t_0}, \mathbf{Z}) \to H_2(X, \mathbf{Z})$ is surjective, there is an element $v \in H_2(S_{t_0}, \mathbf{Z})$ (not necessarily a Hodge class) with $i_* v = \alpha$.

Let $u_0 = \beta - Nv$ in $H_2(S_{t_0}, \mathbf{Z})$. Then $i_* u_0 = 0$; that is, u_0 is in $H_2(S_{t_0}, \mathbf{Z})_{\text{van}}$. Let $T = u_0 + N \cdot H_2(S_{t_0}, \mathbf{Z})_{\text{van}} \subset H_2(S_{t_0}, \mathbf{Z})_{\text{van}}$. Since T is a translate of a subgroup of finite index in $H_2(S_{t_0}, \mathbf{Z})_{\text{van}}$, T has nonempty intersection with the open cone C in $H_2(S_{t_0}, \mathbf{R})_{\text{van}}$. Let u be an element of $C \cap T$. Because U is contractible, we can canonically identify $H_2(S_t, \mathbf{Z})$ with $H_2(S_{t_0}, \mathbf{Z})$ for all t in U. By our assumption on C, u becomes a Hodge class on $H_2(S_t, \mathbf{Z})$ for some t in U. By definition of T, we can write $u = u_0 + Nw$ for some w in $H_2(S_{t_0}, \mathbf{Z})_{\text{van}}$. We know that β in $H_2(S_{t_0}, \mathbf{Z})$ is a Hodge class in $H^2(S_t, \mathbf{Z})$ for all nearby surfaces S_t. Since u becomes a Hodge class in $H^2(S_t, \mathbf{Z})$, $\beta - u$ is a Hodge class in $H_2(S_t, \mathbf{Z})$, and $\beta - u = \beta - (u_0 + Nw) = \beta - (\beta - Nv + Nw) = N(v - w)$. So $v - w$ is a Hodge class in $H_2(S_t, \mathbf{Z})$. By the Lefschetz $(1,1)$ theorem, $v - w$ is algebraic on S_t. And we have $i_*(v - w) = i_* v = \alpha$. So α in $H_2(Y, \mathbf{Z})$ is algebraic. QED

We will prove the hypothesis of Lemma 4.2 as Proposition 5.3. Given that, we now finish the proof of Theorem 4.1.

Let u be an element of $H_2(X, \mathbf{Z}) \cap H_{1,1}(X)$. Topologically, Y is obtained from Y by identifying the fibers $E_1, \ldots, E_r$ over singular points of Y to points. So we have an exact sequence

$$H_3(Y, \mathbf{Z}) \to \oplus_i H_2(E_i, \mathbf{Z}) \to H_2(X, \mathbf{Z}) \to H_2(Y, \mathbf{Z}) \to \oplus_i H_1(E_i, \mathbf{Z}).$$

By Lemma 3.1, $H_2(E_i, \mathbf{Z})$ is spanned by algebraic curves on E_i, for each i. The image of u in $H_2(Y, \mathbf{Z})$ is in $H_{1,1}(Y)$, and hence is in the image of the Chow group $CH_1(Y)$ by Lemma 4.2 and Proposition 5.3. (We use here that the integral Hodge conjecture holds on every smooth projective surface, by the Lefschetz $(1,1)$ theorem.) Since $X \to Y$ is an isomorphism outside a 0-dimensional subset of Y, it is clear that $CH_1(X) \to CH_1(Y)$ is surjective. Therefore there is a 1-cycle α on X whose image in $H_2(X, \mathbf{Z})$ has the same image in $H_2(Y, \mathbf{Z})$ as u does. By the exact sequence above, $\alpha - u$ in $H_2(X, \mathbf{Z})$ is the image of some element of $\oplus H_2(E_i, \mathbf{Z})$. But $\oplus H_2(E_i, \mathbf{Z})$ is spanned by algebraic cycles on $\cup_i E_i$ by Lemma 3.1. Therefore u is algebraic. QED

5 The variation of Hodge structure associated to a family of surfaces

To complete the proof of Theorem 4.1, we need to show that the variation of Hodge structures on the family of surfaces in the 3-fold is as nontrivial as possible (Proposition 5.3). The first step is to rephrase the conclusion we want in terms of a cup product on a general surface in the family (Corollary 5.1), generalizing Proposition 1 in Voisin [39].

Let Y be a terminal complex projective 3-fold. (We will eventually assume that K_Y is trivial, but it seems clearer to formulate the basic arguments in greater generality.) Let H be a very ample line bundle on Y, and S a smooth surface in the linear system $|H|$. (It follows that S is contained in the smooth locus of Y.) We are free to replace H by a large multiple in the course of the argument.

By the proof of Proposition 2.1, the Hodge structure on $H^2(Y, \mathbf{Q})$ is pure of weight 2, and the graded pieces of the Hodge filtration on $H^2(Y, \mathbf{C})$ are $H^2(Y, O)$, $H^1(Y, \underline{\Omega}^1_Y)$, and $H^0(Y, \underline{\Omega}^2_Y)$.

Define the vanishing cohomology $H^2(S,\mathbf{Z})_{\text{van}}$ to be the kernel of the pushforward homomorphism $i_*\colon H^2(S,\mathbf{Z})\cong H_2(S,\mathbf{Z})\to H_2(Y,\mathbf{Z})$. Likewise, write

$$\begin{aligned} H^2(S,O)_{\text{van}} &= \ker(H^2(S,O)\to H^0(Y,\underline{\Omega}^2_Y)^*)\\ H^1(S,\Omega^1_S)_{\text{van}} &= \ker(H^1(S,\Omega^1)\to H^1(Y,\underline{\Omega}^1_Y)^*)\\ H^0(S,\Omega^2_S)_{\text{van}} &= \ker(H^0(S,\Omega^2)\to H^2(Y,\underline{\Omega}^0_Y)^*). \end{aligned}$$

These maps are Hodge-graded pieces of the surjection $H_2(S,\mathbf{C})\to H_2(Y,\mathbf{C})$ (dual to the pullback $H^2(Y,\mathbf{C})\to H^2(S,\mathbf{C})$), and so they are also surjective. By Serre duality, $H^2(Y,\underline{\Omega}^0_Y)^*=H^2(Y,O)^*\cong H^1(Y,K_Y)$. So we can also describe $H^0(S,\Omega^2_S)_{\text{van}}$ as the kernel of the pushforward $H^0(S,K_S)\to H^1(Y,K_Y)$.

