Fibre Bundles

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Lecture 1

Goals of algebraic topology: Try to understand all "shapes", for example all manifolds. The approach of algebraic topology is:

$$\text{(topological spaces)} \to \text{(Abelian groups)} \to \text{(numbers)}$$

This should make it easier to distinguish two topological spaces.

Examples (supposed to be understood).

1. Fundamental group. $x \in X$, $\pi_1(X, x)$ = loops modulo homotopy, in general not Abelian.

2. Homology groups. $H_i(X)$ are a sequence of Abelian groups associated to a space. Roughly, $H_i(X)$ "measures" how many i-dimensional submanifolds are in X.

The homology groups $H_i(X)$ are easy to compute. I shall define a generalisation of the fundamental group, the ***homotopy groups***. They are easy to define but hard to compute. If you can compute these groups, you know a lot about the spaces.

Fibre bundles are a way to break up a space into simpler spaces. Fibre bundles are "twisted products": A map $\pi\colon E \to B$ between spaces is a ***fibre bundle*** with fibre F if, for every point $b \in B$ (the "base space") there is an open set $U \ni b$ such that $\pi^{-1}(U)$ is homeomorphic to $U \times F$. More precisely, we require that the homeomorphism $\pi^{-1}(U) \to U \times F$ give a commuting diagram:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\cong"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\mathrm{proj}_1"] \\ & U \end{tikzcd}$$

In particular, if $\pi\colon E \to B$ is a fibre bundle, then every "fibre" $\pi^{-1}(b)$ is homeomorphic to F.

Example. A fibre bundle whose fibre F is a discrete space is exactly a ***covering space***. In that case,

$$U \times F = \coprod \text{copies of } U\,.$$

For example, $\mathbb{R} \to \mathbb{S}^1$, $\pi(t) = e^{2\pi i t}$ is a covering map with fibre $\mathbb{Z}$. "It is not trivial."

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Definition. A ***trivial*** fibre bundle $\pi\colon E \to B$ is one such that $E \cong B \times F$, with $\pi\colon E \to B$ being the projection map.

Example. E =The Möbius strip is the total space of a fibre bundle, $\pi\colon E \to \mathbb{S}^1$ with fibre $[0,1]$. Again, this is ***not*** a trivial fibre bundle.

Homotopy groups

(For the relation to fibre bundles see lectures 8–9.)

Let $I = [0,1]$ be the closed unit interval. For $n \geq 0$, let I^n be the product space $\underbrace{I \times \cdots \times I}_{n}$, the n-cube.

$$I^n = \{(x_1,\dots,x_n)\colon 0 \leq x_i \leq 1 \quad \forall i\} \subset \mathbb{R}^n$$
$$\partial I^n = \{(x_1,\dots,x_n) \in I^n\colon \text{at least one } x_i \text{ is 0 or 1}\}.$$

Definition. Given any topological space X and a point $x_0 \in X$, the n^{th} ***homotopy group of*** X is

$$\begin{aligned}\pi_n(X,x_0) &= [(I^n,\partial I^n),(X,x_0)]\\ &:= \text{set of homotopy classes of map } (I^n,\partial I^n) \to (X,x_0)\end{aligned}$$

Recall. A ***pair*** (X,A) means a space X and a subspace $A \subseteq X$.

A ***map of pairs*** $f\colon (X,A) \to (Y,B)$ is a (continuous) map $f\colon X \to Y$ such that $f(A) \subseteq B$.

Two maps $f_0, f_1\colon (X,A) \to (Y,B)$ are ***homotopic*** if there is a continuous map $F\colon X \times I \to Y$ such that

$$\begin{aligned}&F(x,0) = f_0(x)\ ,\\ &F(x,1) = f_1(x)\ ,\\ &F(x,t) \in B \quad \forall x \in A,\ t \in I\ .\end{aligned}$$

Homotopy is an equivalence relation, so it makes sense to consider $[(X,A),(Y,B)]$, the set of homotopy classes of maps $(X,A) \to (Y,B)$.

Example. Let $n = 0$; then $I^0 = \{pt\}$, $\partial I^0 = \varnothing$ (I insist). Then

$$\pi_0(X,x) = [(pt,\varnothing),(X,x)]$$

is the set of "path components" of X. So $|\pi_0(X,x)| = 1$ iff X is path-connected.

Example. $\pi_1(X,x)$ is the fundamental group. $\pi_n(X,x)$ for $n \geq 2$ are called the "higher" homotopy groups of X.

Lecture 2

For any space X, $\pi_0(X,x_0)$ is just a set, or more precisely, a "pointed set". It is the set of all path-connected components in X, with the component of x_0 picked out.

But $\pi_n(X,x_0)$ is a group for any $n \geq 1$: For $f,g\colon I^n \to X$, define

$$\boxed{\quad f \quad} \cdot \boxed{\quad g \quad} := \begin{array}{|c|c|}\hline f & g \\ \hline\end{array}$$

That is,

$$(fg)(x_1,\ldots,x_n) = \begin{cases} f(2x_1, x_2, \ldots, x_n) & x_1 \leq \frac{1}{2} \\ g(2x_1 - 1, x_2, \ldots, x_n) & x_1 \geq \frac{1}{2} \end{cases}, \quad x_1,\ldots,x_n \in I$$

As in the case $n = 1$, this defines a group structure on $\pi_n(X,x_0)$, for $n \geq 1$.

First note: fg, as above, is a continuous map $I^n \to X$ with $fg(\partial I^n) = x_0$, because f and g have those properties. (This uses the "gluing lemma": Given topological spaces X and Y with $X = A \cup B$ closed subsets, a function $f\colon X \to Y$ is continuous iff $f|A$ and $f|B$ are continuous.)

If you replace f or g by a map homotopic to it, then you change fg by a homotopy. Therefore this "product" gives a well-defined function $\pi_n \times \pi_n \to \pi_n$.

To check that this operation makes $\pi_n(X,x_0)$ into a group for $n \geq 1$, you have to check associativity, identity and inverses. For example, associativity is proved by writing down a homotopy

$$(fg)h = \begin{array}{|c|c|c|}\hline f & g & \quad h \quad \\ \hline\end{array} \sim \begin{array}{|c|c|c|}\hline \quad f \quad & g & h \\ \hline\end{array} = f(gh)$$

The identity element of $\pi_n(X,x_0)$ is the map $x_0\colon I^n \to X$ with $x_0(x_1,\ldots,x_n) = x_0$ for all $x_1,\ldots,x_n$. You check

$$\begin{array}{|c|c|}\hline x_0 & f \\ \hline\end{array} \sim \boxed{\quad f \quad}$$

he inverse of $f\colon I^n \to X$ is

$$(-f)(x_1,\ldots,x_n) = f(1 - x_1, x_2, \ldots, x_n)$$

$$\begin{array}{|c|c|}\hline f & -f \\ \hline\end{array} \sim \boxed{\quad x_0 \quad}$$

Thus $\pi_n(X,x_0)$ are groups for all $n \geq 1$.

First difference from π_1:

For ***any*** group G, there is a space X with $\pi_1 \cong G$, but $\pi_n(X,x_0)$ is Abelian for all $n \geq 2$.

We want to show that $fg = gf$ in π_n for $n \geq 2$. That is, given $f, g\colon (I^n, \partial I^n) \to (X, x_0)$, you need to show $fg \sim gf$.

$$\begin{array}{|c|c|}\hline f & g \\ \hline \end{array} \sim \begin{array}{|c|c|}\hline f & x_0 \\ \hline x_0 & g \\ \hline \end{array} \sim \begin{array}{|c|c|}\hline x_0 & f \\ \hline g & x_0 \\ \hline \end{array} \sim \begin{array}{|c|c|}\hline g & f \\ \hline \end{array}$$

Another Interpretation of homotopy groups

Lemma. *For any topological space X and nonempty subspace $A \subset X$, let X/A be the "identification space" (="quotient space") defined by identifying A to a point. Then there is a one-to-one correspondence between continuous maps $X/A \to Y$ and continuous maps $X \to Y$ with A mapping to a point.*

Proof. Easy □

So map of pairs $(I^n, \partial I^n) \to (X, x_0)$ are in one-to-one correspondence with maps $I^n/\partial I^n \to X$ with the base point of $I^n/\partial I^n$ mapping to x_0.

Recall.

$$\mathbb{S}^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \colon x_0^2 + \cdots + x_n^2 = 1\}$$
$$\mathbb{D}^{n+1} := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \colon x_0^2 + \cdots + x_n^2 \leq 1\}$$

So $\pi_n(X, x_0)$ is the set of based homotopy classes of maps $\mathbb{S}^n \to X$.

Remark. A map $f\colon (\mathbb{S}^n, p_0) \to (X, x_0)$ is zero in $\pi_n(X, x_0)$ iff it extends to a map $\mathbb{D}^{n+1} \to X$.

Suppose, for example, that $f\colon \mathbb{S}^n \to X$ extends to a map $F\colon \mathbb{D}^{n+1} \to X$. Then shrinking the sphere fixing the base point gives the required homotopy.

Given any map $f\colon (X, x_0) \to (Y, y_0)$ of pointed spaces, we get a function

$$f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0).$$

If $n \geq 1$, this is a group homomorphism.

f_* is defined in an obvious way: given a map $\alpha\colon (I^n, \partial I^n) \to (X, x_0)$, we define

$$f_*\alpha = f \circ \alpha\colon I^n \xrightarrow{\alpha} X \xrightarrow{f} Y.$$

You check that, if $\alpha_1 \sim \alpha_2$ then $f \circ \alpha_1 \sim f \circ \alpha_2$; so we get a function $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$. The fact that this is a group homomorphism for $n \geq 1$ is easy:

$$f_*\left(\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \end{array}\right) = \begin{array}{|c|c|}\hline f_*\alpha & f_*\beta \\ \hline \end{array}$$

Easy to check that f_* makes π_n into a ***functor*** from pointed spaces to

$$\begin{cases} \text{pointed sets} & \text{if } n=0 \\ \text{groups} & \text{if } n=1 \\ \text{Abelian groups} & \text{if } n\geq 2 \end{cases}$$

This means that

$$\begin{cases} (fg)_* = f_* g_* & \text{if } f\colon (Y,y_0)\to(Z,z_0),\ g\colon (X,x_0)\to(Y,y_0) \\ (\mathrm{id})_* = \mathrm{id} & \text{if } \mathrm{id}\colon (X,x_0)\to(X,x_0) \text{ is the identity map.} \end{cases}$$

Lecture 3

Definition. Two based maps $f_0, f_1\colon (X,x_0)\to(Y,y_0)$ are ***homotopic*** if there is a homotopy $F\colon X\times I\to Y$ from f_0 to f_1 such that $F(x_0,t)=y_0$ for all $t\in[0,1]$.

This is an equivalence relation, so you can talk about $[(X,x_0),(Y,y_0)]$, the set of homotopy classes.

Lemma. *If f_0 and f_1 are homotopic maps of pointed spaces, then $(f_0)_*=(f_1)_*$ as homomorphism $\pi_n(X,x_0)\to\pi_n(Y,y_0)$ for all n.*

Proof. Obvious □

So homotopy groups form a functor

$$\begin{pmatrix}\text{Category of}\\ \text{pointed spaces}\end{pmatrix}\to\begin{pmatrix}\text{Homotopy category}\\ \text{of pointed spaces}\end{pmatrix}\xrightarrow[n\geq 2]{\pi_n}\begin{pmatrix}\text{Category of}\\ \text{Abelian groups}\end{pmatrix}$$

where the morphism in the homotopy category of pointed spaces between (X,x_0) and (Y,y_0) are the set of homotopy classes of continuous maps.

Note that two pointed spaces are isomorphic in the homotopy categories iff they are ***homotopy equivalent***:

Recall. If X,Y are objects in a category, we say that X and Y are isomorphic (write $X\cong Y$) if $\exists f,g$ such that

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \ \text{ with } fg=\mathrm{id}_Y \text{ and } gf=\mathrm{id}_X$$

So isomorphic in the homotopy category means there are continuous pointed maps $f\colon X\to Y$ and $g\colon Y\to X$ such that $fg\sim\mathrm{id}_Y$ and $gf\sim\mathrm{id}_X$.

Therefore two homotopy equivalent pointed spaces have isomorphic homotopy groups.

Exercise. $\pi_n(point)=0$ for $n>0$

Definition. A space is ***contractible*** if it it homotopy equivalent to a point. [Same notion for pointed spaces or spaces.]

Example. $\mathbb{R}^n$, $\mathbb{D}^n$, $\mathrm{int}\,\mathbb{D}^n$ are contractible.

Given any based map $\alpha\colon (\mathbb{S}^n, s_0) \to (X, x_0)$ and any path $p\colon [0,1] \to X$ with $p(0) = x_0$, we can define a map $p(\alpha)\colon (\mathbb{S}^n, s_0) \to (X, p(1))$.

(Draw nice pictures)

The construction $p(\alpha)$ gives, for any path $p\colon [0,1] \to X$, an isomorphism $p_\#\colon (X, p(0)) \to \pi_n(X, p(1))$. So the homotopy groups of a path-connected spaces are isomorphic at all basepoints (clearly false if X is not connected).

