Projective resolutions of representations of $GL(n)$

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The representations of the algebraic group $GL(n)$ are well understood over a field of characteristic 0, but they are much more complicated in characteristic p (or over the integers), essentially because there can be nontrivial extensions between two representations. Akin-Buchsbaum [2] and Donkin [10] improved matters by showing that the category $P(n, r)$ of polynomial $GL(V)$-modules ($n = \dim V$) which are homogeneous of fixed degree r, such as $V^{\otimes r}$, $\Lambda^r V$, $S^r V$, the irreducible subquotients of these modules, and so on, has finite global dimension. That is, there is an integer N depending on r and $n = \dim V$ such that every polynomial $GL(V)$-module of degree r has a resolution of length $\leq N$ by projective objects in the category of degree-r representations of $GL(V)$. An equivalent statement is that for polynomial representations A, B of degree r, $\mathrm{Ext}^i_{GL(V)}(A, B) = H^i(GL(V), \mathrm{Hom}\ (A, B))$ (cohomology of $GL(V)$ as an algebraic group) vanishes for $i > N$. (For this restatement, I use that if A and B are polynomial representations of $GL(V)$ of degree r, then $\mathrm{Ext}^i(A, B)$ is the same whether computed in the category of degree-r polynomial representations or in the category of all representations of the algebraic group $GL(V)$, by Donkin [10].)

Donkin gave an explicit upper bound for the global dimension N: N is at most 2 times the length of the maximal chain in the partially ordered set of partitions of r into at most n pieces, with respect to the dominance ordering [10]. We explain the statement and proof in section 1. This bound for the global dimension is fairly large; for example, for $n \geq r$ ("V sufficiently large") the bound is asymptotically a constant times $r^{3/2}$. In fact, the polynomial representations of $GL(V)$ are much simpler than this bound suggests, as we show in this paper: for all $n \geq r$, the category $P(n, r)$ over a field k of characteristic $p > 0$ has global dimension exactly $2(r - \alpha_p(r))$, where $\alpha_p(r)$ is the sum of the digits in the base-p expansion of r. So the global dimension is at most $2(r - 1)$. Over $\mathbf{Z}$, still for $n \geq r$, the category of degree-r polynomial representations of $GL(n)$ has global dimension exactly one more than the maximum of the numbers $2(r - \alpha_p(r))$ over the prime numbers p. Since the category of degree-r representations of $GL(n)$ can be viewed as the category of modules over a certain finite-dimensional algebra, the Schur algebra $S(n, r)$ (as explained in section 1), these results can be viewed as computations of the global dimension of the Schur algebra.

In the case $n \geq r$ which our main result considers, the category $P(n, r)$ is in fact independent of $n = \dim V$, by Green [13]. It can be described without mentioning V as the category of polynomial functors over k, assuming that the base field k is infinite. This case is especially important in various applications, such as Friedlander and Suslin's proof that the cohomology ring of a finite group scheme is finitely generated [12]; section 8 of their paper asks for an upper bound for the injective dimension of a certain polynomial functor, and our main result of course gives such a bound. For $n < r$, that is, when V is not "sufficiently large," we do not succeed in computing the global dimension. We just give a reasonable upper bound, which is a little smaller than $2(r - \alpha_p(r))$.

One of the many problems left open is to define explicit projective resolutions of the most interesting modules, starting with Weyl modules. See Buchsbaum and Rota for a beginning [4], [5].

This paper was inspired by Akin and Buchsbaum's work, especially [2] and [1]. Thanks to Joseph Gubeladze, Greg Kuperberg, and Richard Stanley for their comments.

1 Background

Our calculation of the global dimension of the category $P(n,r)$ depends on the known result that this global dimension is finite. In this section we explain one proof of the finiteness: it follows from the formal properties of $P(n,r)$ discovered by Donkin ($P(n,r)$ is a "highest weight category" in the sense of [7]). It is worth mentioning that Donkin's proof of these properties relies on a paper by Cline, Parshall, Scott, and van der Kallen which also includes an early version of the "finite global dimension" theorem ([8], Theorem 2.4).

We begin by defining the category $P(n,r)$ of finite-dimensional polynomial representations of $GL(n)$ of degree r over a field k. For k infinite, there is an elementary definition of $P(n,r)$ as the category of finite-dimensional representations over k of the abstract group $GL(n,k)$ such that the matrix coefficients of the representation are homogeneous polynomials of degree r in the n^2 matrix entries of $GL(n)$. To cover arbitrary fields k, we have to define $P(n,r)$ as the category of finite-dimensional representations of $GL(n)$ as an algebraic group over k which satisfy the same condition.

For example, $V^{\otimes r}$, the symmetric power S^rV, the exterior power Λ^rV, the divided power $D^rV = S^r(V^*)^*$, and their subquotients are polynomial representations of $GL(V)$ of degree r. (We remind the reader that S^rV denotes the maximal quotient of $V^{\otimes r}$ on which the symmetric group S_r acts trivially, while D^rV is the subspace of $V^{\otimes r}$ on which S_r acts trivially.) By contrast, V^* is a representation of $GL(V)$ as an algebraic group which is not a polynomial representation; in terms of matrices, it is given by $g \mapsto (g^t)^{-1}$, so the matrix coefficients are polynomials in the matrix entries of g *and* in $1/\det(g)$.