The cohomology sheaves of $\underline{\Omega}^j_Y$ are in degrees ≥ 0, and the 0th cohomology sheaf is the sheaf $\Omega^{[j]}_Y$ of reflexive differentials, because Y is klt [20, Theorems 5.4 and 7.12]. Also, $\underline{\Omega}^j_Y$ is concentrated in degrees from 0 to $3-j$, since Y has dimension 3 [16, Théorème V.6.2]. It follows that $\underline{\Omega}^3_Y$ is the canonical sheaf K_Y.

We assume that the Hilbert scheme $\mathcal{H}$ of smooth surfaces in Y in the homology class of S is smooth, which holds if H is sufficiently ample. Then $H^0(S,N_{S/Y})$ is the tangent space to $\mathcal{H}$ at S. Let δ be the class of the extension $0\to TS\to TY|_S\to N_{S/Y}\to 0$ in $H^1(S,N^*_{S/Y}\otimes TS)$. Then the product with δ is the Kodaira-Spencer map $H^0(S,N_{S/Y})\to H^1(S,TS)$, which describes how the isomorphism class of S changes as S moves in Y.

For u in $H^1(S,TS)$, the product with u is a linear map

$$u\cdot\colon H^1(S,\Omega^1)\to H^2(S,O),$$

The dual map

$$u\cdot\colon H^0(S,K_S)\to H^1(S,\Omega^1)$$

can also be described as the product with u. For λ in $H^1(S,\Omega^1)$, define

$$\mu_\lambda\colon H^0(S,N_{S/Y})\to H^2(S,O)$$

by $\mu_\lambda(n)=(\delta n)\lambda$. This map describes the failure of $\lambda\in H^2(S,\mathbf{C})$ to remain a $(1,1)$ class when the surface S is deformed in X. For λ in $H^1(S,\Omega^1)_{\text{van}}$, the map μ_λ lands in the vanishing subspace $H^2(S,O)_{\text{van}}$, because $H^2(S,\mathbf{Q})=H^2(Y,\mathbf{Q})\oplus H^2(S,\mathbf{Q})_{\text{van}}$ as Hodge structures, where the Hodge structure on $H^2(Y,\mathbf{Q})$ is unchanged as S is deformed.

Corollary 5.1. *Let Y be a terminal projective complex 3-fold. Let H be a very ample line bundle on Y, and S a smooth surface in the linear system $|H|$. Suppose that there is an element λ in $H^1(S,\Omega^1)_{\text{van}}$ such that the linear map*

$$\mu_\lambda\colon H^0(S,N_{S/Y})\to H^2(S,O)_{\text{van}}$$

is surjective. Then there is a nonempty open cone C in $H^2(S_{t_0},\mathbf{R})_{\text{van}}$ and a contractible open neighborhood U of t_0 in $\mathcal{H}$ such that every element of $H^2(S_{t_0},\mathbf{Z})_{\text{van}}\cap C$ becomes a Hodge class on S_t for some t in U.

For Y smooth and $H^2(Y,O)=0$ (which implies that every element of $H_2(Y,\mathbf{Z})$ is a Hodge class), this was proved by Voisin [39, Proposition 1]. Voisin also formulated a statement similar to Corollary 5.1 in the case of uniruled 3-folds Y with $H^2(Y,O)\neq 0$ [39, Proposition 4].

Proof. The groups $H^2(S_t, \mathbf{Z})$ form a weight-2 variation of Hodge structures on the Hilbert scheme $\mathcal{H}$ of smooth surfaces $S_t \subset Y$. Let U be a contractible open neighborhood of the given point t_0 in $\mathcal{H}$. We can canonically identify $H^2(S_t, \mathbf{C})$ with $H^2(S_{t_0}, \mathbf{C})$ for all $t \in U$. The surjectivity of μ_λ implies that the map from $\cup_{t \in U} H^{1,1}(S_t, \mathbf{R})_{\text{van}}$ to $H^2(S_{t_0}, \mathbf{R})_{\text{van}}$ is a submersion at λ, I claim.

To prove that this map is a submersion, we follow the argument of [39, Proposition 1], modified so as not to assume that $H^2(Y, O)$ is zero. Let $\pi\colon S_U \to U$ be the universal family of surfaces S_t, restricted to $t \in U$. Write H^2_{van} for the total space of the vector bundle $(R^2\pi_*\mathbf{C})_{\text{van}}$ over U, with fibers $H^2(S_t, \mathbf{C})_{\text{van}}$. The Gauss-Manin connection gives a trivialization of this bundle, and hence a projection map from the total space to one fiber, $\tau\colon H^2_{\text{van}} \to H^2(S_{t_0}, \mathbf{C})_{\text{van}}$. Let $F^1 H^2_{\text{van}}$ be the submanifold of H^2 whose fiber over each point $t \in U$ is the Hodge filtration

$$F^1 H^2(S_t, \mathbf{C})_{\text{van}} = H^{2,0}(S_t)_{\text{van}} \oplus H^{1,1}(S_t)_{\text{van}} \subset H^2(S_t, \mathbf{C})_{\text{van}}.$$

Let $\tau_1\colon F^1 H^2_{\text{van}} \to H^2(S_{t_0}, \mathbf{C})_{\text{van}}$ be the restriction of τ to $F^1 H^2_{\text{van}}$. Let λ be an element of $H^1(S_{t_0}, \Omega^1)_{\text{van}}$, and $\widetilde{\lambda}$ any lift of λ to $F^1 H^2(S_{t_0}, \mathbf{C})_{\text{van}}$. By the proof of Voisin [39, Lemma 2] (modified since we are allowing $H^2(Y, O)$ to be nonzero), we have the following equivalence:

Lemma 5.2. *The map*

$$\mu_\lambda\colon H^0(S_{t_0}, N_{S_{t_0}/Y}) \to H^2(S_{t_0}, O)_{\text{van}}$$

is surjective if and only if τ_1 is a submersion at $\widetilde{\lambda}$.