Lemma. *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space. Then $p_*\colon \pi_n(\tilde{X}) \to \pi_n(X)$ is an isomorphism if $n \geq 2$.*

Example. Use the covering map $\mathbb{R} \to \mathbb{S}^1$ mapping t to $e^{2\pi i t}$. By the lemma,

$$\pi_n(\mathbb{S}^1) \cong \pi_n(\mathbb{R}) = 0 \text{ for } n \geq 2$$

since $\mathbb{R}$ is contractible. But of course

$$\pi_1(\mathbb{S}^1) \cong \mathbb{Z} \text{ and } \pi_1(\mathbb{R}) \cong 0$$

This looks much like $H_*\mathbb{S}^1$:

$$H_0(\mathbb{S}^1;\mathbb{Z}) \cong \mathbb{Z}; \quad H_1(\mathbb{S}^1;\mathbb{Z}) \cong \mathbb{Z}; \quad H_n(\mathbb{S}^1;\mathbb{Z}) = 0 \text{ for } n \geq 2$$

Example. Look at the n-***torus*** $(\mathbb{S}^1)^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n}$.

Here, the universal covering space is $\mathbb{R}^n$. We have

$$\pi_i((\mathbb{S}^1)^n) = \begin{cases} \{*\} & i = 0 \\ \mathbb{Z}^n & i = 1 \\ 0 & i \geq 2 \end{cases}$$

Compare this with the homology groups:

$$H_i((\mathbb{S}^1)^n; \mathbb{Z}) = \mathbb{Z}^{\binom{n}{i}}$$

so it is nonzero for $i = 0, 1, \ldots, n$.

Lecture 4

Theorem. *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space. Then*

$$p_*\colon \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$$

is an isomorphism for $n \geq 2$.

Proof. I'll use the "homotopy lifting property" for covering spaces:

> Given any based map $f\colon Y \to \tilde{X}$, a homotopy $F\colon Y \times [0,1] \to X$ with $F(-,0) = pf$ "lifts" uniquely to $\tilde{X}$. That is, there is a unique map $G\colon Y \times [0,1] \to \tilde{X}$ such that $G(-,0) = f$ and $pG(y,t) = F(y,t)$ for all $y \in Y$, $t \in [0,1]$.

We use that to prove the theorem. Given an element of $\pi_n(X,x_0)$, let $\alpha\colon (I^n,\partial I^n)\to(X,x_0)$ be a corresponding map. Use this homotopy lifting property first for $Y = point$, then $Y = I,\ldots,I^n$. The homotopy lifting property implies that the given map $\alpha\colon I^n\to X$ lifts to a unique map $\tilde{\alpha}\colon I^n\to\tilde{X}$ with the property that $\tilde{\alpha}(0,\ldots,0)=\tilde{x_0}$.

I claim that if $n\geq 2$, then $\tilde{\alpha}$ maps ∂I^n into $\tilde{x_0}$.

Basic Fact. *Any path in $\tilde{X}$ whose projection to X is constant must be constant.*

So any path in ∂I^n maps using $\tilde{\alpha}$ to a path in $\tilde{X}$ must be constant, because its projection to X is constant.

Notice that if $n\geq 2$ then $\partial I^n\cong\mathbb{S}^{n-1}$ is path-connected. So $\tilde{\alpha}|\partial I$ is constant.

So $\tilde{\alpha}$ represents an element of $\pi_n(\tilde{X},\tilde{x_0})$ and it is clear that $p_*(\tilde{\alpha})=\alpha$. Thus I've shown that $p_*\colon\pi_n(\tilde{X})\to\pi_n(X)$ is onto.

Likewise, p_* is one-to-one, because if $\tilde{\alpha}\in\pi_n\tilde{X}$ maps to 0 in $\pi_n X$, you have a homotopy $p\tilde{\alpha}\sim x_0$ (constant map) and by same argument you can lift that to a homotopy $\tilde{\alpha}\sim\tilde{x_0}$. □

So, for example, $\pi_2((\mathbb{S}^1)^2)=0$, because $\pi_2((S^1)^2)=\pi_2(\mathbb{R}^2)=0$. This contrasts with $H_2((\mathbb{S}^1)^2;\mathbb{Z})\cong\mathbb{Z}$.

Roughly, $\pi_n X$ classifies maps $\mathbb{S}^n\to X$, while $H_n(X;\mathbb{Z})$ classifies maps

$$(\text{any closed oriented } n\text{-manifolds} M^n)\to X$$

Such a manifold comes with a fundamental class $[M^n]\in H_n(M;\mathbb{Z})$, and $f_*[M^n]\in H_n(X;\mathbb{Z})$.

We have shown that any map $\mathbb{S}^2\to(\mathbb{S}^1)^2$ is homotopic to a constant map.

Compare: There are maps $(\mathbb{S}^1)^2\to\mathbb{S}^2$ not homotopic to a constant map.

1. Use algebraic geometry: think of $(\mathbb{S}^1)^2$ as an elliptic curve $E=\{Y^2Z=X(X-Z)(X-2Z)\}\subset\mathbb{C}P^2$. Then projecting to $(X:Z)$ gives a map $(\mathbb{S}^1)^2\to\mathbb{C}P^1=\mathbb{S}^2$ which has degree 2.

   So the map $H_2((\mathbb{S}^1)^2;\mathbb{Z})\to H_2(\mathbb{S}^2,\mathbb{Z})$ is multiplication by 2, so this map is not "null homotopic".

2. Choose a small disk $D^2\subset(\mathbb{S}^1)^2$. Map $(\mathbb{S}^1)^2-D^2$ to a point $s_0\in\mathbb{S}^2$, and map $D^2/\partial D^2\cong\mathbb{S}^2$ with ∂D^2 mapping to s_0.

   This map has degree 1 (not 0), so this map is nontrivial.

Note that these maps $f\colon((\mathbb{S}^1)^2)\to\mathbb{S}^2$ induces the zero homomorphism on π_*. Indeed, the only nonzero $\pi_i((\mathbb{S}^1)^2)$ is $\pi_1((\mathbb{S}^1)^2)\cong\mathbb{Z}^2$ and the map $f_*\colon\pi_1((\mathbb{S}^1)^2)\to\pi_1(\mathbb{S}^2)$ is zero because $\pi_1(\mathbb{S}^2)=0$. But f is not homotopic to the constant map, as we see by looking at H_*.

Lemma. *For any space X and Y,*

$$\pi_i(X\times Y)\cong\pi_i(X)\times\pi_i(Y)$$

(If $i\geq 2$, another notation is $\pi_i(X)\oplus\pi_i(Y)$)

Proof. For any space Z, continuous maps $Z\to X\times Y$ are in one-to-one correspondence[1] with pairs of maps $(Z\to X, Z\to Y)$.

So maps $\mathbb{S}^i\to X\times Y$ corresponds to pairs of maps $(\mathbb{S}^i\to X,\mathbb{S}^i\to Y)$. Similarly, homotopies of maps correspond to maps $\mathbb{S}^i\times[0,1]\to X\times Y$. □

[1] by the universal property of products

Again, $H_*(X \times Y)$ behaves differently. For example, suppose $H_*(X,\mathbb{Z})$ are torsion free. Then

$$H_i(X \times Y;\mathbb{Z}) = \bigoplus_{j=0}^{i} H_j(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H_{i-j}(Y;\mathbb{Z})$$

To define the relative homotopy group $\pi_n(X,A,x_0)$, where X is a space, $A \subset X$, $x_0 \in A$, for $n \geq 1$: Think of $I^{n-1} \subset I^n$ as $\{(x_1,\ldots,x_{n-1},0)\} \subset I^n$, $x_i \in [0,1]$. Let J^{n-1} be the closure of $\partial I^n - I^{n-1}$.

Definition.

$$\pi_n(X,A,x_0) = [(I^n,\partial I^n,J^{n-1}),(X,A,x_0)]$$

That is, homotopy classes of maps $I^n \to X$ with ∂I^n mapping into $A \subset X$ and J^{n-1} mapping into $x_0 \in A \subset X$.

Equivalently, because $I^n/J^{n-1} \cong \mathbb{D}^n$ with $\partial I^n/J^{n-1} \cong \mathbb{S}^{n-1}$, $\pi_n(X,A,x_0)$ is the homotopy classes of maps $(\mathbb{D}^n,\mathbb{S}^{n-1},s_0) \to (X,A,x_0)$.

(pretty pictures of elements of relative groups)

Lecture 5

$\pi_n(X,A,x_0)$ is a group if $n \geq 2$, and an Abelian group if $n \geq 3$.

The product on $\pi_n(X,A,x_0)$ for $n \geq 2$ is defined by:

$$(fg)(x_1,\ldots,x_n) = \begin{cases} f(2x_1,x_2,\ldots,x_n) & \text{if } x_1 \in [0,\frac{1}{2}] \\ g(2x_1-1,x_2,\ldots,x_n) & \text{if } x_1 \in [\frac{1}{2},1] \end{cases}$$

(pretty pictures)

Remark. A map $f\colon (\mathbb{D}^n,\mathbb{S}^{n-1},s_0) \to (X,A,x_0)$ defines the zero element in $\pi_n(X,A,x_0)$ iff f is homotopic rel. $\mathbb{S}^{n-1}$ to a map into A.

Proof. Suppose, first, that f is homotopic rel. $\mathbb{S}^{n-1}$ to a map g into A. Clearly $[f]=[g] \in \pi_n(X,A,x_0)$. To show that $[g]=0$ in $\pi_n(X,A,x_0)$, homotop g to a constant map using a shrinking family of discs.

(pretty pictures)

Conversely, suppose that $f\colon (\mathbb{D}^n,\mathbb{S}^{n-1},s_0) \to (X,A,x_0)$ is zero in $\pi_n(X,A,x_0)$. That means we have a homotopy $F\colon \mathbb{D}^n \times I \to X$ such that $F(x,0)=f(x)$, $F(x,1)=x_0$, $F(\mathbb{S}^{n-1},t) \subseteq A$ and $F(s_0,t)=x_0$ for all t.

(pretty picture)

Note that the top face of this cylinder $\mathbb{D}^n \to \mathbb{D}^n \times I$ is homotopic rel. $\mathbb{S}^{n-1}$ to a map $\mathbb{D}^n \to \mathbb{D}^n \times I$ whose image is

$$(\mathbb{D}^n \times \{1\}) \cup (\mathbb{S}^{n-1} \times I)\ .$$

Compose that homotopy with the map F, you get a homotopy rel. $\mathbb{S}^{n-1}$ from $f\colon \mathbb{D}^n \to X$ to a map into A. □

A map $f: (X, A, x_0) \to (Y, B, y_0)$ determines homomorphisms $f_*: \pi_n(X, A, x_0) \to \pi_n(Y, A, y_0)$. Two maps that are homotopic (as a map of triples $(X, A, x_0) \to (Y, B, y_0)$) gives the same homomorphism.

Definition. Given maps of pointed sets

$$(A, 0) \xrightarrow{f} (B, 0) \xrightarrow{g} (C, 0),$$

this is exact at B iff $gf = 0$ and any element of B that maps to $0 \in C$ is in the image of A.

Theorem. *Given a triple (X, A, x_0), there is a long exact sequence*

$$\cdots \to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \cdots \to \pi_1(X, A) \to \pi_0(A) \to \pi_0(X)$$

The homomorphism $\pi_n A \to \pi_n X$ comes from the map $A \to X$.

The homomorphism $\pi_n X \to \pi_n(X, A)$ comes from a map $(X, x_0) \to (X, A)$ (really, $(X, x_0, x_0) \to (X, A, x_0)$).

The "boundary map" $\pi_n A \to \pi_{n-1} A$ is defined by

$$\partial f := \text{the map } I^{n-1} \to X \text{ given as } f(x_1, \ldots, x_{n-1}, 0)$$

(This maps ∂I^{n-1} to x_0, so it gives an element of $\pi_{n-1} A$.)

Proof. I'll just prove that the sequence is exact at $\pi_n(X, A)$.

To show the composition $\pi_n(X) \to \pi_n(X, A) \to \pi_{n-1} A$ is zero:

(pretty picture)

$f|I^{n-1}$ is clearly a constant map x_0, so it is $0 \in \pi_{n-1} A$.

To prove exactness: given an element of $\pi_n(X, A)$, represented by a map

$$f: (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0).$$

Suppose that $\partial f = 0$ in $\pi_{n-1} A$. This means that $f|I^{n-1} \sim x_0$ with ∂I^n staying fixed at x_0.

(pretty pictures)

Glue this two maps, and you get a map $I^n \to X$ that looks like

(pretty picture)

This defines an element of $\pi_n X$.

I claim that this element of $\pi_n X$ maps to the given element of $\pi_n(X, A)$. To do this, use the homotopy from f to g defined by using more and more of the map F. □

CW complexes

Also called cell complexes.

Attaching a cell to a topological space: Let X be any topological space and let $f\colon \mathbb{S}^{n-1} \to X$ be a map. Then we define a new space $X \cup_f e_n$ "X with an n-cell attached", by:

$$(X \amalg \mathbb{D}^n) \big/ (f(x) \sim x \quad \forall x \in \mathbb{S}^{n-1} = \partial \mathbb{D}^n)$$

Note that f need not be injective.