Let us briefly describe what polynomial representations over an arbitrary commutative ring are. (This paragraph can be omitted at first reading.) Over any commutative ring k, a representation of $GL(V)$ as an algebraic group over k, where $V = k^n$, can be defined to be a comodule over the coalgebra $O(GL(V))$ of regular functions on $GL(V)$, and, analogously, we define a polynomial representation of $GL(V)$ over k to be a comodule over the coalgebra $O(\text{End } V) = \oplus_{r\geq 0} S^r(\text{End } V)$; that is, it is a representation of the algebraic monoid of the $n\times n$ matrices. Here the subspaces $S^r(\text{End } V)$ are in fact sub-coalgebras, and we define a polynomial representation of $GL(V)$ over k of degree r to be a comodule over the coalgebra $S^r(\text{End } V)$. Since the coalgebra $S^r(\text{End } V)$ is a finite free k-module, its dual space $D^r(\text{End } V)$ is an algebra which is a finite free k-module, called the Schur algebra. We see that degree-r polynomial representations of $GL(V)$ over any commutative ring k are precisely equivalent to modules over the Schur algebra $S(n,r)_k := D^r(\text{End } V)$.

We now return to considering finite-dimensional polynomial representations of $GL(n)$ over a field k. To prove that the category $P(n,r)$ has finite global dimension, we use the following facts, for which we refer to Martin's exposition [16]. The proof there of the crucial fact (3) incorporates Parshall's and Green's simplifications of Donkin's original proof [9]. Let $\Lambda^+(n,r)$ denote the set of partitions of r into at most n parts, that is, sequences $(\lambda_1, \ldots, \lambda_n)$ of integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\lambda_1 + \lambda_2 + \cdots = r$. We can picture $\Lambda^+(n,r)$ as the set of Young diagrams with r boxes in at most n rows. It is partially ordered by the "dominance ordering": for partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ of r, we say that $\lambda \geq \mu$ if for all j,

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i.$$

In terms of Young diagrams, this ordering means that a Young diagram A is bigger than a Young diagram B if you can get from A to B by repeatedly moving boxes from the end of one row to the end of some lower row.

(1) The simple modules $L(\lambda)$ in the category $P(n,r)$ are indexed by $\lambda \in \Lambda^+(n,r)$. Each simple module $L(\lambda)$ has a projective cover $P(\lambda)$; that is, $P(\lambda)$ is projective, maps onto $L(\lambda)$, and any submodule of $P(\lambda)$ which maps onto $L(\lambda)$ must be all of $P(\lambda)$. There is also a representation called the Weyl module $\Delta(\lambda)$ for $\lambda \in \Lambda^+(n,r)$.

(2) The Weyl module $\Delta(\lambda)$ maps onto the simple module $L(\lambda)$, with kernel an extension of the simple modules $L(\alpha)$ with $\alpha < \lambda$ in the dominance ordering.

(3) The projective indecomposable module $P(\alpha)$ maps onto the Weyl module $\Delta(\alpha)$, with kernel an extension of the Weyl modules $\Delta(\lambda)$ with $\lambda > \alpha$.

In fact, the multiplicities of the pieces in (2) are exactly the same as the multiplicities in (3), in the sense that

$$(P(\alpha) : \Delta(\lambda)) = (\Delta(\lambda) : L(\alpha))$$

([16], p. 118). We don't need this fact, however.

Examples: The Weyl module $\Delta(r)$ is the module $D^r V$, which is also the projective indecomposable module $P(r)$, as follows from (3). The Weyl module $\Delta(1, \ldots, 1)$ is $\Lambda^r V$, which is also the simple module $L(1, \ldots, 1)$, as follows from (2).

Given (1)–(3), the finiteness of the global dimension of $P(n,r)$ is proved as follows. Statement (3) implies that for a maximal element α in the finite poset $\Lambda^+(n,r)$, the Weyl module $\Delta(\alpha)$ is projective. For α not maximal, (3) implies that the projective dimension of the Weyl module $\Delta(\alpha)$ is at most one more than the maximum of the projective dimensions of $\Delta(\lambda)$ for $\lambda > \alpha$. So every Weyl module $\Delta(\alpha)$ has projective dimension at most equal to the length k of the maximal chain $\alpha < \lambda_1 < \cdots < \lambda_k$ starting at α in $\Lambda^+(n,r)$, and in particular every Weyl module has projective dimension at most the length of the maximal chain in $\Lambda^+(n,r)$.

Applying (2) the same way then shows that every simple module in $P(n,r)$ has projective dimension at most twice the length of a maximal chain in $\Lambda^+(n,r)$. Since every module in $P(n,r)$ is an extension of simple modules, this means that we have the same upper bound for the projective dimension of every module in $P(n,r)$. That is, we have proved Donkin's upper bound for the global dimension of $P(n,r)$:

Theorem 1 *Let n and r be positive integers. Over any field, the category $P(n, r)$ has global dimension at most twice the length of a maximal chain in the poset $\Lambda^+(n, r)$.*

For example, when $n \geq r$, the poset $\Lambda^+(n, r)$ is the set $\Lambda^+(r)$ of all partitions of r. The length of a maximal chain in $\Lambda^+(r)$ has been computed exactly; it is asymptotic to $(2r)^{3/2}/3$ as $r \to \infty$ [14]. That gives an upper bound for the global dimension of $P(n, r)$, but the specific upper bound here is not important for our argument; in the following sections we will give a precise calculation of the global dimension of $P(n, r)$ for $n \geq r$, and a much better upper bound for the global dimension when $n < r$. The only information we need from this section is that the global dimension is finite.