To relate Lemma 5.2 to cohomology with real coefficients, note that surjectivity of μ_λ is a Zariski open condition on λ in $H^1(S_{t_0}, \Omega^1)_{\text{van}}$. The vector space $H^1(S_{t_0}, \Omega^1)_{\text{van}}$ has a real structure, given by

$$H^{1,1}(S_{t_0})_{\mathbf{R},\text{van}} = H^1(S_{t_0}, \Omega^1)_{\text{van}} \cap H^2(S_{t_0}, \mathbf{R})_{\text{van}} \subset H^2(S_{t_0}, \mathbf{C})_{\text{van}}.$$

Since we assume in this Corollary that μ_λ is surjective for one λ in $H^1(S_{t_0}, \Omega^1)_{\text{van}}$, it is surjective for some λ in $H^{1,1}(S_{t_0})_{\mathbf{R},\text{van}}$.

In Lemma 5.2, take the lifting $\widetilde{\lambda}$ to be λ itself. Then $\widetilde{\lambda}$ is real, and so is $\tau_1(\widetilde{\lambda})$. By our assumption on λ, Lemma 5.2 gives that τ_1 is a submersion at $\widetilde{\lambda}$, and so the restriction

$$\tau_{1,\mathbf{R}}\colon H^{1,1}_{\mathbf{R},\text{van}} \to H^2(S_{t_0}, \mathbf{R})_{\text{van}}$$

of τ_1 to $\tau_1^{-1}(H^2(S_{t_0}, \mathbf{R})_{\text{van}})$ is also a submersion. Here $\tau_1^{-1}(H^2(S_{t_0}, \mathbf{R})_{\text{van}})$ is identified with

$$\cup_{t \in U} F^1 H^2(S_t, \mathbf{C})_{\text{van}} \cap H^2(S_t, \mathbf{R})_{\text{van}} = \cup_{t \in U} H^{1,1}(S_t, \mathbf{R})_{\text{van}} =: H^{1,1}_{\mathbf{R},\text{van}}.$$

Since $\tau_{1,\mathbf{R}}$ is a submersion at $\widetilde{\lambda}$ on the real manifold $H^{1,1}_{\mathbf{R},\text{van}}$ (a real vector bundle over U), the image of $\tau_{1,\mathbf{R}}$ contains a nonempty open subset of $H^2(S_{t_0}, \mathbf{R})_{\text{van}}$, as we wanted.

The image of $\tau_{1,\mathbf{R}}$ is a cone, and so it contains an open cone C in $H^2(S_{t_0}, \mathbf{R})_{\text{van}}$. Therefore, all elements of $H^2(S_{t_0}, \mathbf{Z})_{\text{van}}$ in the open cone C become Hodge classes on S_t for some t in U. Corollary 5.1 is proved. QED

Proposition 5.3. *Let Y be a terminal projective complex 3-fold with trivial canonical bundle. Write S_{t_0} for the surface in Y corresponding to a point t_0 in $\mathcal{H}$, with inclusion $i\colon S_{t_0} \to Y$. Write $H_2(S_{t_0}, \mathbf{Z})_{\text{van}} = \ker(i_*\colon H_2(S_{t_0}, \mathbf{Z}) \to H_2(Y, \mathbf{Z}))$. Then there is a nonempty open cone C in $H^2(S_{t_0}, \mathbf{R})_{\text{van}}$ and a contractible open neighborhood U of t_0 in $\mathcal{H}$ such that every element of $H^2(S_{t_0}, \mathbf{Z})_{\text{van}} \cap C$ becomes a Hodge class on S_t for some t in U.*

Proof. Let H be a very ample line bundle on Y and let S be a smooth surface in $|nH|$ for a positive integer n. (We will eventually take n big enough and S general in $|nH|$.) Let $V = H^0(S, K_S)_{\text{van}}$ and $V' = H^0(Y, O(S))/H^0(Y, O)$. By the exact sequence of sheaves on Y

$$0 \to O_Y \to O(S) \to O(S)|_S \to 0,$$

we can view V' as a subspace of $H^0(S, O(S)|_S) = H^0(S, N_{S/Y})$. For n sufficiently large, the long exact sequence of cohomology gives an exact sequence

$$0 \to V' \to H^0(S, N_{S/Y}) \to H^1(Y, O) \to 0.$$

Likewise, by definition of V, we have an exact sequence

$$0 \to V \to H^0(S, K_S) \to H^1(Y, K_Y) \to 0,$$

where the pushforward map shown is the boundary map from the exact sequence of sheaves on Y:

$$0 \to K_Y \to K_Y(S) \to K_S \to 0.$$

We will only need to move S in its linear system (although $H^1(Y, O)$ need not be zero). That is, we will show that for a general $\lambda \in H^1(S, \Omega^1_S)_{\text{van}}$ the restriction of μ_λ to $V' \subset H^0(S, N_{S/Y})$ maps onto $H^2(S, O)_{\text{van}} = V^*$; by Corollary 5.1, that will finish the proof of Proposition 5.3. We will see that these two vector spaces have the same dimension, using that K_Y is trivial, and so the argument just barely works. (For K_Y more positive, it would not work at all.)