Example. Let $f\colon \mathbb{S}^{n-1} \to \text{point}$ be the only possible map, namely the constant map. Then

$$(\text{point}) \cup_f e_n = (\text{point} \cup \mathbb{D}^n) \big/ \sim \text{ as above} \cong \mathbb{S}^n$$

Definition. A space X is a ***CW-complex*** if it is obtained by the following procedure: We have subspaces $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$, where X_n is called the n-skeleton of X, such that

- X_0 is a space with the discrete topology.
- X_n is obtained from X_{n-1} by attaching n-cells along some collection of maps $f_\alpha\colon \mathbb{S}^{n-1} \to X_{n-1}$.
- $X = \bigcup_{n \geq 0} X_n$, and a subset of X is closed iff its intersection with each X_n is closed.

Lecture 6

Every simplicial complex is a CW-complex.

(pretty picture)

Conversely, every CW-complex is homotopy equivalent to a simplicial complex.

Simplicial complexes are "combinatorial", but you can define a given space as a CW-complex with fewer cells.

Example. $\mathbb{S}^1$ as a simplicial complex:

(pretty picture)

$\mathbb{S}^1$ as a CW-complex:

(pretty picture)

A finite CW-complex is one with finitely many cells.

Example. Every compact manifold is a finite CW-complex ("Morse theory")

The homotopy extension property for CW-complexes

Let X be a CW-complex. $B \subset X$ closed is called a ***subcomplex*** if $B \cap X_n$ is a union of some of the cells of X. We say (X, B) is a ***CW-pair***.

Lemma. *Let (X, A) be a CW-pair. Suppose given a map $f\colon X \to Y$, Y any topological space. Suppose we are also given a homotopy $F\colon A \times I \to Y$ from $f|A$ to some other map. Then this extends to a homotopy $G\colon X \times I \to Y$ from f to some other map.*

Proof. Suppose we have constructed a homotopy

$$G_n\colon (A \cup X_n) \times I \to Y$$

from $f|A \cup X_n$ to some other map extending $F\colon A \times I \to Y$. We have to do this for the $(n+1)$-skeleton, $A \cup X_{n+1}$.

It suffices to define such a homotopy on each $(n+1)$-cell in X not contained in A.

We have a continuous map $f\colon \mathbb{D}^{n+1} \to Y$ (the composition of $\mathbb{D}^{n+1} \to X \to Y$), a homotopy from $f|\mathbb{S}^n$ to some other map into Y (given by $\mathbb{S}^n \times I \to X_n \times I \to Y$). We want to extend f to a homotopy $G\colon \mathbb{D}^{n+1} \times I \to Y$ from f to another map.

You can choose such a homotopy G by

$$G(x,t) = \begin{cases} f\left(\frac{x}{1-\frac{t}{2}}\right) & \text{if } |x| \le 1 - \frac{t}{2} \\ F\left(\frac{x}{|x|}, 2(|x| - (1 - \frac{t}{2}))\right) & \text{if } 1 - \frac{t}{2} \le |x| \le 1 \end{cases}$$

□

Lemma (Compression Lemma)**.** *Let (X, A) be a CW-pair, and let (Y, B) be* ***any*** *pair of spaces. Suppose that for any $n \ge 0$ such that X has an n-cell not in A, $\pi_n(Y, B, b_0) = 0$ for all $b_0 \in B$. Suppose given a map $f\colon (X, A) \to (Y, B)$. Then f is homotopic* ***rel.*** *A to a map of X into B.*

Proof. Suppose that we have already constructed a homotopy from f rel. A to a map $f_{n-1}\colon X \to Y$ with $f_{n-1}(X_{n-1}) \subseteq B$. Then we will show that you can do this for X_n.

So try to define the homotopy we want on a given n-cell in X, say not contained in A.

We have a continuous map $\alpha\colon \mathbb{D}^n \to Y$ $(\mathbb{D}^n \to X \xrightarrow{f_{n-1}} Y)$. We know that α maps $\mathbb{S}^{n-1}$ into B. So $\alpha\colon (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (Y, B)$ gives an element of $\pi_n(Y, B, b_0) = 0$. This means that α is homotopic rel. $\mathbb{S}^{n-1}$ to a map $\mathbb{D}^n \to B$. This lets us extend our homotopy from X_{n-1} to X_n. □

Theorem (Whitehead's Theorem)**.** *Let $f\colon X \to Y$ be a continuous map of connected CW-complexes. Suppose that f induces isomorphisms $f_*\colon \pi_n X \to \pi_n Y$ for all n. Then f is a homotopy equivalence.*

Corollary. *A CW-complex X is contractible iff $\pi_n X = 0$ for all $n \ge 0$.*

Proof of Corollary. Apply Whitehead's Theorem to the map $X \to pt$. □

Proof of Whitehead's Theorem. First, suppose $f\colon X \to Y$ is the inclusion of a subcomplex.

We have the LES

$$\cdots \to \pi_n X \xrightarrow{\cong} \pi_n Y \to \underbrace{\pi_n(Y, X)}_{=0} \to \pi_{n-1} X \xrightarrow{\cong} \dots$$

So $\pi_n(Y, X) = 0$ for all $n \ge 1$.

Apply the compression lemma to the identity map $(Y, X) \to (Y, X)$, we find that id_Y is homotopic to a map into X without changing the map on X, i.e. X is a "deformation retraction" of Y. This easily implies that X is homotopic to Y. QED so far…

Lecture 7

Proof of Whitehead's Theorem (cont'd). I checked this in the special case where $X \to Y$ is the inclusion of a closed sub-CW-complex. In that case, I checked that Y "deformation retracts" onto X, which means that there is a homotopy rel. X from id_Y to a retraction $Y \to X$.

Lemma. *If $X \subseteq Y$ and Y deformation retracts to X, then the inclusion map $X \hookrightarrow Y$ is a homotopy equivalence.*

Proof. We have maps $X \hookrightarrow Y$ and $Y \twoheadrightarrow X$.

The composition $X \hookrightarrow Y \twoheadrightarrow X$ is id_X, by definition of a retraction, and the composition $Y \twoheadrightarrow X \hookrightarrow Y$ is homotopic to id_Y.

So $X \hookrightarrow Y$ is a homotopy equivalence. □

Definition. A map $f\colon X \to Y$ of CW-complexes is ***cellular*** if $f(X_n) \subseteq Y_n$ for all $n \geq 0$.

Theorem (Cellular Approximation)**.** *Any map between CW-complexes is homotopic to a cellular map.*

Proof. Similar to the proofs of

- Any continuous map between smooth (C^∞) manifolds is homotopic to a smooth map.
- Any continuous map between simplicial complexes is homotopic to a simplicial map, after possibly subdividing the domain.

□

By the cellular approximation theorem, it suffices to prove Whitehead's theorem for a cellular map $f\colon X \to Y$.

To prove that, we replace f by an inclusion map:

Let $f\colon X \to Y$ be a map of spaces. Then we define the ***mapping cylinder***

$$M_f = \frac{X \times [0,1] \amalg Y}{(x,1) \sim f(x) \quad \forall x \in X}$$

For example, if $f\colon \mathbb{S}^1 \to \mathbb{R}^2$ is the constant map, then M_f is

(pretty picture)

There is a deformation retract from M_f to $Y \subseteq M_f$ by

$$\begin{aligned} F(y,t) &= y & \forall y \in Y, t \in I \\ F((x,u),t) &= (x,(1-t)u+t) & \forall x \in X, u, t \in I \end{aligned}$$

Note that we can think of X as a subspace of M_f, and the composition $X \hookrightarrow M_f \simeq Y$ is the f we started with. Thus, every map is "equivalent" to an inclusion map.

To finish Whitehead's theorem, we have to show: if $f\colon X \to Y$ is a cellular map of connected CW-complexes and if $f_*\colon \pi_n X \to \pi_n Y$ is an isomorphism for all $n \geq 0$, then f should be a homotopy equivalence.

Instead of f, look at the inclusion map $X \hookrightarrow M_f (\simeq Y)$. This map is an isomorphism on homotopy groups, and it's the inclusion of a closed subspace. Since f is a cellular map, you can check that M_f is a CW-complex with X a subcomplex. So we know that $X \hookrightarrow M_f$ is a homotopy equivalence, so $X \xrightarrow{\sim} Y$ also. □

Example (Peano). $\exists$ continuous map I onto I^2. This is not cellular, no matter how you decompose these spaces into cells.

Example. Whitehead's theorem is not true for arbitrary topological spaces. (Any "reasonable" topological space will be homotopy equivalent to a CW-complex, so the theorem applies). For example, let X be the quasi-circle. Then $\pi_n X = 0$ for all $n \geq 0$, but it is not contractible.

Remark. There are connected CW-complexes X and Y with $\pi_n X \cong \pi_n Y$ as a group for all $n \geq 1$, but which are not homotopy equivalent.

For example, take $\mathbb{R}P^2$ and $\mathbb{S}^2 \times \mathbb{R}P^\infty$.

We know that there is a double covering $\mathbb{S}^2 \to \mathbb{R}P^2$, so $\pi_i(\mathbb{R}P^2) \cong \pi_i(\mathbb{S}^2)$ for all $i \geq 2$, $\mathbb{S}^\infty \to \mathbb{R}P^\infty$ (and $\mathbb{S}^\infty$ is contractible) gives $\pi_i(\mathbb{R}P^\infty) = 0$ for $i \geq 2$.

Also, π_1 of the two spaces is isomorphic to $\mathbb{Z}/2$.

But $\mathbb{R}P^2 \not\simeq \mathbb{S}^2 \times \mathbb{R}P^\infty$, for example

$$H_i(\mathbb{R}P^2; \mathbb{Z}) = 0 \text{ for } i \geq 3$$
$$H_i(\mathbb{S}^2 \times \mathbb{R}P^\infty; \mathbb{Z}) \neq 0 \text{ for } i \geq 3 \text{ odd}$$

Lecture 8

Recall

Definition. A fibre bundle $\pi\colon E \to B$ is a (continuous) map such that every point of B has an open neighbourhood U such that there is a homeomorphism $\varphi\colon \pi^{-1}U \to U \times F$, where F is the "fibre" of π, such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}U \arrow[rr, "\cong", "\varphi"'] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ & U & \end{tikzcd}$$

In particular, $\pi^{-1}(\text{any point in } B) \cong F$.

One writes $F \to E \to B$ or

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$$

to denote a fibre bundle.

Example. Covering spaces are fibre bundles with F discrete.

Example. $[0,1] \to \text{Möbius strip} \to \mathbb{S}^1$.

Example. Complex projective spaces

$$\begin{aligned}\mathbb{C}P^n &= \{\text{complex lines (1-dim. } \mathbb{C}\text{-linear subspace) in } \mathbb{C}^{n+1}\} \\ &= (\mathbb{C}^{n+1} - \{0\}) \big/ ((z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n) \quad \forall \lambda \in \mathbb{C} - \{0\})\end{aligned}$$

This is actually a fibre bundle

$$\begin{tikzcd} \mathbb{C} - \{0\} \arrow[r] & \mathbb{C}^{n+1} - \{0\} \arrow[d] \\ & \mathbb{C}P^n \end{tikzcd}$$

Also, $\mathbb{S}^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \colon \sum_{i=0}^n |z_i|^2 = 1\} \subset \mathbb{C}^{n+1}$ maps onto $\mathbb{C}P^n$. In fact,

$$\mathbb{C}P^n = \mathbb{S}^{2n+1} \big/ ((z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n) \quad \forall |\lambda| = 1)$$

i.e. $\mathbb{C}P^n = \mathbb{S}^{2n+1}/\mathbb{S}^1$. This is a fibre bundle

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & \mathbb{S}^{2n+1} \arrow[d, "\pi"] \\ & \mathbb{C}P^n \end{tikzcd}$$

Let $U_i = \{[z_0, \dots, z_n] \in \mathbb{C}P^n \colon z_i \neq 0\}$ for $i = 0, \dots, n$. I claim that $\pi^{-1}(U_i) \cong U_i \times \mathbb{S}^1$ and preserves fibres:

Proof. Say $i = 0$, WLOG.

As a start, let's find a ***section*** of $\pi \colon \pi^{-1}U_0 \to U_0$. (A section of any map $\pi \colon E \to B$ is a map $s \colon B \to E$ such that $\pi s = \mathrm{id}_B$.)

To define such a section, map

$$\underbrace{[z_0, \dots, z_n]}_{\in U_0 \subset \mathbb{C}P^n} \sim [1, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}] \mapsto \frac{(1, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0})}{\left(1 + \sum_{i=1}^n \left|\frac{z_i}{z_0}\right|^2\right)^{\frac{1}{2}}} \in \mathbb{S}^{2n+1}$$

Then we define a homeomorphism $\pi^{-1}(U_0) \cong U_0 \times \mathbb{S}^1$ by

$$U_0 \times \mathbb{S}^1 \ni (x, \lambda) \mapsto \lambda s(x) \in \pi^{-1} U_0,$$

where s is the section I defined. □

Example. Take $n = 1$ above. We have a fibre bundle

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & \mathbb{S}^3 \arrow[d] \\ & \mathbb{C}P^1 \arrow[r, equal] & \mathbb{S}^2 \end{tikzcd}$$

called the Hopf fibration.

The map $\mathbb{S}^3 \to \mathbb{S}^2$ is given by

$$\{(z_0, z_1) \in \mathbb{C}^2 \colon\ |z_0|^2 + |z_1|^2 = 1\} \mapsto \frac{z_0}{z_1} \in \mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$$

This map takes

$$(r_0 e^{i\theta_0}, r_1 e^{i\theta_1}) \mapsto \frac{r_0}{r_1} e^{i(\theta_0 - \theta_1)} \in \mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$$

For a given value of $r_0/r_1 \in (0, \infty)$, the set of points in $\mathbb{S}^3$ with that value is $\cong (\mathbb{S}^1)^2$. For $r_0/r_1 = 0$ or ∞, the set of corresponding point in $\mathbb{S}^3$ is $\cong \mathbb{S}^1$, either $\{(0, \lambda) \colon \lambda \in \mathbb{S}^1\}$ or $\{(\lambda, 0) \colon \lambda \in \mathbb{S}^1\}$.