As an aside, we mention that the above upper bound for the global dimension could be improved by combining the argument above with Donkin's description of the blocks of the Schur algebra [11]. That is, the global dimension is bounded by twice the length of the maximal chain *in a block*, viewed as a subset of $\Lambda^+(n, r)$. But in a sense the resulting improvement is small: for a fixed prime p, and, say, $n \geq r$, the length of the longest chain in a block in $\Lambda^+(n, r)$ in characteristic p is still at least a constant times $r^{3/2}$ as r goes to infinity. The elementary methods of the following section will give a much better result.

2 The main theorem

Let k be a field of characteristic $p > 0$, and consider the category $P(n, r)$ of finite-dimensional polynomial representations of $GL(V)$ over k of degree r, where V is a vector space of dimension n. (We only need to consider fields of positive characteristic, since $P(n, r)$ has global dimension 0 over fields of characteristic 0.) We write $\alpha_p(r)$ for the sum of the digits in the base-p expansion of r.

Theorem 2 *For $n \geq r$, the global dimension of the category $P(n, r)$ is $2(r - \alpha_p(r))$.*

We begin by working over any field k. (Of course, for k of characteristic 0, representations of $GL(V)$ are completely reducible, so the category $P(n, r)$ has global dimension 0. So the interesting case is where k has characteristic $p > 0$.) At first, n and r can be any positive integers.

We will use the natural "contravariant duality" $M \mapsto M^0$ on the category $P(n, r)$ ([16], p. 79). Here, for any representation M of $GL(V)$, viewed as a functor $W \mapsto M(W)$ from n-dimensional vector spaces to vector spaces, M^0 denotes the functor $W \mapsto M(W^*)^*$. This is an equivalence of categories from $P(n, r)$ to itself, reversing arrows. In particular, it takes projective modules to injective ones. The dual of $\Lambda^r V$ in this sense is $\Lambda^r(V^*)^* \cong \Lambda^r V$, the dual of $V^{\otimes r}$ is $V^{\otimes r}$, and the dual of the symmetric power $S^r V$ is the divided power $D^r V$.

Since every module in $P(n, r)$ has a finite projective resolution (this being all we need from the previous section), contravariant duality implies that every module also has a finite injective resolution. Now let M and N be any modules in $P(n, r)$ such that $\mathrm{Ext}^i(M, N) \neq 0$ for some i. (Throughout this section, Ext will mean Ext in the category $P(n, r)$ of degree-r polynomial representations of $GL(V)$.) Let

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_j \to 0$$

be an injective resolution of M; then the long exact sequence for $\mathrm{Ext}^*(\cdot, N)$ implies that $\mathrm{Ext}^{i'}(I, N) \neq 0$ for some injective I and some $i' \geq i$. Likewise, if we then take a projective resolution of N,

$$0 \to P_l \to \cdots \to P_0 \to N \to 0,$$

we find that $\mathrm{Ext}^{i''}(I, P) \neq 0$ for some injective I, projective P, and some $i'' \geq i$.

Thus, to compute the global dimension of the category $P(n, r)$, viewed as the maximum i such that $\mathrm{Ext}^i(M, N) \neq 0$ for some M, $N \in P(n, r)$, it suffices to compute the maximal i such that $\mathrm{Ext}^i(I, P) \neq 0$ for some injective I and projective P. (A more explicit proof of this is to use mapping cones to construct a projective resolution of length at most d of any module M, starting from a finite right resolution of M into modules of projective dimension at most d. See [3], p. 28.)

Moreover, we have good information on the projectives and injectives in $P(n, r)$. Let $D^r(V) = S^r(V^*)^*$ denote the rth divided power module. Then the modules

$$D^\lambda V := D^{\lambda_1} V \otimes \cdots \otimes D^{\lambda_j} V$$

are projective for $\lambda \in \Lambda^+(n, r)$, and every indecomposable projective in $P(n, r)$ is a direct summand of one of these. (This follows by identifying $P(n, r)$ with the category of modules over the Schur algebra $S(n, r) := D^r(\mathrm{End}(V))$, since one can write $S(n, r)$, viewed as a left $S(n, r)$-module and thus as an object in $P(n, r)$, as a direct sum of the modules $D^\lambda V$ for $\lambda \in \Lambda^+(n, r)$ [2].) By contravariant duality, the modules

$$S^\lambda V := S^{\lambda_1} V \otimes \cdots \otimes S^{\lambda_j} V$$

are injective for $\lambda \in \Lambda^+(n, r)$, and every indecomposable injective is a direct summand of one of them.

Thus the global dimension of $P(n, r)$ is the maximum i such that

$$\mathrm{Ext}^i(S^\lambda V, D^\mu V) \neq 0$$

for some $\lambda, \mu \in \Lambda^+(n, r)$.