Fix a trivialization of the canonical bundle K_Y. This gives an isomorphism between the two short exact sequences of sheaves above, in particular an isomorphism $K_S \cong N_{S/Y}$ of line bundles on S. So we have an isomorphism between the two exact sequences of cohomology, including an isomorphism $V \cong V'$. In terms of this identification, μ_λ for λ in $H^1(S, \Omega^1)_{\text{van}}$ is a linear map $V' \to V^*$. For varying λ, this is equivalent to the pairing

$$\mu\colon V \times V' \to H^1(S, \Omega^1)_{\text{van}}$$
$$\mu(v, v') = v(v'\delta),$$

which is symmetric. (Recall that δ is the class of the extension $0 \to TS \to TY|_S \to N_{S/Y} \to 0$ in $H^1(S, N^*_{S/Y} \otimes TS)$.) (Proof: this pairing is the restriction of a symmetric pairing $H^0(S, N_{S/Y}) \otimes H^0(S, N_{S/Y}) \to H^1(S, \Omega^1)$, given by $u \otimes v \mapsto uv\gamma\delta$, where $u, v \in H^0(S, N_{S/Y})$, and $\gamma \in H^0(S, N^*_{S/Y} \otimes (TS)^* \otimes \Omega^1_S)$ is the natural map $N_{S/Y} \otimes TS \cong \Omega^2_S \otimes TS \to \Omega^1_S$ of bundles on S.) Serre duality $H^1(S, \Omega^1)^*_{\text{van}} \cong H^1(S, \Omega^1)_{\text{van}}$ gives a dual map

$$q = \mu^*\colon H^1(S, \Omega^1)_{\text{van}} \to S^2 V^*.$$

We can think of q as a linear system of quadrics in the projective space $P(V^*)$ of lines in V. The condition that μ_λ from $V' \subset H^0(S, N_{S/Y})$ to $V^* = H^2(S, O)_{\text{van}}$ is surjective for generic $\lambda \in H^1(S, \Omega^1)_{\text{van}}$ is equivalent to the condition that the quadric defined by $q(\lambda)$ is smooth for generic λ. Thus, by Corollary 5.1, Proposition 5.3 will follow if we can show that the quadric $q(\lambda)$ is smooth for generic λ.

Note that we lose nothing by restricting the pairing μ to the subspaces $V \subset H^0(S, K_S)$ and $V' \subset H^0(S, N_{S/Y})$. Indeed, as discussed in the proof of Proposition 2.1, the Hodge structure $H^2(S, \mathbf{Q})$ is polarized by the intersection form, and the restriction $H^2(Y, \mathbf{Q}) \to H^2(S, \mathbf{Q})$ is injective, with image a sub-Hodge structure. Therefore $H^2(S, \mathbf{Q})$ is the orthogonal direct sum of $H^2(Y, \mathbf{Q})$ and its orthogonal complement. This gives a splitting of each Hodge-graded piece of $H^2(S, \mathbf{C})$. For example, for $H^0(S, K_S)$, this gives the decomposition

$$H^0(S, K_S) \cong H^0(Y, \underline{\Omega}^2_Y) \oplus H^0(S, K_S)_{\text{van}}.$$

Therefore, we also have a canonical decomposition of the isomorphic vector space $H^0(S, N_{S/Y})$. This is the decomposition

$$H^0(S, N_{S/Y}) \cong H^0(Y, TY) \oplus H^0(Y, O(S))/H^0(Y, O).$$

Thinking of $H^0(S, N_{S/Y})$ as the first-order deformation space of S in Y, these two subspaces correspond to: moving S by automorphisms of Y, and moving S in its linear system. The first type of move does not change the Hodge structure of S, and so it is irrelevant to our purpose (trying to make a given integral cohomology class on S into a Hodge class).

We use the following consequence of Bertini's theorem from Voisin [39, Lemma 15], which we apply to our space V (identified with V') and $W = H^1(S, \Omega^1)$. (Note that we follow the numbering of statements in the published version of [39], not the preprint.)

Lemma 5.4. *Let $\mu \colon V \otimes V \to W$ be symmetric and let $q \colon W^* \to S^2V^*$ be its dual. For v in V, write $\mu_v \colon V \to W$ for the corresponding linear map. Think of q as a linear system of quadrics in $\mathbf{P}(V^*)$. Then the generic quadric in im(q) is smooth if the following condition holds. There is no closed subvariety $Z \subset \mathbf{P}(V^*)$ contained in the base locus of im(q) and satisfying:*

$$\mathit{rank}(\mu_v) \leq \dim(Z)$$

for all $v \in Z$.

We have to show that such a subvariety Z does not exist for a general surface $S \in |nH|$ with n sufficiently divisible. We follow the outline of Höring and Voisin's argument ([19, after Lemma 3.35], extending [39, after Lemma 7] in the smooth case). After replacing the very ample line bundle H by a multiple if necessary, we can assume that $H^i(Y, O(lH)) = 0$ for $i > 0$ and $l > 0$. We degenerate the general surface S to a surface with many nodes, as follows. Consider a general symmetric $n \times n$ matrix A with entries in $H^0(Y, O(H))$. Let S_0 be the surface in Y defined by the determinant of A in $H^0(Y, O(nH))$. By the assumption of generality, S_0 is contained in the smooth locus of Y. By Barth [1], the singular set of S_0 consists of N nodes, where

$$N = \binom{n+1}{3} H^3.$$

Let $\mathcal{S} \to \Delta$ be a Lefschetz degeneration of surfaces $S_t \in |nH|$ over the unit disc Δ such that the central fiber S_0 has nodes $x_1, \ldots, x_N$ as singularities. The map q above makes sense for any smooth surface S in a 3-fold. Voisin showed that the limiting space

$$\lim_{t \to 0} \operatorname{im}(q_t \colon H^1(S_t, \Omega^1_{S_t}) \to (H^0(S_t, K_{S_t}) \otimes H^0(S_t, O(nH)))^*),$$

which is a linear subspace of $(H^0(S_0, K_{S_0}) \otimes H^0(S_0, O(nH)))^*$, contains for each $1 \leq i \leq N$ the multiplication-evaluation map which is the composite

$$H^0(S_0, K_{S_0}) \otimes H^0(S_0, O(nH)) \to H^0(S_0, K_{S_0}(nH)) \to K_{S_0}(nH)|_{x_i}$$

[39, Lemma 7].