Identify $\mathbb{S}^3 - \{(0, 1)\} \cong \mathbb{R}^3$ by stereographic projection. Those two circles in $\mathbb{S}^3$ correspond to the z-axis and the circle in the (x, y)-plane.

(pretty picture)

The fibres of the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ are $(1, 1)$-circles inside those tori.

(pretty picture)

Any two fibres of the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ are linked to each other.

Definition. A ***(Serre) fibration*** is a map $\pi \colon E \to B$ which has the homotopy lifting property for all discs $\mathbb{D}^n$, $n \geq 0$.

Recall

Definition. A map $\pi \colon E \to B$ has the homotopy lifting property (HLP) for a space X if, for every map $f \colon X \to E$ and every homotopy from $\pi f \colon X \to B$, the homotopy lifts (not necessarily uniquely) to E (i.e. there is a map $F \colon X \times [0, l] \to E$ such that $F(x, 0) = f(x)$ and $\pi F \colon X \times [0, 1] \to B$ is the given homotopy).

Example. $X = point$.

(pretty pictures)

Lemma. *Every fibre bundle is a fibration.*

Proof. This HLP for X says something about the pairs $(X \times [0, 1], X \times \{0\})$.

Let $\pi \colon E \to B$ be a fibre bundle, we want to prove that π has HLP for $\mathbb{D}^n$, $n \geq 0$. That is, suppose we have maps $f \colon I^n \to E$ (remember $I^n \cong \mathbb{D}^n$) and a homotopy $G \colon I^n \times I \to B$ with $G|I^n \times \{0\} = \pi f$. We want to find a map $F \colon I^n \times I \to E$ with $F|I^n \times \{0\} = f$ and $\pi F = G$.

We have a map $I^{n+1} \to B$, and a lift of a map to E on $I^n = I^n \times \{0\} \subset I^{n+1}$, we want a lift on all of I^{n+1}.

Since $\pi \colon E \to B$ is a fibre bundle, there is an open covering by opens U_α such that $\pi^{-1} U_\alpha \cong U_\alpha \times F$ (over U_α).

So I^{n+1} is covered by the open subsets $G^{-1} U_\alpha$. Since I^{n+1} is compact (and metric), there is an $\epsilon > 0$ such that every ϵ-ball in I^{n+1} maps into some U_α.

So if we divide I^{n+1} into small enough cubes, each little cube will map under G into some $U_\alpha \subset B$.

(pretty pictures)

We lift G up one little cube at a time.

We are given a map (a little) $I^{n+1} \to U_\alpha$, which we want to lift to $U_\alpha \times F$. We are already given[2] on $I^n \times \{0\} \subset I^{n+1}$. So we can make such a lift by

$$I^{n+1} \to U_\alpha \times F; \quad (x_1, \ldots, x_{n+1}) \mapsto (G(x), f(x_1, \ldots, x_n)).$$

where f here is the composition $I^n \times \{0\} \to U_\alpha \times F \to F$. □

Lecture 9

Theorem. *Let $\pi\colon E \to B$ be a fibration. Let e_0 be a point in E, $b_0 = \pi(e_0) \in B$, and let $F = \pi^{-1}(b_0)$. Then the natural homomorphism*

$$\pi_n(E, F, e_0) \to \pi_n(B, b_0)$$

is an isomorphism $\forall\, n \geq 0$. So we have a long exact sequence

$$\pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \cdots \to \pi_0 F \to \pi_0 E \to 0$$

assuming B is path connected.

Proof. First, observe that $\pi_n(E,F) \to \pi_n B$ is surjective: An element of $\pi_n B$ is given by a map $(I^n, \partial I^n) \to (B, b_0)$.

(pretty picture)

$$\begin{tikzcd} & E \arrow[d] \\ I^n \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

I know that the map $E \to B$ has the the HLP for discs. That is, given a map $I^n \to B$ and a lift of this map on $I^{n-1} \times \{0\} \subset I^n$ to E, then I can extend this to a lift of the whole map $I^n \to E$.

(pretty picture)

Notice that $(I^n, I^{n-1}) \cong (I^n, J^{n-1})$.

(pretty picture)

Therefore this lifting property is true for I^n and J^{n-1} (instead of I^{n-1}).

We choose a lift of $\alpha | J^{n-1}$ to E by mapping $J^{n-1} \subset I^n$ to $e_0 \in E$.

$$\begin{tikzcd} & E \arrow[d] \\ I^n \arrow[r, "\alpha"'] \arrow[ur, "\tilde{\alpha}"] & B \end{tikzcd}$$

[2] Always given on a subset homomorphic to $I^n \times \{0\}$, if you do this sensibly.

So $[\tilde{\alpha}] \in \pi_(E, F, e_0)$. Clearly this element maps to $[\alpha] \in \pi_n(B, b_0)$, as we want.

Next, show that $\pi_n(E, F) \to \pi_n B$ is injective:

Given two elements $\tilde{\alpha_0}, \tilde{\alpha_1} \in \pi_n(E, F)$ with the same image in $\pi_n B$, I want to show $\tilde{\alpha_0} = \tilde{\alpha_1}$ in $\pi_n(E, F)$.

We have continuous maps $\tilde{\alpha_0}, \tilde{\alpha_1} \colon (I^n, \partial I^n, J^{n-1}) \to (E, F, e_0)$ such that $\pi\tilde{\alpha_0} \sim \pi\tilde{\alpha_1}$ as maps $(I^n, \partial I^n) \to (B, b_0)$. That homotopy is given by a map $F \colon I^n \times I \to B$ such that

$$F|I^n \times \{0\} = \pi\tilde{\alpha_0}$$
$$F|I^n \times \{1\} = \pi\tilde{\alpha_1}$$
$$F|\partial I^n \times I = b_0$$

(pretty pictures)

We choose a lift of $F|J^n$ to $\tilde{F}$ on J^n, where J^n is the union of all faces of I^{n+1} except the top one. Namely, take this lift to be, on J^n:

(pretty picture)

By HLP, we can extend this partial lift to a lift $\tilde{F} \colon I^{n+1} \to E$ of $F \colon I^{n+1} \to B$. This lift must map the "top" fact of I^{n+1} into F. This homotopy shows that $[\tilde{\alpha_0}] = [\tilde{\alpha_1}] \in \pi_n(E, F)$.

Finally, if B is path-connected, I claim that $\pi_0(F) \to \pi_0 E$ is onto:

Pick any element of $\pi_0 E$, i.e. any path-component of E, say containing the point e_1. Since B is path-connected, $\exists$ a path from $\pi(e_1)$ to b_0. You can lift this to a path in E, from e_1 to some point. Clearly that endpoint $\in F$. □

Example. Let $\pi \colon E \to B$ be a covering space. That is, π is a fibre bundle with discrete fibre F.

Then $\pi_n(F) = 0$ for $n \geq 1$, so this LES simplifies to (if E is connected):

$$0 = \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F = 0 \text{ if } n \geq 2$$
$$0 = \pi_1 F \to \pi_1 E \to \pi_1 B \to \pi_0 F = F \to \pi_0 E = 0$$

So $\pi_n E \cong \pi_n B$ for $n \geq 2$, and $F \cong \pi_1 B / \pi_1 E$. This agrees with the theorem on covering spaces.

Example. Let $E = B \times F$, for any space B and F. The LES just splits up into SESs

$$0 \to \pi_n F \to \pi_n E = \pi_n F \times \pi_n B \to \pi_n B \to)$$

Example. Take the Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$. We know that $\pi_1 \mathbb{S}^1 \cong \mathbb{Z}$ and $\pi_i \mathbb{S}^1 = 0$ for $i \geq 2$. So, for $i \geq 3$, this LES shows that $\pi_n \mathbb{S}^3 \cong \pi_n \mathbb{S}^2$. In low dimensions, we have

$$0 = \pi_2 \mathbb{S}^1 \to \pi_2 \mathbb{S}^3 = 0 \to \pi_2 \mathbb{S}^2 \to \pi_1 \mathbb{S}^1 \cong \mathbb{Z} \to \pi_1 \mathbb{S}^3 = 0$$

($\pi_2 \mathbb{S}^3 = 0$ is by the same argument as $\pi_1 \mathbb{S}^3 = 0$) So $\pi_2 \mathbb{S}^2 \cong \mathbb{Z}$. (This looks like $H_2(\mathbb{S}^2; \mathbb{Z}) \cong \mathbb{Z}$)

Remark. For any space X, any $n \geq 1$, there is a natural homomorphism $\pi_n X \to H_n(X, \mathbb{Z})$, given by

$$(f \colon \mathbb{S}^n \to X) \mapsto f_*[\mathbb{S}^n]$$

where $f_* \colon H_n \mathbb{S}^n = \mathbb{Z}[\mathbb{S}^n] \to H_n X$ ($H_n \mathbb{S}^n \cong \mathbb{Z}$, generated by $[\mathbb{S}^n]$).

Fact. *For any n, the map $\pi_n \mathbb{S}^n \to H_n(\mathbb{S}^n, \mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism.*

The Hopf fibration shows, in particular (using the fact),

$$\pi_3\mathbb{S}^2 \cong \pi_3\mathbb{S}^3 \cong \mathbb{Z}$$

(Hopf, 1930s).

If you look at where this isomorphism comes from, you conclude that $\pi_3\mathbb{S}^2 \cong \mathbb{Z}$, generated by the ***Hopf map*** $\mathbb{S}^3 \to \mathbb{S}^2$.

Thus the Hopf map is a continuous map between CW-complexes that induces the zero homomorphism $H_i\mathbb{S}^3 \to H_i\mathbb{S}^2$ for all $i \geq 1$, but is ***not*** homotopic to a constant map.

Lecture 10

For the fibration $\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{C}P^n$, we know that

$$\pi_i(\mathbb{S}^1) = \begin{cases} \mathbb{Z} & i = 1 \\ 0 & i \geq 2 \end{cases}$$

So the LES gives

$$0 = \pi_2\mathbb{S}^{2n+1} \to \pi_2\mathbb{C}P^n \to \pi_1\mathbb{S}^1 = \mathbb{Z} \to \pi_1\mathbb{S}^{2n+1} = 0$$

I mentioned last time that $\pi_i\mathbb{S}^n = 0$ for $0 < i < n$. This is very easy by cellular approximation: write $\mathbb{S}^n = e^0 \cup e^n$. Any map $\mathbb{S}^i \to \mathbb{S}^n$, $i < n$, is homotopic to a cellular map, so a map into the i-skeleton of $\mathbb{S}^n$. Such a map is constant, in this case.

Theorem (Hurewicz)**.** *Let X be any $(n-1)$-connected space (i.e. $\pi_i = 0$ for $i \leq n-1$). Then $H_0(X;\mathbb{Z}) = \mathbb{Z}$, $H_i(X;\mathbb{Z}) = 0$ for $1 \leq i \leq n-1$, and $\pi_n X \cong H_n(X;\mathbb{Z})$.*

We also have a fibration $\mathbb{S}^1 \to \mathbb{S}^\infty \to \mathbb{C}P^\infty := \bigcup_{n\geq 0}\mathbb{C}P^n$, and $\mathbb{S}^\infty \simeq pt$. We read off from the LES:

$$\pi_2\mathbb{C}P^\infty \cong \mathbb{Z}$$

and

$$0 = \pi_i\mathbb{S}^\infty \to \pi_i\mathbb{C}P^\infty \to \pi_{i-1}\mathbb{S}^1 = 0 \text{ for } i \geq 3$$

So $\pi_i\mathbb{C}P^\infty = 0$ for all $i \neq 2$.

Definition. An ***Eilenberg-MacLane space*** $K(G,n)$ is a space X with $\pi_n X \cong G$ and $\pi_i X = 0$ for $i \neq n$.

So $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$ space.

A few other examples:

- $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2,1)$ space
- $(\mathbb{S}^1)^n$ is a $K(\mathbb{Z}^n,1)$ space
- Any closed surface X of genus $g \geq 1$ is a $K(G,1)$ space, where $G = \pi_1 X$, because the universal cover of X is homeomorphic to $\mathbb{R}^2$.

Remark. The fundamental group of a closed surface of genus $g \geq 1$ is

$$\pi_1 X_g = \langle a_1, b_1, \ldots, a_g, b_g | [a_1,b_1][a_2,b_2]\ldots[a_g,b_g]\rangle$$

You can replace $\mathbb{C}$ by the quaternions $\mathbb{H}$

$$\mathbb{H} = \{a + bi + cj + dk \colon a, b, c, d \in \mathbb{R}\}$$

with multiplication $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$. This makes $\mathbb{H}$ into an associative $\mathbb{R}$-algebra, but in fact a division ring.

Definition.

$$\mathbb{H}P^n = \{\text{left one-dimensional sub-}\mathbb{H}\text{-vector spaces of } \mathbb{H}^{n+1}\}$$

We have a map $\mathbb{S}^3 \to \mathbb{S}^{4n+3} \to \mathbb{H}P^n$, which is a fibre bundle, where $\mathbb{S}^3 = \{x \in \mathbb{H} \colon |x| = 1\}$.