So we want to find projective resolutions of the injective modules $S^\lambda V$. This is most naturally done in two steps: first we will resolve the modules $S^\lambda V$ into tensor products of exterior powers of V, and then resolve those into tensor products of divided powers of V. We will assume $n \geq r$ at this point, which implies that all tensor products of divided powers of V with total degree r are projective. (For $n < r$, we only claimed, above, that tensor products of at most n modules $D^i V$ are projective, and indeed, tensor products of more than n $D^i V$'s need not be projective. For example, in characteristic p, $V^{\otimes p}$ is not projective for $1 < n < p$. Akin and Buchsbaum proved this for $n = 2$, $p = 3$ ([2], pp. 183-184).)

The resolution of $\Lambda^r V$ into divided powers which we will use was found by Akin [1]. This is obtained simply by writing down the normalized bar resolution [15] which computes $\mathrm{Ext}_{S(V^*)}(k, k) = \Lambda(V)$, where we write $S(V)$ for the polynomial algebra on V and $\Lambda(V)$ for the exterior algebra on V. The normalized bar resolution for the algebra $S(V^*)$ is a complex of graded vector spaces:

$$\cdots \to (S(V^*)^*)^{\otimes 2} \to S(V^*)^* \to k \to 0.$$

And the cohomology of this complex is $\Lambda(V)$, with $\Lambda^i V$ placed in degree i with respect to the grading and in the ith group from the right. So, writing $D^i V$ in place of $S^i(V^*)^*$, we have exact sequences, for all $r \geq 1$:

$$0 \to D^r V \to \oplus_{i_1+i_2=r} D^{i_1} V \otimes D^{i_2} V \to \cdots \to \oplus_{i_1+\cdots+i_{r-1}=r} D^{i_1} V \otimes \cdots \otimes D^{i_{r-1}} V \to V^{\otimes r} \to \Lambda^r V \to 0.$$

Here the numbers $i_1, i_2, \ldots$ which occur in the direct sums are positive integers. By the naturality of the construction, this is an exact sequence of polynomial representations of $GL(V)$. For $n \geq r$, it is a projective resolution of $\Lambda^r V$ of length $r-1$.

We can resolve $S^r V$ into a tensor product of exterior powers by the same method. Here we use the normalized bar resolution which computes $\mathrm{Ext}_{\Lambda(V^*)}(k,k) = S(V)$ to produce exact sequences, for all $r \geq 1$:

$$0 \to \Lambda^r V \to \oplus_{i_1+i_2=r} \Lambda^{i_1} V \otimes \Lambda^{i_2} V \to \cdots \to \oplus_{i_1+\cdots+i_{r-1}=r} \Lambda^{i_1} V \otimes \cdots \otimes \Lambda^{i_{r-1}} V \to V^{\otimes r} \to S^r V \to 0.$$

Again, the indices $i_1, i_2, \ldots$ are positive integers.

We can combine the resolution of $S^r V$ with that of $\Lambda^r V$ to get a projective resolution of $S^r V$ of length $2(r-1)$, assuming $n \geq r$. A first step is to tensor together copies of the resolution of $\Lambda^i V$ for different values of i: this gives a projective resolution of $\Lambda^{i_1} V \otimes \cdots \otimes \Lambda^{i_m} V$ of length $r-m$ for $i_1 + \cdots + i_m = r$, $i_j \geq 1$. Thus we have projective resolutions of all the modules in the resolution of $S^r V$, and by choosing maps between them we get a projective resolution of $S^r V$ of length $2(r-1)$, of which we just write out a few terms here:

$$0 \to D^r V \to \oplus_{i_1+i_2=r} D^{i_1} V \otimes D^{i_2} V \to \cdots \to V^{\otimes r} \to S^r V \to 0.$$

The important thing for what follows is that the first map here is just the first map in the resolution of $\Lambda^r V$, which we know explicitly (it does not depend on any choices). This is because our projective resolution of $\Lambda^r V$ has length $r-1$, while all the other modules in the Λ-resolution of $S^r V$, such as $\oplus \Lambda^{i_1} V \otimes \Lambda^{i_2} V$, have projective resolutions of length strictly less than $r-1$ (e.g., in the case of $\Lambda^{i_1} V \otimes \Lambda^{i_2} V$, the length is $(i_1 - 1) + (i_2 - 1) = r-2$).

Now, for any positive integer r, $S^r V$ is a direct summand of the module

$$V^{\otimes a_0} \otimes (S^p V)^{\otimes a_1} \otimes (S^{p^2} V)^{\otimes a_2} \otimes \cdots,$$

where $r = a_0 + a_1 p + a_2 p^2 + \cdots$ is the base-p expansion of r. (For any $i_1 + i_2 + \cdots = r$, the composition of the natural map $S^r(V) \to S^{i_1} V \otimes S^{i_2} V \otimes \cdots$ with the product surjection in the other direction is multiplication by $r!/(i_1! i_2! \cdots)$ on $S^r V$, and if we take the i_j's here to be powers of p as in the base-p expansion of r, we have $r!/(i_1! i_2! \cdots) \neq 0 \pmod{p}$.)