Recall that we have identified $V = H^0(S, K_S)_{\text{van}}$ with $V' = H^0(Y, O(S))/H^0(Y, O)$. When we degenerate a general surface S to the nodal surface S_0, the base locus $B \subset P(V^*)$ of $\operatorname{im}(q)$ specializes to a subspace of the base locus B_0 of $\operatorname{im}(q_0) \subset P(V_0^*)$, where

$$V_0 = H^0(S_0, K_{S_0}) \cong V_0' = H^0(S_0, O_{S_0}(nH)).$$

Let W be the set of nodes of S_0. By Voisin's lemma just mentioned, B_0 is contained in

$$C_0 := \{v \in P(V_0^*) : v^2|_W = 0\}.$$

As a set, C_0 is a linear subspace:

$$\begin{aligned} C_0 &= \{v \in P(V_0^*) : v|_W = 0\} \\ &= P(H^0(S_0, K_{S_0} \otimes I_W)^*). \end{aligned}$$

By [19, eq. 3.36], extending [39, Corollary 3] in the smooth case (using only that $H^i(Y, O(lH)) = 0$ for $i > 0$ and $l > 0$), we have $h^0(Y, K_Y(nH) \otimes I_W) \leq cn^2$ for some constant c independent of n. Thus the base locus B_0 of $\operatorname{im}(q_0)$ has dimension at most cn^2, for some constant c independent of n. By specializing, the base locus B of $\operatorname{im}(q)$ also has dimension at most cn^2 for general surfaces S in $|nH|$.

By our assumption on the subvariety Z of B, for $v \in Z$ we have

$$\begin{aligned} \operatorname{rank}(\mu_v \colon V \to H^1(S, \Omega^1)) &\leq \dim(Z) \\ &\leq cn^2. \end{aligned}$$

By the following lemma, it follows that $\dim(Z) \leq A$ for some constant A independent of n. This is Höring-Voisin's [19, Lemma 3.37], extending [39, Lemma 12] in the smooth case. (As before, we follow the numbering from the published version of [39]. In our case, $H^2(Y, O)$ need not be zero, but that is not used in these proofs.)

Lemma 5.5. *Let Y be a Gorenstein projective 3-fold with isolated canonical singularities, H as above. For each positive integer n, let S be a general surface in $|nH|$, and define V, V', μ associated to S as above. Let c be any positive constant. Then there is a constant A such that the sets*

$$\Gamma = \{v \in V : \mathit{rank}(\mu_v) \leq cn^2\}$$

and

$$\Gamma' = \{v' \in V' : \mathit{rank}(\mu_{v'}) \leq cn^2\}$$

both have dimension bounded by A, independent of n.

By our assumption on the subvariety Z of B again, for $v \in Z$ we have

$$\begin{aligned}\operatorname{rank}(\mu_v \colon V \to H^1(S, \Omega^1)) &\leq \dim(Z) \\ &\leq A.\end{aligned}$$

This implies that Z is empty by Lemma 5.6, to be proved next. But Z is a variety, so we have a contradiction. This completes the proof that the generic quadric in the linear system $\operatorname{im}(q)$ is smooth. Proposition 5.3 is proved. QED

To complete the proof of Proposition 5.3 and hence Theorem 4.1, it remains to prove the following lemma.

Lemma 5.6. *Let Y be a terminal projective 3-fold with K_Y trivial, H as above. Let A be a positive integer. Let $S \in |nH|$ be general, with n large enough (depending on A). Let $V = H^0(S, K_S)_{\text{van}}$ and $\mu_v \colon V' \to H^1(S, \Omega^1)$ the product with an element $v \in V$, as defined above. Then the set*

$$W = \{v \in V : \mathit{rank}(\mu_v) < A\}$$

is equal to 0.

Proof. We have to modify the proof of Voisin's Lemma 13 [39] to allow Y to be singular and also to have $H^2(Y, O)$ not zero. We use Höring and Voisin's ideas on how to deal with Y being singular, by working on the smooth surface S as far as possible [19, proof of Proposition 3.22].

Let S be a smooth surface in $|nH|$. Consider the following exact sequences of vector bundles on S, constructed from the normal bundle sequence of S in Y:

$$0 \to \Omega^1_S(nH) \to \Omega^2_Y|_S(2nH) \to K_S(2nH) \to 0$$

and

$$0 \to O_S \to \Omega^1_Y|_S(nH) \to \Omega^1_S(nH) \to 0.$$

Let δ_1 and δ_2 be the resulting boundary maps:

$$\delta_1 \colon H^0(S, K_S(2nH)) \to H^1(S, \Omega^1_S(nH))$$

and

$$\delta_2 \colon H^1(S, \Omega^1_S(nH)) \to H^2(S, O).$$

Let $\delta = \delta_2 \circ \delta_1 \colon H^0(S, K_S(2nH)) \to H^2(S, O)$.

Lemma 5.7. *The image of δ is $H^2(S, O)_{\text{van}}$, for large enough n and any S as above.*

Proof. We first show that δ_1 is surjective. By the long exact sequence of cohomology associated to the first exact sequence above, it suffices to show that $H^1(S, \Omega^2_Y|_S(2nH))$ is zero. In terms of the sheaf $\Omega^{[2]}_Y$ of reflexive differentials, we have an exact sequence of sheaves on Y:

$$0 \to \Omega^{[2]}_Y(nH) \to \Omega^{[2]}_Y(2nH) \to \Omega^2_Y|_S(2nH) \to 0.$$

By Serre vanishing on Y, both $H^1(Y,\Omega_Y^{[2]}(2nH))$ and $H^2(Y,\Omega_Y^{[2]}(nH))$ vanish for large n, and so $H^1(S,\Omega^2_Y|_S(2nH))=0$ for all smooth S in $|nH|$ with n large.