Example. For $n = 1$, this is a fibre bundle $\mathbb{S}^3 \to \mathbb{S}^7 \to \mathbb{H}P^1 = \mathbb{H} \cup \{\infty\} \cong \mathbb{S}^4$. This map $\mathbb{S}^7 \to \mathbb{S}^4$ is another "Hopf map".

Fact. *$\pi_7\mathbb{S}^4 \cong \mathbb{Z}$, generated by this Hopf map.*

Example. We can take $n = \infty$, $\mathbb{S}^3 \to \mathbb{S}^\infty \to \mathbb{H}P^\infty$, and this gives

$$\pi_i \mathbb{H}P^\infty \cong \begin{cases} \pi_{i-1}\mathbb{S}^3 & i \geq 1 \\ 0 & i = 0 \end{cases}$$

Remark. The groups $\pi_i\mathbb{S}^3 \cong \mathbb{S}^2$ for $i \geq 4$, are know for $i \leq 30$ or so, but not in general. At least we know (Serre, 1950)

$$\pi_i\mathbb{S}^3 = \begin{cases} \mathbb{Z} & i = 3 \\ \text{finite Abelian group} & i \geq 4 \end{cases}$$

Some Lie groups

- $\mathbb{S}^1 = \{x \in \mathbb{C} \colon |x| = 1\}$ is an Abelian group under multiplication.
- $\mathbb{S}^3 = \{x \in \mathbb{H} \colon |x| = 1\}$ is a non-Abelian group under multiplication.
- The general linear group

  $$GL(n, \mathbb{R}) = \{A \in M_n\mathbb{R} \colon \det A \neq 0\} = \{\text{linear isomorphism} f \colon \mathbb{R}^n \to \mathbb{R}^n\}.$$

- The orthogonal group

  $$\begin{aligned} O(n) &= \{\text{linear maps } f \colon \mathbb{R}^n \to \mathbb{R}^n \text{ with } \left|f(x)\right| = |x| \quad \forall x \in \mathbb{R}^n\} \\ &= \{A \in GL(n, \mathbb{R}) \colon AA^T = I\} \end{aligned}$$

  is a compact Lie group.

Lemma. *$O(n) \overset{\sim}{\hookrightarrow} GL(n, \mathbb{R})$, i.e. $O(n)$ and $GL(n, \mathbb{R})$ are homotopy equivalent.*

Proof. (Gram-Schmidt)

$A \in GL(n,\mathbb{R})$ corresponds to n linearly independent vectors in $\mathbb{R}^n$ (the rows of A), and $A \in O(n)$ corresponds to n orthonormal vectors in $\mathbb{R}^n$.

There is a retraction $GL(n,\mathbb{R}) \to O(n)$, by the Gram-Schmidt process

$$(v_1,\dots,v_n) = \left(\frac{v_1}{|v_1|}, \frac{v_2 - av_1}{|v_2 - av_1|}, \dots\right)$$

This is a deformation retraction (there is an obvious homotopy...) □

We have a fibre bundle $O(n-1) \to O(n) \to \mathbb{S}^{n-1}$. [Check!]

Roughly, you can say: $O(n)$ is built by fibre bundles from $\mathbb{S}^0, \mathbb{S}^1, \dots, \mathbb{S}^{n-1}$.

So, for example, what is $\pi_0 O(n) = \pi_0 GL(n,\mathbb{R})$?

We know $O(1) = \pm 1$ has two components. From the fibre bundle $O(n-1) \to O(n) \to \mathbb{S}^{n-1}$, we get, if $n \geq 3$,

$$0 = \pi_1 \mathbb{S}^{n-1} \to \pi_0 O(n-1) \to \pi_0 O(n) \to \pi_0 \mathbb{S}^{n-1} = 0$$

So $\pi_0 O(n-1) \cong \pi_0 O(n)$ if $n \geq 3$.

To describe $O(2)$: there is an exact sequence

$$1 \to SO(n) \to O(n) \to \{\pm 1\} \to 1$$

and $SO(2)$, the group of rotations of $\mathbb{R}^2$, is isomorphic to $\mathbb{S}^1$. This group is connected, so $O(2)$ has two connected components. By our fibre bundles, it follows that $O(n)$ has two connected components for all n. (Equivalently, $SO(n) \overset{\sim}{\hookrightarrow} GL(n,\mathbb{R})^{>0}$ is connected for all $n \geq 1$.)

Next, look at $\pi_1(O(n), \mathrm{id}) = \pi_1(SO(n), \mathrm{id})$. We have an exact sequence

$$\pi_2 \mathbb{S}^{n-1} \to \pi_1 SO(n-1) \to \pi_1 SO(n) \to \pi_1 \mathbb{S}^{n-1}$$

So, if $n \geq 4$, we have $\pi_1 SO(n-1) \cong \pi_1 SO(n)$. So it is enough to compute $\pi_1 SO(3)$.

Lecture 11

$SO(3) \cong \mathbb{R}P^3$, so $\pi_1 SO(3) \cong \mathbb{Z}/2$.

It follows that for $n \geq 3$ the universal cover of $SO(n)$ is a double cover. It is called $\mathrm{Spin}(n)$.

Finally, we compute $\pi_2 SO(n)$:

$$\pi_3 \mathbb{S}^{n-1} \to \pi_2 SO(n-1) \to \pi_2 SO(n) \to \pi_2 \mathbb{S}^{n-1}$$

So for $n \geq 4$, the map $\pi_2 SO(n-1) \to \pi_2 SO(n)$ is surjective. We know

$$\pi_2 SO(2) \cong \pi_2 SO(3) = 0,$$

so $\pi_2 SO(n) = 0$ for all n.

Loop spaces

For a pointed space (X, x_0) its ***loop space*** is

$$\Omega X = \{\text{continuous } p\colon [0,1] \to X \text{ such that } p(0) = p(1) = x_0\}$$

This has a natural topology, namely the compact-open topology: A subset of ΩX is open iff it is a union of finite intersections of sets of the form

$$M(K,U) = \{p \in \Omega X\colon\ p(K) \subseteq U\}$$

where K is compact and $U \subseteq X$ is open. (Can put the compact-open topology on $\mathrm{Map}(X,Y)$ in general).

ΩX is a pointed space: the selected element is the constant loop at x_0.

One basic fact about ΩX is that

$$\pi_n \Omega X \cong \pi_{n+1}(X, x_0)$$

Maps $I^n \to \Omega X$ can be mapped to maps $I^{n+1} \to X$ in the obvious way, and this induces the isomorphism.

The path space of X is the space

$$PX = \{p\colon [0,1] \to X \text{ such that } p(0) = x_0\}$$

with the compact open topology.

Define a map $PX \to X$, $p \mapsto p(1)$. This is a fibration, with fibres identified with ΩX. Note that $P(X)$ is contractible, so the long exact sequence gives

$$\underbrace{\pi_i PX}_{=0} \to \pi_i X \to \pi_{i-1}\Omega X \to \underbrace{\pi_{i=1} PX}_{=0}$$

so $\pi_i X \xrightarrow{\cong} \pi_{i-1}\Omega X$, as we know.

Previously, we showed that any map is "equivalent" to an inclusion map using the mapping cylinder. Now we show that any map is "equivalent" to a fibration.

Let $f\colon X \to Y$ be continuous. Let

$$E_f = \left\{(x,p)\colon x \in X, [0,1] \xrightarrow{p} Y \text{ such that } p(0) = f(x)\right\}$$

Note that $X \subset E_f$ (by identifying X with constant paths), and that E_f deformation retracts to X.

There is a map $\alpha\colon E_f \to Y$, $(x,p) \mapsto p(1)$. This is a fibration.

Proof. We have to show that for any map $g\colon I^{n+1} \to Y$ and a lift to E_f on $I^n \subset I^{n+1}$, we can extend it to a lift on all of I^{n+1}.

g gives a map $I^n \to \{\text{paths in } Y\}$. Let the given lift $I^n \to E_f$ be $h(x_1,\ldots,x_n) = (h_X(x_i), h_P(x_i))$. Define

$$H_X(x_1,\ldots,x_{n+1}) = h_X(x_1,\ldots,x_n)$$

and let $H_P(x_1,\ldots,x_{n+1})$ be the path $H_p(xs_1,\ldots,x_n)$ followed by the path

$$g|\{(x_1,\ldots,x_n)\} \times [0,1].$$

$\square$

Lecture 12

Definition. The ***homotopy fibre*** F_f of $f\colon X \to Y$ is a fibre of the fibration $E_f \to Y$.

Remark. Homotopic maps have homotopy equivalent homotopy fibres. More generally, if

$$F_f \to X_1 \xrightarrow{f} Y_1 \quad \text{and} \quad F_g \to X_2 \xrightarrow{g} Y_2$$

with $X_1 \simeq X_2$ and $Y_1 \simeq Y_2$, then $F_f \simeq F_g$

Examples. (i) $\mathrm{fibre}(X \to pt) = X$

(ii) $\mathrm{fibre}(pt \to X) = \mathrm{fibre}(PX \to X) = \Omega X$

(iii) If $F \to E \to B$ is a fibration then $\mathrm{fibre}(E \to B) = F$.

(iv) What is $\mathrm{fibre}(F \to E)$?

Let

$$E_f = \left\{(e,p)\colon e \in E, [0,1] \xrightarrow{p} B, p(0) = f(e)\right\}$$

$E_f \to B$, $(e,p) \mapsto p(1)$ is a fibration. Its fibre over b is

$$F_f = \left\{(e,p)\colon e \in E, [0,1] \xrightarrow{p} B, p(0) = f(e), p(1) = b\right\}$$

$F_f \to E$, $(e,p) \mapsto e$ is a fibration, $F_f \simeq F$, so

$$\mathrm{fibre}(F \to E) = \mathrm{fibre}(F_f \to E) = \left\{[0,1] \xrightarrow{p}\colon p(0) = f(e_0) = p(1)\right\} = \Omega B$$

So we obtain a fibration $\Omega B \to F \to E$. Its LES is the same as for the original fibration.

We can repeat the process to get an infinite sequence of fibrations

$$\cdots \to \Omega^2 B \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B.$$

Lemma. *For any space X and positive integer n we can construct a space $X_{\le n}$ and a map $X \to X_{\le n}$ such that*

$$\pi_i(X_{\le n}) = \begin{cases} \pi_i X & i \le n \\ 0 & i > n \end{cases}.$$

(Actually we assume X a CW-complex).

Proof. We first kill $\pi_{n+1}X$. Pick a set of generators for this group, with representatives $\alpha\colon S^{n+1} \to X$. Define

$$X_{n+1} = X \cup_\alpha (n+2 \text{ cells}).$$

There is an obvious map $X \to X_{n+1}$. By cellular approximation theorem, $\pi_i X \xrightarrow{\cong} \pi_i X_{n+1}$ for $i \le n$.

Furthermore, $\pi_{n+1}X \twoheadrightarrow \pi_{n+1}X_{n+1}$ by cellular approximation, and by construction the map is 0. So $\pi_{n+1}X_{n+1} = 0$. Repeat. □

Example. $(\mathbb{S}^n)_{\le n}$ is $K(\mathbb{Z}, n)$.

Lemma. *For any positive n and group G (Abelian if $n \geq 2$), any two $K(G, n)$ spaces are homotopy equivalent.*

Proof. We assume $G = \mathbb{Z}$ and that the spaces are CW-complexes. Then it suffices to consider $X_1 = (\mathbb{S}^n)_{\leq n}$.

We show the result by constructing a map $X_1 \to X_2$ and applying Whitehead's theorem.

Define $\alpha \colon X_1^{(n)} = \mathbb{S}^n \to X_2$ by taking it to represent a generator for $\pi_n X_2$. Then $\alpha_* \colon \pi_i X_1^{(n)} \to \pi_i X_2$ are isomorphisms for all $i \leq n$.

Now extend α to a map $X_1 \to X_2$. We do this one cell at a time. Since $\pi_i X_2 = 0$ for $i > n$, we can extend any map $\mathbb{S}^i \to X_2$ to a map $\mathbb{D}^{i+1} \to X_2$.

The resulting $\alpha \colon X_1 \to X_2$ induces isomorphisms $\alpha_* \colon \pi_i X_1 \to \pi_i X_2$ for all i, so α is a homotopy equivalence by Whitehead's theorem. □

Lecture 13

Lemma.

$$\Omega K(G, n) \simeq K(G, n-1)$$

Proof.

$$\pi_i \Omega K(G, n) = \pi_{i+1} K(G, n) = \begin{cases} G & i+1 = n \\ 0 & \text{otherwise} \end{cases}$$

□

Example. $\Omega(\mathbb{C}P^\infty) \simeq \mathbb{S}^1$

Definition. For a space X, let $X_{>n} = \mathrm{fibre}(X \to X_{\leq n})$.

The fibration $X_{>n} \to X \to X_{\leq n}$ gives the long exact sequence

$$\cdots \to \pi_i X_{>n} \to \pi_i X \to \pi_i X_{\leq n} \to \pi_{i-1} X_{>n} \to \ldots$$

So

$$\pi_i X_{>n} = \begin{cases} 0 & i \leq n \\ \pi_i X & i > n \end{cases}$$

For X connected, $X_{>1}$ is the universal cover of X.