So any tensor product of modules $S^i V$ is a direct summand of a tensor product of modules $S^{p^a} V$, $a \geq 0$. For $r \leq n$, we showed above that the global dimension of the category $P(n,r)$ is the maximum projective dimension of the tensor products of modules $S^i V$ of total degree r; now we see that it suffices to consider tensor products of the modules $S^{p^a} V$. Now our resolution above (just applied to $S^{p^a} V$) shows that $S^{p^a} V$ has projective dimension at most $2(p^a - 1)$; so a tensor product $S^{p^{a_1}} V \otimes \cdots S^{p^{a_l}} V$ of total degree r has projective dimension at most $2(r-l)$, l being the number of factors. Now any expression of r as a sum of powers of p will involve at least $\alpha_p(r)$ terms, where $\alpha_p(r)$ is the sum of the digits in the base-p expansion of r; so we have shown that the global dimension of $P(n,r)$ is at most $2(r-\alpha_p(r))$ for $r \leq n$.

We now prove that the global dimension is exactly $2(r-\alpha_p(r))$ for $r \leq n$. It suffices to find a single module with projective dimension $2(r-\alpha_p(r))$; following the above ideas, we will prove that the module $V^{\otimes a_0} \otimes S^pV^{\otimes a_1} \otimes \cdots$ has the desired projective dimension, where $r = a_0 + a_1 p + \cdots$ is the base-p expansion of r.

Indeed, we have an explicit projective resolution of this module, by tensoring together the resolution of each S^iV, and this resolution has length $2(r-\alpha_p(r))$. To prove that this resolution has minimal length, it suffices to show that the first map in the resolution (clearly an injection) is not split. In fact, if this is so, then repeated application of the long exact sequence for Ext shows that the resolution represents a nonzero element of the group

$$\mathrm{Ext}^{2(r-\alpha_p(r))}(V^{\otimes a_0} \otimes (S^pV)^{\otimes a_1} \otimes \cdots, V^{\otimes a_0} \otimes (D^pV)^{\otimes a_1} \otimes \cdots).$$

The first map in the resolution of $S^{p^a}V$ is

$$0 \to D^{p^a}V \to \oplus_{i=1}^{p^a-1} D^iV \otimes D^{p^a-i}V.$$

So the first map in the resolution of the tensor product module $V^{\otimes a_0} \otimes (S^pV)^{\otimes a_1} \otimes \cdots$ has the form

$$0 \to \otimes_j D^{p^{i_j}}V \to \oplus_j (\otimes_{k\neq j} D^{p^{i_k}}V) \otimes (\otimes_{i=1}^{p^{i_j}-1} D^iV \otimes D^{p^{i_j}-i}V),$$

where the sequence $i_1, i_2, \ldots$ consists of a_0 0's, a_1 1's, and so on. We need to show that this first inclusion is not split.

This is elementary, since one knows explicitly the space of maps between any two of the projective modules $D^\lambda V$, $\lambda \in \Lambda^+(n,r)$. (For this, we don't need to assume $r \leq n$.) Let $x_1, x_2, \ldots, x_n$ be a basis for V, and for any representation M of $GL(n)$ and any $\lambda \in \mathbf{Z}^n$, let M_λ denote the $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$-weight space of M. By Akin and Buchsbaum [2], for any module M in $P(n,r)$ and $\lambda \in \Lambda^+(n,r)$, the map

$$\mathrm{Hom}(D^\lambda V, M) \to M_\lambda$$

given by evaluation on the element $x_1^{\otimes \lambda_1} \otimes x_2^{\otimes \lambda_2} \otimes \cdots \in D^\lambda V$ is a bijection. (Here, and below, Hom means Hom in the category $P(n,r)$ of degree-r polynomial representations of $GL(V)$. Since $P(n,r)$ is a full subcategory of the category of all representations of $GL(V)$ as an algebraic group, we can also think of Hom, below, as the space of $GL(V)$-linear maps.)

It follows that a basis for $\mathrm{Hom}(D^\lambda V, D^\mu V)$, for $\lambda, \mu \in \Lambda^+(n,r)$ (actually μ could be any partition of r), is given (over any field) by the maps of the following form.

$$\begin{aligned} D^{\lambda_1}V \otimes D^{\lambda_2}V \otimes \cdots &\hookrightarrow (D^{i_{11}}V \otimes D^{i_{12}}V \otimes \cdots) \otimes (D^{i_{21}}V \otimes D^{i_{22}}V \otimes \cdots) \otimes \cdots \\ &= (D^{i_{11}}V \otimes D^{i_{21}}V \otimes \cdots) \otimes (D^{i_{12}}V \otimes D^{i_{22}}V \otimes \cdots) \otimes \cdots \\ &\to D^{\mu_1}V \otimes D^{\mu_2}V \otimes \cdots, \end{aligned}$$

where i_{ab} is a natural number for all $a, b \geq 1$ such that $\lambda_a = \sum_b i_{ab}$ and $\mu_b = \sum_a i_{ab}$. Here the first map is the tensor product of the natural inclusions, and the last map is the tensor product of the natural product maps. In particular, although it is not directly relevant to our problem, it is interesting to observe that these maps are defined over $\mathbf{Z}$: the same maps give a basis for the space of maps $D^\lambda V \to D^\mu V$ over any field.