Next, the long exact sequence involving δ_2 shows that the cokernel of δ_2 is contained in $H^2(S,\Omega^1_Y|_S(nH))$. Since $K_S=nH|_S$ by the adjunction formula, the dual of that H^2 space is $H^0(S,TY|_S)=H^0(S,\Omega^2_Y|_S)$.

By the exact sequence

$$0\to \Omega_Y^{[2]}(-nH)\to \Omega_Y^{[2]}\to \Omega^2_Y|_S\to 0$$

of sheaves on Y, we have an exact sequence

$$H^0(X,\Omega_Y^{[2]}(-nH))\to H^0(Y,\Omega_X^{[2]})\to H^0(S,\Omega_Y^{[2]}|_S)\to H^1(Y,\Omega_Y^{[2]}(-nH)).$$

Since Y is normal, the sheaf $\Omega_Y^{[2]}$ is reflexive, and $\dim(Y)>1$, the groups on the left and right are zero for n large. (Consider an embedding of Y into some $\mathbf{P}^N$ and use Serre vanishing and Serre duality on $\mathbf{P}^N$, as in [17, proof of Corollary III.7.8].) So the map $H^0(X,\Omega^{[2]})\to H^0(S,\Omega^2_Y|_S)$ is an isomorphism. By the results above on δ_1 and δ_2, this gives an exact sequence

$$H^0(S,K_S(2nH))\xrightarrow[\delta]{} H^2(S,O)\to H^0(Y,\Omega^{[2]})^*.$$

Finally, we need to rephrase this in terms of du Bois's object $\underline{\Omega}^2_Y$ in the derived category of Y. The cohomology sheaves of $\underline{\Omega}^j_Y$ are in degrees ≥ 0, and the 0th cohomology sheaf is $\Omega_Y^{[j]}$ because Y is klt [20, Theorems 5.4 and 7.12]. So there is a natural map $\Omega_Y^{[2]}\to\underline{\Omega}^2_Y$ in $D(Y)$. Because the other cohomology sheaves of $\underline{\Omega}^2_Y$ are in degrees >0, it is immediate that the map $H^0(Y,\Omega^{[2]})\to H^0(Y,\underline{\Omega}^2_Y)$ is an isomorphism. So the previous paragraph yields an exact sequence:

$$H^0(S,K_S(2nH))\xrightarrow[\delta]{} H^2(S,O)\to H^0(Y,\underline{\Omega}^2_Y)^*.$$

Equivalently, the image of δ is $H^2(S,O)_{\mathrm{van}}$. QED

Assume that $v\in V$ satisfies the condition that $\operatorname{rank}(\mu_v)<A$. Using that n is sufficiently large, Höring and Voisin show that $\delta(H^0(S,O_S(3nH)))$ is orthogonal to v with respect to Serre duality [19, after Proposition 3.40]. By Lemma 5.7, $H^2(S,O)_{\mathrm{van}}$ is orthogonal to v. Since $V=H^0(S,K_S)_{\mathrm{van}}$ is dual to $H^2(S,O)_{\mathrm{van}}$, it follows that $v=0$. Lemma 5.6 is proved. This also completes the proofs of Proposition 5.3 and Theorem 4.1. QED

6 The integral Tate conjecture for 3-folds

We now prove the integral Tate conjecture for 3-folds in characteristic zero that are rationally connected or have Kodaira dimension zero with $h^0(X,K_X)>0$ (Theorem 6.1). In any characteristic, we will prove the integral Tate conjecture for abelian 3-folds (Theorem 7.1).

Theorem 6.1. *Let X be a smooth projective 3-fold over the algebraic closure of a finitely generated field of characteristic zero. If X is rationally connected or it has Kodaira dimension zero with $h^0(X,K_X)>0$ (hence equal to 1), then X satisfies the integral Tate conjecture.*

Proof. We start by proving the following known lemma.

Lemma 6.2. *Let X be a smooth projective variety over the separable closure k_s of a finitely generated field k. For codimension-1 cycles on X, the Tate conjecture implies the integral Tate conjecture.*

Proof. For a prime number l invertible in k and a positive integer r, the Kummer sequence

$$0 \to \mu_{l^r} \to G_m \underset{l^r}{\to} G_m \to 0$$

of étale sheaves on X gives a long exact sequence of cohomology, and hence an exact sequence involving the Picard and Brauer groups:

$$0 \to \text{Pic}(X)/l^r \to H^2_{\text{et}}(X, \mu_{l^r}) \to \text{Hom}(\mathbf{Z}/l^r, \text{Br}(X)) \to 0.$$

Writing $NS(X)$ for the group of divisors modulo algebraic equivalence, we have $\text{Pic}(X)/l^r = NS(X)/l^r$, because the group of k_s-points of an abelian variety is l-divisible. Since $NS(X)$ is finitely generated, taking inverse limits gives an exact sequence:

$$0 \to NS(X) \otimes \mathbf{Z}_l \to H^2(X, \mathbf{Z}_l(1)) \to \text{Hom}(\mathbf{Q}_l/\mathbf{Z}_l, \text{Br}(X)) \to 0.$$

The last group is automatically torsion-free. It follows that the Tate conjecture implies the integral Tate conjecture in the case of codimension-1 cycles. QED

Lemma 6.3. *Let X be a smooth projective 3-fold over the algebraic closure $\overline{k}$ of a finitely generated field k of characteristic zero. Suppose that the Tate conjecture holds for codimension-1 cycles on X, and that the integral Hodge conjecture holds on $X_{\mathbf{C}}$ for some embedding $\overline{k} \hookrightarrow \mathbf{C}$. Then the integral Tate conjecture holds for X (over $\overline{k}$).*