Postnikov towers

$$\begin{tikzcd} X \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\ X_{\leq 1} & X_{\leq 2} \arrow[l] & X_{\leq 3} \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

$X_{\leq n}$ becomes "closer" to X as n increase. And the "difference" between $X_{\leq n}$ and $X_{\leq n-1}$ is an Eilenberg-MacLane space: there is a fibration

$$K(\pi_n X, n) \to X_{\leq n} \to X_{\leq n-1}$$

In this sense, X is built from Eilenberg-MacLane spaces by a sequence of fibrations.

This suggests why a CW-complex is not in general determined up to homotopy by its homotopy groups. Any space X has the same homotopy groups as $\prod_{n\geq 1} K(\pi_n X, n)$. X is in general an iterated fibration of these pieces, but not necessarily trivial.

Definition. A map $i\colon A \to X$ is a ***cofibration*** if it has the homotopy extension property:

Given any map $f_0\colon X \to Y$ and a homotopy from $f_0|A$ to some other map $A \to Y$ we can extend to a homotopy from f_0 to some map $X \to Y$.

Example. The inclusion of a subcomplex A into a CW-complex X is a cofibration.

Fact (Easy)**.** *Any cofibration is injective.*

Given a cofibration $A \hookrightarrow X$, the sequence $A \to X \to X/A$ is called a cofibre sequence.

Any map is equivalent to a cofibration using the mapping cylinder: for $X \xrightarrow{f} Y$, let

$$M_f = (X \times [0,1]) \amalg Y \big/ ((x,1) \sim f(x))$$

$M_f \simeq Y$, and the inclusion $X \hookrightarrow M_F$, $x \mapsto (x,0)$ is a cofibration. We can therefore define the cofibre of $f\colon X \to Y$ to be M_f/X (up to homotopy equivalence).

Examples. (i) cofibre$(pt \to X) = X$, since $pt \to X$ is a cofibration.

(ii) For $f\colon X \to pt$, $M_f = CX$, the cone on X. So cofibre$(X \to pt) = CX/X = \Sigma X$, the suspension of X.

(iii) For $f\colon \mathbb{S}^n \to X$, cofibre$(f) = X \cup_f \mathbb{D}^{n+1}$.

(iv) If $A \hookrightarrow X$ is a cofibration then $X/A \simeq X \cup_A CA$. $X \to X \cup_A CA$ is a cofibration, so

$$\text{cofibre}(X \to X/A) = \text{cofibre}(X \to (X \cup_A CA)) = \Sigma A$$

Thus we get a cofibration $X \to X/A \to \Sigma A$. We can repeat to get an infinite sequence of cofibrations

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \Sigma(X/A) \to \Sigma^2 A \to \dots$$

This is "dual" to a result about fibrations and loop spaces (see Lecture 12). This is called Eckmann-Hilton duality

Homotopy groups	(Co)homology groups
Eilenberg-MacLane spaces	Spheres
Fibrations	Cofibrations
Loop spaces	Suspensions

Lecture 14

Vector Bundles

Definition. A vector bundle over B is a space E with a map $\pi\colon E \to B$ such that for each $b \in B$, the fibre $\pi^{-1}(b)$ is a real dimension n vector space. We require that every point in B has a neighbourhood U such that there is a homeomorphism $\varphi\colon \pi^{-1}U \to U \times \mathbb{R}^n$ such that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "\mathrm{proj}_1"] \\ & U & \end{tikzcd}$$

commutes, and φ is a linear isomorphism on each fibre.

In particular, a vector bundle is a fibre bundle. The dimension n of the fibre is the rank of E.

Definition. A map $\phi\colon E \to F$ of vector bundles over B is a bundle morphism if it is continuous, linear on each fibre, and

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & F \arrow[dl] \\ & B & \end{tikzcd}$$

commutes.

A bundle morphism that is also a homeomorphism is an isomorphism.

A vector bundle of rank 1 is called a line bundle.

Examples. (i) $B \times \mathbb{R}^n$ is the trivial vector bundle over B, denoted $\mathbb{R}^n_B$ (or just $\mathbb{R}^n$).

(ii) If M is a smooth manifold then TM is a vector bundle over M.

(iii) $E = \{(x, v)\colon x \in \mathbb{R}P^n, v \in [x]\}$ is a line bundle over $\mathbb{R}P^n$, called the universal or tautological line bundle.

If E is trivial then $E - \{\text{zero sections}\} \cong \mathbb{R}P^n \times (\mathbb{R} - \{0\})$ (disconnected). But $E - \{\text{zero sections}\} \cong \mathbb{R}^{n-1} - \{0\}$, so E is not trivial.

Remark. For any vector bundle $\pi\colon E \to B$ the projection π is a homotopy equivalence.

Lemma. *A vector bundle $E \to B$ is trivial iff there are n sections $s_1, \ldots, s_n\colon B \to E$ that are linearly independent at each point.*

In particular, any Lie group has trivial tangent bundle.

Lecture 15

Definition. A matrix on a vector bundle $E \to B$ is a symmetric positive definite bilinear form on each fibre, continuous in the sense that it gives a continuous function $E \times_B E \to \mathbb{R}$.

Lemma. *Every vector bundle E on a paracompact space B has a metric.*

Proof. B paracompact is equivalent to existence of a partition of unity on B:

For any open cover $\{U_\alpha\}$ of B there are continuous functions $f_\alpha\colon B \to \mathbb{R}_{\geq 0}$ such that $\operatorname{supp}(f_\alpha) \subseteq U_\alpha$, at any point only finitely many f_α are nonzero, and $\sum f_\alpha = 1$.

So we can define a metric on E by pulling back metrics via local trivialisations and weighting by a partition of unity. □

Lemma. *If $E \to B$ is a trivial vector bundle with a metric then E is also trivial as a metrized vector bundle, i.e. there is a bundle isomorphism $E \cong B \times \mathbb{R}^n$ that is an isometry on each fibre.*

Proof. Apply Gram-Schmidt to a trivialisation to make it a metric trivialisation. □

Let $\pi\colon E \to Y$ a vector bundle, $f\colon X \to Y$ continuous. Then the pullback bundle f^*E over X is the set

$$\{(x,e) \in X \times E\colon f(x) = \pi(e)\}$$

The bundle projection $\pi_2\colon f^*E \to X$ is $(x,e) \mapsto x$.

If $U \subseteq Y$, $\varphi = (\pi, \varphi_2)\colon \pi^{-1}U \to U \times \mathbb{R}^n$ is a local trivialisation of E, then

$$\varphi'\colon \pi_2^{-1}(f^{-1}(U)) \to f^{-1}U \times \mathbb{R}^n, \quad (x,e) \mapsto (x, \varphi_2(e))$$

is a local trivialisation of f^*E.

The fibre of f^*E over $x \in X$ is $\pi^{-1}(f(x))$.

The map $F\colon f^*E \to E, (x,e) \mapsto e$ is a bundle morphism covering f

$$\begin{tikzcd} f^*E \arrow[r, "F"] \arrow[d, "\pi_2"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

If $X \subseteq Y$, $i\colon X \hookrightarrow Y$ is the inclusion map, then i^*E is also denoted $E|_X$.

Example. If M is a submanifold of a smooth manifold N then TM, $TN|_M$ are both vector bundles on M, and there is an obvious inclusion $TM \hookrightarrow TN|_M$

If E is a subbundle of F then we can define a quotient bundle F/E. Its fibre over $x \in X$ is $(F/E)_x = F_x/E_x$.

Example. The quotient bundle $TN|_M/TM$ is the normal bundle of M in N.

For vector bundles E, F over X we can define bundles $E \oplus F$, $E \otimes_{\mathbb{R}} F$, E^* whose fibres are $E_x \oplus F_x$, $E_x \otimes_{\mathbb{R}} F_x$, $(E_x)^*$ respectively.

Classification of real vector bundles over X

For a line bundle L over X choose a metric on L. Then

$$Y = \{e \in L\colon \|e\| = 1\}$$

is a double cover of X. Conversely, for a double cover Y we can define a line bundle by $L = Y \times \mathbb{R}/(\mathbb{Z}/2)$, where $\mathbb{Z}/2$ identifies (y, v) with $(\sigma(y), -v)$ ($\sigma(y)$ is the other point of y lying over $\pi(y)$).

The double cover of a path-connected space X are classified by $\operatorname{Hom}(\pi_1 X, \mathbb{Z}/2)$. Since $\mathbb{Z}/2$ is Abelian any homomorphism $\pi_1 X \to \mathbb{Z}/2$ factors through $\pi_1 X \big/ [\pi_1 X, \pi_1 X] \cong H_1(X)$. So the double covers are classified also by

$$\operatorname{Hom}(H_1(X;\mathbb{Z}), \mathbb{Z}/2) \cong H^1(X;\mathbb{Z}/2)\ .$$

Cohomology groups

For any commutative ring R we have (singular) cohomology groups $H^i(X;R)$. If R is a field, then

$$H^i(X;R) = \mathrm{Hom}_R(H_i(X;R),R)\ .$$

There is a product

$$H^i(X;R) \times H^j(X;R) \to H^{i+j}(X;R)$$

which makes $H^*(X;R)$ into an associative, graded commutative ring.

For $X \to Y$ continuous, we have pullbacks $f^*\colon H^i(Y;R) \to H^i(X;R)$ which give a ring homomorphism $H^*(Y;R) \to H^*(X;R)$.

Lecture 16

Characteristic classes

Stiefel-Whitney classes

For any real vector bundle $E \to X$ on any space X, we can define $w_i(E) \in H^i(X;\mathbb{Z}/2)$ for all $i \geq 0$. These satisfy:

(1) Dimension: $w_0(E) = 1$, $w_i(E) = 0$ for $i > \mathrm{rank}(E)$.

(2) Naturality: If $f\colon X \to Y$ and $E \to Y$ is a vector bundle, then $w_i(f^*(E)) = f^*(w_i(E))$ for all i.

(3) Whitney sum formula: For vector bundles E, F over X,

$$w_i(E \oplus F) = \sum_{j=0}^{i} w_j(E)\, w_{i-j}(F)\ .$$

(4) Non-triviality: Let L be the universal line bundle on $\mathbb{R}P^1$. Then

$$w_1(L) \neq 0 \in H^1(\mathbb{R}P^1;\mathbb{Z}/2) \cong \mathbb{Z}/2$$

The Stiefel-Whitney classes are uniquely determined by these properties. We will prove this later.

Properties

(i) Let E be the trivial bundle $X \times \mathbb{R}^n$. Then

$$E \cong f^*(pt \times \mathbb{R}^n) \text{ for a map } f\colon X \to pt$$

$H^i(pt, \mathbb{Z}/2) = 0$ for all $i > 0$, so

$$w_i(E) = f^*(w_i(pt \times \mathbb{R}^n)) = 0$$

(ii) For any $E \to X$ we have $w_i(E \oplus \mathbb{R}^n) = w_i(E)$. For

$$w_i(E \oplus \mathbb{R}^n) = \sum_{j=0}^{i} w_j(E) w_{i-j}(\mathbb{R}^n) = w_i(E) w_0(\mathbb{R}^n) = w_i(E)$$

by (i).

(iii) $T\mathbb{R}^{n+1}|\mathbb{S}^n = T\mathbb{S}^n \oplus N_{\mathbb{R}^{n+1}/\mathbb{S}^n}$. $T\mathbb{R}^{n+1}|\mathbb{S}^n$ and $N_{\mathbb{R}^{n+1}/\mathbb{S}^n}$ are both trivial, so for $i > 0$, $w_i(T\mathbb{S}^n) = 0$ by (i).

We say that E is stably trivial if $E \oplus \mathbb{R}^a \cong \mathbb{R}^{e+a}$ for some a. So $T\mathbb{S}^n$ is stably trivial, but not trivial in general.

Lemma. *Let $E \to X$ be a rank-n vector bundle. Suppose E has a non-zero section. Then $w_n(E) = 0$.*

More generally, if E has sections $s_1, \dots, s_a$, linearly independent at each point, then $w_n E = \cdots = w_{n-a+1}(E) = 0$.

Proof. If E has linear independent sections $s_1, \dots, s_a$ this gives a map $f\colon \mathbb{R}^a_X \hookrightarrow E$. Choose a metric on E. Then E splits orthogonally as

$$E \cong \mathbb{R}^a_X \oplus (E/\mathbb{R}^a_X)$$

So for $i > \operatorname{rank}(E/\mathbb{R}^a_X) = n - a$, $w_i(E) = w_i(E/\mathbb{R}^a_X) = 0$. □

Lecture 17

If $E \oplus F = \mathbb{R}^n_X$ and we know the classes $w_i E$ then we find $w_i F$ inductively:

$$\begin{aligned} 0 &= w_1(E) + w_1(F) & &\Rightarrow & w_1(F) &= -w_1(E) \\ 0 &= w_2(E) + w_1(E) w_1(F) + w_2(F) & &\Rightarrow & w_2(F) &= (w_1 E)^2 - w_2(E) \\ & & &\cdots & & \end{aligned}$$

For a vector bundle E we define the total Stiefel-Whitney class of E to be

$$w(E) = 1 + w_1(E) + w_2(E) + \cdots \in H^*(X; \mathbb{Z}/2)$$

With this notation the Whitney sum formula becomes

$$w(E \oplus F) = w(E) w(F)$$

So in the particular case $E \oplus F = \mathbb{R}^N_X$ we get $w(F) = (w(E))^{-1}$, which can be computed from a series formula (same computation as before).