Now, returning to the case $r \leq n$, we can show that the inclusion at the beginning of our projective resolution of $V^{\otimes a_0} \otimes (S^p V)^{\otimes a_1} \otimes \cdots$, $0 \to P_N \to P_{N-1}$, is not split. It suffices to show that any map $P_{N-1} \to P_N$ of $GL(V)$-modules is not surjective. Since such a map is also a map of T-modules where T is a maximal torus in $GL(V)$, it respects the decomposition of P_{N-1} and P_N into weight spaces, and in particular maps the x_1^r-weight space in P_{N-1} to the 1-dimensional x_1^r-weight space of $P_N = \otimes_j D^{p^{i_j}} V$, where $x_1, \ldots, x_n$ is a basis for V. Clearly, to show that no map $P_{N-1} \to P_N$ is surjective, it suffices to show that every such map is 0 on the x_1^r-weight space of P_{N-1}.

Here

$$P_{N-1} = \oplus_j (\otimes_{k \neq j} D^{p^{i_k}} V) \otimes (\otimes_{i=1}^{p^{i_j}-1} D^i V \otimes D^{p^{i_j}-i} V).$$

In particular, P_{N-1} is a direct sum of tensor products of $\alpha_p(r)+1$ modules $D^i V$, while P_N is a tensor product of just $\alpha_p(r)$ modules $D^i V$. We described above a basis for the space of $GL(V)$-maps between such tensor products, as a composition of inclusions $D^b V \hookrightarrow D^{b_1} V \otimes \cdots$ and product maps $D^{b_1} V \otimes \cdots \to D^b V$; since each summand of P_{N-1} involves more factors $D^i V$ than P_N does, each of our maps from that summand to P_N must involve some nontrivial product map $D^{b_1} V \otimes \cdots \to D^b V$. But all the factors $D^b V$ that occur in P_N have $b = p^a$ for some a, and any nontrivial product map $D^{b_1} V \otimes \cdots \to D^{p^a} V$ is 0 on the $x_1^{p^a}$-weight subspace. (It takes the element $(x_1)^{\otimes b_1} \otimes \cdots$ to $(p^a)!/((b_1)! \cdots)(x_1)^{\otimes p^a}$, which is 0 (mod p).)

This means that any map $P_{N-1} \to P_N$ is 0 on the x_1^r-weight subspace. So the inclusion $0 \to P_N \to P_{N-1}$ at the beginning of our projective resolution of $V^{\otimes a_0} \otimes (S^p V)^{\otimes a_1} \otimes \cdots$ is not split, where $N = 2(r - \alpha_p(r))$, and we have proved that the category $P(n, r)$ has global dimension exactly $2(r - \alpha_p(r))$ for $r \leq n$. QED

3 Representations over Z

Over any commutative ring k, the category of degree-r polynomial representations of $GL(n)$ is equivalent to the category of modules over the Schur algebra $S(n, r)$ $(= S(n, r)_{\mathbf{Z}} \otimes_{\mathbf{Z}} k)$, which is a finite free k-module. (Explicitly, $S(n, r) = D^r(\text{End } V)$, for a free k-module V of rank n.) We have shown above that the Schur algebra $S(n, r)$ for $r \leq n$ has global dimension $2(r - \alpha_p(r))$ over a field of characteristic $p > 0$, and of course it has global dimension 0 over a field of characteristic 0 for all n and r. Thus the following result determines the global dimension of the Schur algebra $S(n, r)$ over $\mathbf{Z}$ for $r \leq n$, and (using the next section's results) gives an upper bound for the global dimension of $S(n, r)$ over $\mathbf{Z}$ for $r > n$. To see that the following lemma actually applies to the Schur algebra $S(n, r)$ over $\mathbf{Z}$ for all n and r, we must observe that $S(n, r) \otimes \mathbf{Z}/p$ has global dimension 0 for $p > r$, either by our Theorems 2 and 3, or, more simply, by identifying the Schur algebra with the endomorphism ring of $V^{\otimes r}$ as a representation of the symmetric group S_r ([13], p. 29).

Lemma 1 *Let R be a $\mathbf{Z}$-algebra which is a finite free $\mathbf{Z}$-module. Suppose that $R \otimes \mathbf{Q}$ is semisimple (that is, it has global dimension 0) and the algebras $R \otimes \mathbf{Z}/p$ have finite* *global dimension, 0 for p sufficiently large. Then the global dimension of R is exactly one more than the maximum over all prime numbers p of the global dimensions of $R \otimes \mathbf{Z}/p$.*

The proof is mostly in Akin-Buchsbaum ([2], pp. 195-196), although they only gave an upper bound for the global dimension of $S(n,r)$ over $\mathbf{Z}$, and the upper bound they give is the maximum of the global dimensions of $S(n,r) \otimes \mathbf{Z}/p$ plus two rather than one.