Proof. By Lemma 6.2, the integral Tate conjecture holds for codimension-1 cycles on X. It remains to prove integral Tate for 1-cycles on X. Let $u \in H^4(X, \mathbf{Z}_l(2))$ be a Tate class; that is, u is fixed by $\text{Gal}(\overline{k}/l)$ for some finite extension l of k. Let H be an ample line bundle on X. Multiplication by the class of H is an isomorphism from $H^2(X, \mathbf{Q}_l(1))$ to $H^4(X, \mathbf{Q}_l(2))$, by the hard Lefschetz theorem. So there is a positive integer N with $Nu = Hv$ for some $v \in H^2(X, \mathbf{Z}_l(1))$. Because the isomorphism from $H^2(X, \mathbf{Q}_l(1))$ to $H^4(X, \mathbf{Q}_l(2))$ is Galois-equivariant, v is a Tate class (this works even if there is torsion in $H^2(X, \mathbf{Z}_l(1))$, because we are considering Tate classes over $\overline{k}$). By our assumptions, v is algebraic, that is, a $\mathbf{Z}_l$-linear combination of classes of subvarieties of X. So $Hv = Nu$ is algebraic and thus a $\mathbf{Z}_l$-linear combination of classes of curves on X.

In particular, Nu is a $\mathbf{Z}_l$-linear combination of Hodge classes in $H^4(X_{\mathbf{C}}, \mathbf{Z})$. Since the subgroup of Hodge classes is a summand in $H^4(X_{\mathbf{C}}, \mathbf{Z})$, it follows that u is a $\mathbf{Z}_l$-linear combination of Hodge classes in $H^4(X_{\mathbf{C}}, \mathbf{Z})$. Since the integral Hodge conjecture holds for $X_{\mathbf{C}}$, u is a $\mathbf{Z}_l$-linear combination of classes of curves on X. QED

We prove Theorem 6.1 using Lemma 6.3. The integral Hodge conjecture holds for rationally connected 3-folds by Voisin [39, Theorem 2] and for 3-folds X with Kodaira dimension zero and $h^0(X, K_X) = 1$ by Theorem 4.1, generalizing Voisin

[39, Theorem 2]. It remains to check the Tate conjecture in codimension 1 for X over $\overline{k}$. That is clear if $h^{0,2}(X) = 0$; then all of $H^2(X_{\mathbf{C}}, \mathbf{Z})$ is algebraic by the Lefschetz $(1,1)$ theorem, and so all of $H^2(X, \mathbf{Z}_l(1))$ is algebraic. That covers the case where X is rationally connected.

It remains to prove the Tate conjecture in codimension 1 for a 3-fold X over $\overline{k}$ of Kodaira dimension zero with $h^{0,2}(X) > 0$. Let Y be a minimal model of X; then Y is terminal and has torsion canonical bundle. By Höring and Peternell, generalizing the Beauville-Bogomolov structure theorem to singular varieties, there is a projective variety Z with canonical singularities and a finite morphism $Z \to Y$, étale in codimension one, such that Z is a product of an abelian variety, (singular) irreducible symplectic varieties, and (singular) Calabi-Yau varieties in a strict sense [18, Theorem 1.5]. Their theorem is stated over $\mathbf{C}$, but that implies the statement over $\overline{k}$. Höring and Peternell build on earlier work by Druel and Greb-Guenancia-Kebekus [8, 14].

Since Y has dimension 3 and $h^0(Z, \Omega^{[2]}) \geq h^0(Y, \Omega^{[2]}) = h^0(X, \Omega^2) > 0$, the only possibilities are: Z is an abelian 3-fold or the product of an elliptic curve and a K3 surface with canonical singularities. (A strict Calabi-Yau 3-fold Z has $h^0(Z, \Omega^{[2]}) = 0$, by definition.) So there is a resolution of singularities Z_1 of Z which is either an abelian 3-fold or the product of an elliptic curve and a smooth K3 surface. Since we have a dominant rational map $Z_1 \dashrightarrow X$, the Tate conjecture in codimension 1 for X will follow from the same statement for Z_1 [36, Theorem 5.2].

It remains to prove the Tate conjecture in codimension 1 for Z_1, which is either an abelian 3-fold or the product of a K3 surface and an elliptic curve over $\overline{k}$. Faltings proved the Tate conjecture in codimension 1 for all abelian varieties over number fields [10], extended to all finitely generated fields of characteristic zero by Zarhin. Finally, the Tate conjecture holds for K3 surfaces in characteristic zero, by Tankeev [34]. Since $H^2(S \times E, \mathbf{Q}_l) \cong H^2(S, \mathbf{Q}_l) \oplus H^2(E, \mathbf{Q}_l)$ for a K3 surface S and an elliptic curve E, the Tate conjecture in codimension 1 holds for $S \times E$. QED

7 The integral Tate conjecture for abelian 3-folds in any characteristic

We now prove the integral Tate conjecture for abelian 3-folds in any characteristic. In characteristic zero, we have already shown this in Theorem 6.1. However, it turns out that a more elementary proof works in any characteristic, modeled on Grabowski's proof of the integral Hodge conjecture for complex abelian 3-folds [13, Corollary 3.1.9]. More generally, we show that the integral Tate conjecture holds for 1-cycles on all abelian varieties of dimension g if the "minimal class" $\theta^{g-1}/(g-1)!$ is algebraic on every principally polarized abelian variety (X, θ) of dimension g (Proposition 7.2).