Example. Let $L_{\mathbb{R}P^n}$ be the universal line bundle over $\mathbb{R}P^n$. $w_i(L) = 0$ for $i > \operatorname{rank} L = 1$.

Let $i\colon \mathbb{R}P^1 \to \mathbb{R}P^n$ be the standard inclusion. $L_{\mathbb{R}P^n}|\mathbb{R}P^1 = L_{\mathbb{R}P^1}$, so

$$i^*(w_1(L_{\mathbb{R}P^1})) = w_1(L_{\mathbb{R}P^1}) \neq 0$$

Hence $w_1(L_{\mathbb{R}P^n}) \neq 0$, so $w_1(L_{\mathbb{R}P^n})$ is the generator u of $H^1(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

On $\mathbb{R}P^n$ we can view L as a subbundle of the trivial bundle $\mathbb{R}^{n+1}$.

$$0 \to L \to \mathbb{R}^{n+1} \to L^\perp \to 0$$

So $\mathbb{R}^{n+1} \cong L \oplus L^\perp$. $w(L) = 1 + u$, so

$$w(L^\perp) = (1+u)^{-1} = 1 + u + \cdots + u^n \in H^*(\mathbb{R}P^n; \mathbb{Z}/2) .$$

So $L^\perp$ has all Stiefel-Whitney classes $w_0, \ldots, w_n$ nonzero.

Lemma. $T\mathbb{R}P^n \cong \mathrm{Hom}(L, L^\perp)$

Proof.

$$\mathbb{R}P^n \cong \mathbb{S}^n / \{\pm 1\}$$
$$T_x \mathbb{S}^n = \langle x \rangle^\perp$$

So
$$T_{\pm x}\mathbb{R}P^n = \{\{(x,v),(-x,-v)\} \colon \langle x, v\rangle = 0\} \xleftarrow[\phi]{\cong} \mathrm{Hom}(\langle x\rangle, \langle x\rangle^\perp)$$

where the identification $\phi \colon \alpha \mapsto \{(x, \alpha x), (-x, -\alpha x)\}$ is natural. Hence

$$T\mathbb{R}P^n \cong \mathrm{Hom}(L, L^\perp).$$

□

For a smooth manifold M we write $w_i(M)$ for $w_i(TM)$.
We have
$$\mathrm{Hom}(L, L^\perp) \oplus \mathrm{Hom}(L, L) \cong \mathrm{Hom}(L, \mathbb{R}^{n+1}) \cong (L^*)^{\oplus(n+1)}$$

$\mathrm{Hom}(L,L) \cong \mathbb{R}$, and for any real bundle E choosing a metric gives $E \cong E^*$. So

$$w(\mathbb{R}P^n) = w(\mathrm{Hom}(L, L^\perp)) = w(L^{\oplus(n+1)}) = w(L)^{n+1} = (1+u)^{n+1}$$

Lecture 18

In particular, $w(\mathbb{R}P^n) = 1$ iff $n = 2^k - 1$ for some $k \in \mathbb{N}$.

Corollary. *If $\mathbb{R}^n$ has the structure of a division algebra then n is a power of 2.*

Proof. If $\mathbb{R}^n$ is a division algebra then $\mathbb{R}P^{n-1}$ is parallelisable, so $w(\mathbb{R}P^{n-1}) = 1$. □

Question: For a manifold M of dimension n what is the smallest N such that M can be embedded in $\mathbb{R}^N$?

Whitney's theorem tells us that $N \le 2n$.

Actually it is easier to ask when M can be immersed in $\mathbb{R}^N$.

For a smooth map $f \colon M \to \mathbb{R}^N$ the differential $\mathrm{d}f$ can be considered as a map of bundles over M

$$\mathrm{d}f \colon TM \to f^*(T\mathbb{R}^N)$$

If f is an immersion then (by definition) $\mathrm{d}f$ is injective.

$f^*(T\mathbb{R}^N)$ is the trivial bundle $\mathbb{R}^N_M$. So if $f\colon M \to \mathbb{R}^N$ is an immersion then

$$TM \oplus (\mathbb{R}^N/TM) \cong \mathbb{R}^N$$

and

$$w(M)^{-1} = w(\mathbb{R}^N/TM) ,$$

which vanishes above dimension $N-n$ in $H^*(M;\mathbb{Z}/2)$.

Lemma. $\mathbb{R}P^{2^r}$ *does not immerse in* $\mathbb{R}^{2^{r+1}-2}$.

Proof.

$$w(\mathbb{R}P^{2^r}) = (1+u)^{2^r+1} = (1+u^{2^r})(1+u) = 1+u+u^{2^r}$$
$$w(\mathbb{R}P^{2^r}) = 1+u+\cdots+u^{2^r-1}$$

The result follows by the preceding argument. □

Remark. This is sort of the best possible example: Whitney proved that any M^n immerses in $\mathbb{R}^{2n-1}$.

Cobordism

For a smooth closed manifold M^n we can ask whether $M \cong \partial N^{n+1}$ for some compact N (with boundary). If there is such N we say that M bounds.

The ***Stiefel-Whitney numbers*** of a closed manifold M^n are the elements $\mathbb{Z}/2$ defined by

$$\int_M w_1(M)^{i_1} w_2(M)^{i_2} \cdots w_n(M)^{i_n} \in \mathbb{Z}/2 ,$$

where $i_1 + 2i_2 + \cdots + ni_n = n$. (So $w_1(M)^{i_1} \ldots w_n(M)^{i_n} \in H^n(M;\mathbb{Z}/2)$, and $\int_M$ is the isomorphism $H^n(M;\mathbb{Z}/2) \to \mathbb{Z}/2$ given by Poincaré duality).

Lecture 19

Theorem. *If M bounds then all Stiefel-Whitney numbers of M are* 0.

Proof. Assume $M^n = \partial N^{n+1}$, N compact with boundary.

$$TN|_M = TM \oplus N_{M/N}$$

The normal line bundle is trivial (taking the unit vector pointing "into" N gives a global section). So

$$w_j(TM) = w_j(i^*TN) = i^* w_j(TN) , \quad \forall j .$$

Any Stiefel-Whitney numbers have the form

$$\begin{aligned}\langle w_1(M)^{k_1} \cdots w_n(M)^{k_n}, [M]\rangle &= \langle i^*(w_1(N)^{k_1} \cdots w_n(N)^{k_n}), [M]\rangle \\ &= \langle w_1(N)^{k_1} \cdots w_n(N)^{k_n}, i_*[M]\rangle = 0\end{aligned}$$

since $i_*[M] = \partial[N] = 0 \in H_n(N)$. □

Examples. (i) A compact 0-manifold is a finite set of points and bounds iff the number of points is even.

(ii) A compact 1-manifold is a finite union of circles, so bounds.

(ii) For $\mathbb{R}P^2$, $w_1^2 = w_2 = u^2$, so the Stiefel-Whitney numbers are both 1. So $\mathbb{R}P^2$ does not bound.

Theorem. *If all Stiefel-Whitney numbers of M are 0 then M bounds*

Proof. Hard. □

Example. If $n = 2m - 1$ then $w(\mathbb{R}P^n) = (1+u)^{2m} = (1+u^2)^m$. So all odd Stiefel-Whitney classes are 0. Hence so is any product of degree n. So all Stiefel-Whitney numbers of $\mathbb{R}P^n$ vanish.

If n is even then $w_1(\mathbb{R}P^n) \neq 0$, so $w_1^n \neq 0$.

Grassmannians and universal bundles

Definition. $\mathrm{Gr}(a, \mathbb{R}^n) = \{a$ dimensional linear subspaces of $\mathbb{R}^n\}$

Let $X(a, \mathbb{R}^n) = \{(v_1, \dots, v_a) \in (\mathbb{R}^n)^* : v_1, \dots, v_a$ are linearly independent$\}$. $X(a, \mathbb{R}^n)$ is an open subset of $\mathbb{R}^{na}$, so has an obvious structure as a topological space and manifold. There is an obvious surjection

$$X(a, \mathbb{R}^n) \twoheadrightarrow \mathrm{Gr}(a, \mathbb{R}^n), \quad (v_1, \dots, v_a) \mapsto \mathrm{span}\{v_1, \dots, v_a\}$$

and we give $\mathrm{Gr}(n, \mathbb{R}^n)$ the quotient topology.

Let $Y(a, \mathbb{R}^n) \subseteq X(a, \mathbb{R}^n)$ be the orthonormal a-triples. We have

$$Y(a, \mathbb{R}^n) \twoheadrightarrow \mathrm{Gr}(a, \mathbb{R}^n)$$

and this gives the same quotient topology. Y is compact, so so is $\mathrm{Gr}(a, \mathbb{R}^n)$.

Lemma. $\mathrm{Gr}(a, \mathbb{R}^n)$ *is a closed manifold of dimension $a(n-a)$.*

Proof. Let W be a dimension $n-a$ subspace of $\mathbb{R}^n$, and let

$$U = \{V \subseteq \mathbb{R}^n : \dim V = a, V \cap W = \{0\}\} \subseteq \mathrm{Gr}(a, \mathbb{R}^n).$$

Then we can identify

$$\mathbb{R}^{a(n-a)} \cong \mathrm{Hom}(\mathbb{R}^a, \mathbb{R}^{n-a}) \longleftrightarrow U; \quad f \mapsto \{(x, f(x)) : x \in \mathbb{R}^a\}$$

These charts U have smooth transitional functions. □

The universal bundle E on $\mathrm{Gr}(a, \mathbb{R}^n)$ is a rank-a vector bundle, whose fibre over $V \in \mathrm{Gr}(n, \mathbb{R}^n)$ is precisely $V \subseteq \mathbb{R}^n$, $E \subseteq \mathrm{Gr} \times \mathbb{R}^n$.

Lecture 20

Lemma. *If $f, g\colon X \to Y$ are homotopic and E is a vector bundle on Y then $f^*E \cong g^*E$.*

Proof. We assume that X is a CW-complex. (We have shown that fibre bundles have the homotopy lifting property w.r.t. maps from CW-complexes. In fact this holds for all maps, but the proof is more complicated).

We have a homotopy $F\colon X \times [0,1] \to Y$, which gives a pull-back bundle F^*E on $X \times [0,1]$. We need to show $F^*E|X \times \{0\} \cong F^*E|X \times \{1\}$.

Let $V = \varphi^* f^* E$ on $X \times [0,1]$, where $\varphi\colon X \times [0,1] \to X$ is the obvious projection. $V|X \times \{t\} \cong f^*E$ for all t, so STP $V \cong F^*E$.

Let $I = \{(x,\alpha)\colon x \in X \times [0,1], \alpha \in \mathrm{Iso}(V_x, F^*E_X)\}$. I is a fibre bundle on $X \times [0,1]$ with typical fibre $GL(n,\mathbb{R})$.

Let $i_t\colon X \times [0,1]$, $x \mapsto (x,t)$. There is an obvious homotopy $i_0 \sim i_t$.

Since $V|X \times \{0\} \cong F^*E|X \times \{0\}$, there is an obvious lifting of i_0 to I. By the homotopy lifting property, there is also a lifting of i_1 to I. A lift of i_1 is precisely an isomorphism $V|X \times \{1\} \cong F^*E|X \times \{1\}$, as desired. □

Theorem. *Let X be a compact Hausdorff space. Then there is a natural bijection*

$$[X, \mathrm{Gr}(n,\mathbb{R}^\infty)] \longleftrightarrow \{\textit{Isomorphism classes of rank-}n\textit{ vector bundle over }X\}$$

identifying $[f] \in [X, \mathrm{Gr}(n,\mathbb{R}^\infty)]$ with f^*E_n. Here

$$\mathbb{R}^\infty = \bigcup_{a \geq 0} \mathbb{R}^a = \{(x_1, x_2, \dots)\colon x_i \in \mathbb{R},\ \textit{only finitely many } x_i \neq 0\}$$
$$\mathrm{Gr}(n,\mathbb{R}^\infty) = \{n \textit{ dimensional subspaces of} \mathbb{R}^\infty\} = \bigcup_{a \geq n} \mathrm{Gr}(n,\mathbb{R}^a)$$

and E_n is the universal rank-n bundle on $\mathrm{Gr}(n,\mathbb{R}^\infty)$. $[X, \mathrm{Gr}(n,\mathbb{R}^\infty]$ is the set of homotopy classes of maps $X \to \mathrm{Gr}(n,\mathbb{R}^\infty)$.

Proof. By the previous result, $[f] \mapsto f^*E_n$ is a well-defined function. Let $E \to X$ be a rank-n bundle. Suppose that we can find an injective map of vector bundles $\varphi\colon E \to \mathbb{R}^N$. Define $f\colon X \to \mathrm{Gr}(n,\mathbb{R}^N)$ by $f(x) = \mathrm{im}(\varphi\colon E_x \to \mathbb{R}^N)$. Obviously $f^*E_n \cong E$.

Now we want to find φ. Let U_α be a finite cover of X by trivialising neighbourhoods, and let ρ_α be a partition of unity subordinate to U_α. The local trivialisations $\phi_\alpha\colon E_{U_\alpha} \to \mathbb{R}^n$ can be extended to maps $\rho_\alpha \cdot \phi_\alpha\colon E \to \mathbb{R}^n$, and $\bigoplus_\alpha \rho_\alpha\phi_\alpha\colon X \to \bigoplus_\alpha \mathbb{R}^n$ is injective. It follows that $[f] \mapsto f^*E$ is surjective.