Proof. Let M, N be finitely generated R-modules (which is equivalent to being R-modules which are finitely generated as abelian groups). Tensoring a free R-resolution of M with $\mathbf{Q}$ gives a free $R \otimes \mathbf{Q}$-resolution of $M \otimes \mathbf{Q}$; then the universal coefficient theorem for complexes of abelian groups gives that

$$\mathrm{Ext}^i_R(M,N) \otimes \mathbf{Q} = \mathrm{Ext}^i_{R \otimes \mathbf{Q}}(M \otimes \mathbf{Q}, N \otimes \mathbf{Q}),$$

which is 0 for $i > 0$ since $R \otimes \mathbf{Q}$ is semisimple. That is, the groups $\mathrm{Ext}^i_R(M,N)$ are torsion for $i > 0$. Since M is finitely generated, M has a resolution (possibly infinite) by finite free R-modules, so the abelian groups $\mathrm{Ext}^i_R(M,N)$ are finitely generated. Being torsion, they are finite.

Suppose that M and N are R-modules which are finite free $\mathbf{Z}$-modules. Then tensoring a free R-resolution of M with $\mathbf{Z}/p$, for a prime number p, gives a free R/p-resolution of M/p, and the universal coefficient theorem for complexes of abelian groups gives an exact sequence

$$0 \to \mathrm{Ext}^i_R(M,N) \otimes \mathbf{Z}/p \to \mathrm{Ext}^i_{R/p}(M/p, N/p) \to Tor_1^{\mathbf{Z}}(\mathrm{Ext}^{i+1}_R(M,N), \mathbf{Z}/p) \to 0.$$

Let d denote the maximum of the global dimensions of the algebras R/p over all primes p (which is finite, since all these global dimensions are finite, and they are 0 for p sufficiently large). Then, for all primes p, the middle group in this exact sequence is 0 for $i > d$, and so $\mathrm{Ext}^i_R(M,N)/p$ is also 0 for $i > d$. Since $\mathrm{Ext}^i_R(M,N)$ is a finite abelian group, this means that $\mathrm{Ext}^i_R(M,N) = 0$ for $i > d$.

Now let M and N be finitely generated R-modules which are not assumed to be $\mathbf{Z}$-free. Since the algebra R is a finite free $\mathbf{Z}$-module, every finitely generated R-module M fits into an exact sequence

$$0 \to M_1 \to M_0 \to M \to 0,$$

where M_1 and M_0 are finitely generated $\mathbf{Z}$-free R-modules. Since $\mathrm{Ext}^i_R(M,N) = 0$ for M and N $\mathbf{Z}$-free when $i > d$, the long exact sequence for Ext shows first that $\mathrm{Ext}^i_R(M,N) = 0$ for M $\mathbf{Z}$-free and finitely generated, N finitely generated, and $i > d$, and then that $\mathrm{Ext}^i_R(M,N) = 0$ for M, N finitely generated and $i > d+1$. It follows that every finitely generated R-module has projective dimension at most $d+1$. The global dimension of any ring (the maximum projective dimension of its modules) is also the maximum projective dimension of its finitely generated modules ([15], p. 203); so R has global dimension at most $d+1$.

Finally, we prove that R has global dimension exactly $d+1$. Let p be a prime number such that R/p has global dimension d (it exists by the definition of d). Let M be an R/p-module of projective dimension d; then we can view M as an R-module, and we will show that it has projective dimension $d+1$ as an R-module. This is proved directly in [17], p. 100, but we prefer to deduce it from a change-of-rings spectral sequence which might be useful in computations.

Lemma 2 *Let K be a commutative ring, R a K-algebra which is a free K-module, S any K-algebra. Then, for any $R \otimes_K S$-module M and any R-module N, there is a spectral sequence*

$$E_2^{ij} = Ext^i_{R\otimes S}(M, Ext^j_K(S, N)) \Rightarrow Ext^{i+j}_R(M, N).$$

Proof. In Cartan and Eilenberg [6], p. 345, (2) and (3), take $\Gamma = R$, $\Sigma = K$, $\Lambda = S$, $A = M$, $B = S$, $C = N$. QED

We apply this spectral sequence to our situation, with $K = \mathbf{Z}$ and $S = \mathbf{Z}/p$. Since $\mathbf{Z}$ has global dimension 1, the spectral sequence just becomes a long exact sequence, for any R/p-module M and R-module N:

$$\mathrm{Ext}^i_{R/p}(M, N_p) \to \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^{i-1}_{R/p}(M, N/p) \to \mathrm{Ext}^{i+1}_{R/p}(M, N_p).$$

Here we have identified $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/p, N)$ with the p-torsion subgroup N_p of N and $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/p, N)$ with N/p.

In particular, we can apply this sequence when N as well as M is an R/p-module, so that $N_p = N/p = N$. Since we have chosen a prime number p such that R/p has the maximum global dimension, d, there are R/p-modules M and N with $\mathrm{Ext}^d_{R/p}(M, N) \neq 0$, and the above exact sequence shows that $\mathrm{Ext}^{d+1}_R(M, N) \neq 0$. Thus R has global dimension at least $d + 1$, which completes the proof of Lemma 1. QED

4 Representations of $GL(V)$ of degree greater than dim V

Here we consider the problem of computing the global dimension of the category of degree r polynomial representations of $GL(V)$ for $r > n = \dim V$, over a field k of characteristic $p > 0$. We just give an upper bound in this case.