Theorem 7.1. *Let X be an abelian 3-fold over the separable closure of a finitely generated field. Then the integral Tate conjecture holds for X.*

Proof. The argument is based on Beauville's Fourier transform for Chow groups of abelian varieties, inspired by Mukai's Fourier transform for derived categories. Write $CH^*(X)_{\mathbf{Q}}$ for $CH^*(X) \otimes \mathbf{Q}$. Let X be an abelian variety of dimension g

over a field k, with dual abelian variety $\widehat{X} := \mathrm{Pic}^0(X)$, and let $f\colon X \times \widehat{X} \to X$ and $g\colon X \times \widehat{X} \to \widehat{X}$ be the projections. The *Fourier transform* $F_X\colon CH^*(X)_{\mathbf{Q}} \to CH^*(\widehat{X})_{\mathbf{Q}}$ is the linear map

$$F_X(u) = g_*(f^*(u) \cdot e^{c_1(L)}),$$

where L is the Poincaré line bundle on $X \times \widehat{X}$ and $e^{c_1(L)} = \sum_{j=0}^{2g} c_1(L)^j/j!$. For k separably closed, define the Fourier transform $H^*(X, \mathbf{Q}_l(*)) \to H^*(\widehat{X}, \mathbf{Q}_l(*))$ by the same formula.

By Beauville, the Fourier transform sends $H^j(X, \mathbf{Z}_l(a))$ to $H^{2g-j}(X, \mathbf{Z}_l(a+g-j))$, and this map is an isomorphism [2, Proposition 1]. By contrast, it is not clear whether the Fourier transform can be defined integrally on Chow groups; that actually fails over a general field, by Esnault [26, section 3.1]. Beauville's proof (for complex abelian varieties) uses that the integral cohomology of an abelian variety is an exterior algebra over $\mathbf{Z}$, and the same argument works for the $\mathbf{Z}_l$-cohomology of an abelian variety over any separably closed field.

Next, let $\theta \in H^2(X, \mathbf{Z}_l(1))$ be the first Chern class of a principal polarization on an abelian variety X. Then we can identify $\widehat{X}$ with X, and the Fourier transform satisfies

$$F_X(\theta^j/j!) = (-1)^{g-j}\, \theta^{g-j}/(g-j)!$$

[2, Lemme 1]. Here $\theta^j/j!$ lies in $H^{2j}(X, \mathbf{Z}_l(j))$ (although it is not obviously algebraic, meaning the class of an algebraic cycle with $\mathbf{Z}_l$ coefficients). Finally, let $h\colon X \to Y$ be an isogeny, and write $\widehat{h}\colon \widehat{Y} \to \widehat{X}$ for the dual isogeny. Then the Fourier transform switches pullback and pushforward, in the sense that for $u \in CH^*(Y)_{\mathbf{Q}}$,

$$F_X(h^*(u)) = \widehat{h}_*(F_Y(u))$$

[2, Proposition 3(iii)].

The following is the analog for the integral Tate conjecture of Grabowski's argument on the integral Hodge conjecture [13, Proposition 3.1.8].

Proposition 7.2. *Let k be the separable closure of a finitely generated field. Suppose that for every principally polarized abelian variety (Y, θ) of dimension g over k, the minimal class $\theta^{g-1}/(g-1)! \in H^{2g-2}(Y, \mathbf{Z}_l(g-1))$ is algebraic. Then the integral Tate conjecture for 1-cycles holds for all abelian varieties of dimension g over k.*

Proof. Let X be an abelian variety of dimension g over k, and let u be a Tate class in $H^2(X, \mathbf{Z}_l(1))$ (meaning that u is fixed by some open subgroup of the Galois group). The Tate conjecture holds for codimension-1 cycles on abelian varieties over k, by Tate [35], Faltings [10], and Zarhin. This implies the integral Tate conjecture for codimension-1 cycles on X, by Lemma 6.2. So u is a $\mathbf{Z}_l$-linear combination of classes of line bundles, hence of ample line bundles.

For each ample line bundle L on X, there is a principally polarized abelian variety (Y, θ) and an isogeny $h\colon X \to Y$ with $c_1(L) = h^*\theta$ [28, Corollary 1, p. 234]. Then the Fourier transform of $c_1(L)$ is given by:

$$\begin{aligned} F_X(c_1(L)) &= F_X(h^*\theta) \\ &= \widehat{h}_*(F_Y(\theta)) \\ &= (-1)^{g-1}\widehat{h}_*(\theta^{g-1}/(g-1)!). \end{aligned}$$

By assumption, $\theta^{g-1}/(g-1)!$ in $H^{2g-2}(Y, \mathbf{Z}_l(g-1))$ is algebraic (with $\mathbf{Z}_l$ coefficients). Since the pushforward preserves algebraic classes, the equality above shows that $F_X(c_1(L))$ is algebraic. By the previous paragraph, it follows that the Fourier transform of any Tate class in $H^2(X, \mathbf{Z}_l(1))$ is algebraic in $H^{2g-2}(\widehat{X}, \mathbf{Z}_l(g-1))$.

Since the Fourier transform is Galois-equivariant and is an isomorphism from $H^2(X, \mathbf{Z}_l(1))$ to $H^{2g-2}(\widehat{X}, \mathbf{Z}_l(g-1))$, it sends Tate classes bijectively to Tate classes. This proves the integral Tate conjecture for 1-cycles on $\widehat{X}$, hence for 1-cycles on every abelian variety of dimension g over k. QED

We now return to the proof of Theorem 7.1. Let k be the separable closure of a finitely generated field, and let X be an abelian 3-fold over k. We want to prove the integral Tate conjecture for X.

By Proposition 7.2, it suffices to show that for every principally polarized abelian 3-fold (X, θ) over k, the class $\theta^2/2$ in $H^4(X, \mathbf{Z}_l(2))$ is algebraic. (This is clear for $l \neq 2$.) A general principally polarized abelian 3-fold X over k is the Jacobian of a curve C of genus 3. In that case, choosing a k-point of C determines an embedding of C into X, and the cohomology class of C on X is $\theta^2/2$ by Poincaré's formula [15, p. 350]. (Poincaré proved this for Jacobian varieties over $\mathbf{C}$, but that implies the same formula in l-adic cohomology for Jacobians in any characteristic.) By the specialization homomorphism on Chow groups [12, Proposition 2.6, Example 20.3.5], it follows that $\theta^2/2$ is algebraic for every principally polarized abelian 3-fold over k. Theorem 7.1 is proved. QED

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