Theorem 3 *For $r > n$, the category $P(n, r)$ has global dimension at most $2(r - \sum_j \lceil p^{i_j}/n \rceil)$, where $r = p^{i_1} + p^{i_2} + \cdots$, $i_1 \le i_2 \le \cdots$, is the base-p expansion of r. In particular, this is at most $2(r - \alpha_p(r))$ and so at most $2(r-1)$.*

Proof. We still know that the global dimension is finite, as Akin-Buchsbaum and Donkin proved (see section 1). The argument of section 2 then implies that the global dimension of $P(n, r)$ is the maximum of the projective dimension of the injective modules $S^\lambda V$ for $\lambda \in \Lambda^+(n, r)$.

There is a resolution of $S^\lambda V$ into tensor products of divided power modules, of exactly the same form as we found in section 1 for $n \ge r$; this can be deduced from the resolution for $n \ge r$ by applying Green's "deflating Schur functor" $P(N, r) \to P(n, r)$ for $N \ge n$ [13], also explained in our usual reference [16]. We don't describe this in detail, because it seems not to be useful for our purpose. The problem is that the resolution is no longer a projective resolution in general, because for $r > n$ it can involve tensor products of more than n divided power modules $D^i V$, that is, modules $D^\lambda V$ for partitions λ which are not in $\Lambda^+(n, r)$. Such modules are in general not projective, as we mentioned in section 2.

Fortunately, the resolution of $S^\lambda V$ into tensor products of exterior powers is still useful. (We defined this directly for V of any dimension in section 2.) Some of the modules

$$\Lambda^\lambda V = \Lambda^{\lambda_1} V \otimes \Lambda^{\lambda_2} V \otimes \cdots$$

that occur in the resolution will be 0 for $r > n$, but that only makes the resolution simpler. Precisely, $\Lambda^\lambda V$ is nonzero if and only if the partition λ is the dual (switching rows and columns) of a partition in $\Lambda^+(n,r)$. By keeping track of which modules become 0, we see that our Λ-resolution of $S^{p^a}V$ has length $p^a - \lceil p^a/n \rceil$. By viewing each module $S^\lambda V$, $\lambda \in \Lambda^+(n,r)$, as a direct summand of tensor products of modules $S^{p^a}V$, we see that each module $S^\lambda V$ is a direct summand of a module with Λ-dimension at most $d := r - \sum_j \lceil p^{i_j}/n \rceil$, where $r = p^{i_1} + p^{i_2} + \cdots$, $i_1 \leq i_2 \leq \cdots$, is the base-p expansion of r.

Now, by Akin-Buchsbaum ([2], p. 193), we have $\mathrm{Ext}^i(\Lambda^\lambda V, \Lambda^\mu V) = 0$ for all λ, $\mu \in \Lambda^+(n,r)\tilde{}$ and all $i > 0$. Here Ext refers to Ext in the category $P(n,r)$ of polynomial representations of $GL(V)$, and $\tilde{}$ stands for the dual partition (so that $\Lambda^\lambda V$ is nonzero if and only if $\lambda \in \Lambda^+(n,r)\tilde{}$). (Another proof that $\mathrm{Ext}^i(\Lambda^\lambda V, \Lambda^\mu V) = 0$ is that the modules $\Lambda^\lambda V$ have both Weyl and Schur filtrations, and $\mathrm{Ext}^i(M,N) = 0$ for $i > 0$ when M is a Weyl module and N is a Schur module, by Cline-Parshall-Scott-van der Kallen ([8], [7]). Roughly speaking, Weyl modules are a little like projective modules and Schur modules, defined to be the contravariant duals of Weyl modules, are a little like injective modules. The modules $\Lambda^\lambda V$ are right in the middle.)

Applying this to our Λ-resolution of the modules $S^\lambda V$ shows that $\mathrm{Ext}^i(S^\lambda V, \Lambda^\mu V) = 0$ for all $\lambda \in \Lambda^+(n,r)$, $\mu \in \Lambda^+(n,r)\tilde{}$ and all $i > d$. Moreover, by contravariant duality, it follows that $\mathrm{Ext}^i(\Lambda^\mu V, D^\lambda V) = 0$ for all $\mu \in \Lambda^+(n,r)\tilde{}$, $\lambda \in \Lambda^+(n,r)$ and all $i > d$. But this means that every $\Lambda^\mu V$ has projective dimension at most d. Indeed, since every indecomposable projective module is a direct summand of some $D^\lambda V$ for $\lambda \in \Lambda^+(n,r)$, as we said in section 2, we know that $\mathrm{Ext}^i(\Lambda^\mu V, P) = 0$ for all $i > d$ and all projective modules P; and since the category $P(n,r)$ has finite global dimension, the long exact sequence for Ext applied to a finite projective resolution of an arbitrary module M implies that $\mathrm{Ext}^i(\Lambda^\mu V, M) = 0$ for $i > d$. That is, $\Lambda^\mu V$ has projective dimension at most d as we claimed.

Combining this with our Λ-resolutions of the modules $S^\lambda V$, we have proved that the modules $S^\lambda V$ for $\lambda \in \Lambda^+(n,r)$ have projective dimension at most $2d$. By our basic argument, this means that the category $P(n,r)$ has global dimension at most $2d$. QED

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