It remains to show that the map is injective, i.e. that if $f, g\colon X \to \mathrm{Gr}(n,\mathbb{R}^\infty)$ have $f^*E_n \cong g^*E_n$ then f and g are homotopic. Let $E = f^*E \cong g^*E$. f and g define embeddings $s, t\colon E \to \mathbb{R}^N$ (since X is compact, f, g both map into $\mathrm{Gr}(n,\mathbb{R}^N)$ for some $N < \infty$).

Now think of s as an embedding $E \hookrightarrow \mathbb{R}^N \oplus 0 \subseteq \mathbb{R}^{2N}$ and t as $E \hookrightarrow 0 \oplus \mathbb{R}^N \subseteq \mathbb{R}^{2N}$. We can define a homotopy $F\colon E \times [0,1] \to \mathbb{R}^{2N}$ from s to t by $F(-,u) = us + (1-u)t$. The corresponding homotopy $G\colon X \times [0,1] \to \mathrm{Gr}(n,\mathbb{R}^{2N})$ is a homotopy from f to g. □

Lecture 21

Remark. The proof works with slight modification for X paracompact.

As a particular case of the theorem, we have

$$\{\text{line bundles on } X\} = [X, \mathrm{Gr}(1,\mathbb{R}^\infty)] = [X, \mathbb{R}P^\infty] = [X, K(\mathbb{Z}/2,1)] = H^1(X;\mathbb{Z}/2)$$

Theorem (Leray-Hirsch). *Let $F \to E \to B$ a fibre bundle, R a commutative ring. Suppose that there are elements $x_1,\dots,x_n \in H^*(E;R)$ such that the restrictions of x_i to $H^*(F;R)$ form a basis for $H^*(F;R)$ as a free R-module (i.e. $H^*(F;R) = \bigoplus_i R\cdot x_i|F$) for all fibres F. Then $H^*(E;R)$ is a free module over $H^*(B;R)$ with basis $x_1,\dots,x_n$.*

For B compact, we prove this by induction on the number of open sets in an open cover of B.

In particular, let E be a rank-n vector bundle on X. Let

$$P(E) = \{(x,L)\colon\ x\in X, L\subseteq E_x \text{ dim 1 linear subspace}\}.$$

$P(E)$ is an $\mathbb{R}P^{n-1}$-bundle on X. There is a natural line bundle L on $P(E)$, whose fibre over (x,L) is L.

Intermission. We define the first Stiefel-Whitney class of a line bundle $L\to X$ to be the image of L under the identification

$$\{\text{line bundles}\} \longleftrightarrow H^1(X;\mathbb{Z}/2)$$

Explicitly, if f is the (unique up to homotopy) map $X\to\mathbb{R}P^\infty$ such that $L = f^*E_1$, then $w_1(L) = f^*(u)$ where u is the generator of $H^1(\mathbb{R}P^\infty;\mathbb{Z}/2)\cong\mathbb{Z}/2$.

w_1 thus constructed obviously satisfies the naturality condition

$$w_1(g^*L) = g^*w_1(L)$$

$L|P(E)_x$ is the universal line bundle on $\mathbb{R}P^{n-1}$, so

$$(w_1(L))|P(E)_x = w_1(L|P(E)_x) = u \in H^1(\mathbb{R}P^\infty;\mathbb{Z}/2)$$

Let $w = w_1(L)$. Consider $1, w, \dots, w^{n-1}\in H^*(P(E);\mathbb{Z}/2)$. Their restrictions to a fibre $\mathbb{R}P^{n-1}$ are $1, u,\dots,u^{n-1}$ which form a basis for $H^*(\mathbb{R}P^{n-1};\mathbb{Z}/2)$.

So by the theorem,

$$\begin{gathered} H^*(P(E);\mathbb{Z}/2) \text{ is a free module over} \\ H^*(X;\mathbb{Z}/2) \text{ with basis } 1, w,\dots,w^{n-1}. \end{gathered} \tag{A}$$

Let $\pi\colon P(E)\to X$ be the projection map.

$$\begin{tikzcd} \pi^*E \arrow[r] \arrow[d] & E \arrow[d] \\ P(E) \arrow[r] & X \end{tikzcd}$$

The pull-back map $H^*(X;\mathbb{Z}/2)\to H^*(P(E);\mathbb{Z}/2)$ is injective by (A).

Moreover, L is a subbundle of π^*E (obviously). So (choosing an arbitrary metric)

$$\pi^*E = L\oplus E'$$

where E' is a rank-$(n-1)$ vector bundle on $P(E)$. Repeating, we get

Theorem (Splitting principle)**.** *For any vector bundle $E \to X$, there is a map $f\colon Y \to X$ such that*

(1) $f^*E = L_1 \oplus L_n$ for line bundles L_i on Y.

(2) $f^*\colon H^*(X;\mathbb{Z}/2) \to H^*(Y;\mathbb{Z}/2)$ is injective

Corollary. *The Stiefel-Whitney classes are determined uniquely by the axioms (if they exist).*

Proof. The axioms force the definition of $w_1(L)$ for a line bundle L, since any L is pulled back from $E_1 \to \mathbb{R}P^\infty$ and $w_1(E_1) = u$.

Next, the axioms imply

$$w(L_1 \oplus \cdots \oplus L_n) = w(L_1)\cdots w(L_n)$$

for any line bundles $L_1, \dots, L_n$ (where $w(L_i) = 1 + w_1(L_i)$).

Finally, for an arbitrary bundle $E \to X$, let $f\colon Y \to X$ be the map from the splitting principle. $w(f^*E) = f^*w(E)$ is uniquely determined, and since f^* is injective so is $w(E)$. □

Finally, we show that there really are classes satisfying the axioms.

Let E be a rank-n vector bundle on X. We know that

$$H^*(P(E),\mathbb{Z}/2) = \bigoplus_{i=0}^{n-1} w^i H^*(X;\mathbb{Z}/2)$$

So we must have

$$w^n = \sum_{i=0}^{n-1} a_{n-i} w^i$$

where $a_j \in H^j(X;\mathbb{Z}/2)$. We define

$$w_j(E) = \begin{cases} 1 & j = 0 \\ a_j & j = 1, \dots, n \\ 0 & j > n \end{cases}$$

We need to check that this satisfies the axioms.

If L is a line bundle then $P(L) = X$ and by definition $w \in H^1(P(E),\mathbb{Z}/2)$ is $w_1 L \in H^1(X;\mathbb{Z}/2)$ (for our line bundle definition of w_1). So the new definitions agrees with the definition of w_1 for line bundles. In particular, $w_1(E_1) = u$, as desired.

Naturality is easy.

Lecture 22

RTP that the Whitney sum formula is satisfied. We start by showing that if $L_1, \dots, L_n$ are line bundles on X and $E = L_1 \oplus \cdots \oplus L_n$ then

$$w(E) = w(L_1)\cdots w(L_n)$$

There are maps $f_1, \dots, f_n\colon X \to \mathbb{R}P^\infty$ such that $L_i \cong f_i^* E_1$. Consider $(f_1, \dots, f_n)\colon X \to (\mathbb{R}P^\infty)^n$. Then $L_1, \dots, L_n$ are pulled back from $K_1, \dots, K_n$, where $K_i = \pi_i^* E_1$, the pull-back under the projection $\pi_i\colon (\mathbb{R}P^\infty)^n \to \mathbb{R}P^\infty$.

By naturality it suffices to prove that

$$w(F) = w(K_1)\cdots w(K_n) \in H^*((\mathbb{R}P^\infty)^n;\mathbb{Z}/2)$$

where $F = K_1 \oplus \cdots \oplus K_n$.

We know that

$$H^*((\mathbb{R}P^\infty)^n;\mathbb{Z}/2) = (\mathbb{Z}/2)[x_1,\ldots,x_n]$$

where $x_i = w_1(K_i)$.

Consider the projective bundle $P(F) \to (\mathbb{R}P^\infty)^n$.

We have an element $u \in H^1(P(F);\mathbb{Z}/2)$ which is $w_1(L_{univ})$

$$H^*(P(F);\mathbb{Z}/2) = \bigoplus_{i=0}^{n-1} u^i H^*((\mathbb{R}P^\infty)^n;\mathbb{Z}/2)$$

By definition,

$$u^n = \sum_{i=0}^{n-i} w_{n-i}(F)u^i$$

So

$$\sum_{i=0}^{n} w_{n-i}(F)u^i = 0$$

The bundle $P(F) \to (\mathbb{R}P^\infty)^n$ has n obvious sections s_i corresponding to the n line bundles $K_i \subseteq F$. Pulling back (restricting to the j-th section)

$$\sum_{i=0}^{n} w_{n-i}(F)(s_j^* u)^i = 0 \in H^*((\mathbb{R}P^\infty)^n;\mathbb{Z}/2)$$

Now

$$s_j^* u = w_1(s_j^* L_{univ}) = w_1 K_j = x_j \in H^*((\mathbb{R}P^\infty)^n;\mathbb{Z}/2)$$

So for $j = 1,\ldots,n$,

$$0 = \sum_{i=0}^{n} w_{n-i}(F)x_j^i \in H^*((\mathbb{R}P^\infty)^n;\mathbb{Z}/2)$$

Let

$$f(t) = \sum_{i=0}^{n} w_{n-i}(F)t^i \in (\mathbb{Z}/2)[x_1,\ldots,x_n][t]$$

This is a monic degree n polynomial. We know $f(x_j) = 0 \quad \forall j$, so since $(\mathbb{Z}/2)[x_1,\ldots,x_n]$ is an integral domain

$$f(t) = (t - x_1)\cdots(t - x_n)$$

In particular,

$$w(F) = f(1) = (1 + x_1)\cdots(1 + x_n) = w(K_1)\cdots w(K_n).$$

Finally we show that $w(E \oplus F) = w(E)w(F)$ for all $E, F \to X$.

The splitting principle gives $f\colon Y \to X$ such that f^*E, f^*F are direct sums of line bundles and $f^*\colon H^*(X;\mathbb{Z}/2) \to H^*(Y;\mathbb{Z}/2)$ is injective. So

$$f^* w(E \oplus F) = w(f^*E \oplus f^*F) = w(f^*E)w(f^*F) = f^*(w(E)w(F))$$

and by injectivity, $w(E \oplus F) = w(E)w(F)$. □

Recall that isomorphism classes of rank-n vector bundles over X correspond to elements of $[X, \mathrm{Gr}(n,\mathbb{R}^\infty)]$. Note that

$$\mathrm{Gr}(n,\mathbb{R}^\infty) = \mathrm{Emb}(\mathbb{R}^n,\mathbb{R}^\infty)\big/GL(n,\mathbb{R})$$

$\mathrm{Emb}(\mathbb{R}^n,\mathbb{R}^\infty)$ is contractible (proof similar to contractibility of $\mathbb{S}^\infty$).

Definition. For a topological group G let EG be any contractible space with free G-action. Then the ***classifying space*** of G is $BG = EG/G$.

So $\mathrm{Gr}(n,\mathbb{R}^\infty) = BGL(n,\mathbb{R})$.

We have a fibre bundle

$$G \to EG \to BG$$

By the long exact sequence of homotopy groups it follows that

$$\pi_i BG \cong \pi_{i-1} G$$

(in particular BG is connected)

This gives some information about classifying vector bundles on spheres,

$$[\mathbb{S}^b, \mathrm{Gr}(n,\mathbb{R}^\infty)] = [\mathbb{S}^b, BO(n)] = \pi_b BO(n) = \pi_{b-1} O(n)$$

Example. $\pi_0 O(n) = \mathbb{Z}/2$ for all $n \geq 1$, so there are exactly two isomorphism classes of rank-n vector bundles on $\mathbb{S}^1$. The Möbius line bundle L has $w_1 L \neq 0$, so the trivial bundle $\mathbb{R}^n_{\mathbb{S}^1}$ and $L \oplus \mathbb{R}^{n-1}_{\mathbb{S}^1}$ are not isomorphic, so they represent the two classes.

For a topological group G there always is a contractible EG, and the homotopy type of BG is independent of choice of EG. So BG is well-defined up to homotopy equivalence.

Looping the natural fibration $G \to EG \to BG$ gives

$$\Omega BG \to G \to EG \simeq pt,$$

so $G \simeq \Omega BG$. Apart from details

$$\{\text{connected topological spaces}\} \underset{B}{\overset{\Omega}{\rightleftarrows}} \{\text{topological groups}\}$$

Lecture 23

Definition. A ***principal*** G***-bundle*** over a space B is a space E with a free G-action such that $B = E/G$. (G is a topological group.)

Thus $G \to E \to B$ is a fibre bundle. E is trivial if $E \cong B \times G$ with the obvious G-action. If E is contractible it is called universal, then $B = E/G = BG$.

Definition. Given a principal G-bundle $E \to B$ and an action of G on a space F, the associated F-bundle over B is $(E \times F)/G$, where $g(e,f) = (g(e), g(f))$. This is a fibre bundle

$$F \to (E \times F)/G \to E/G = B\ .$$

Example. Given a principal $GL(n,\mathbb{R})$-bundle the associated $\mathbb{R}^n$-bundle corresponding to the usual representation of $GL(n,\mathbb{R})$ on $\mathbb{R}^n$ is a vector bundle. This gives an equivalence between $GL(n,\mathbb{R})$-bundles on B and rank-n vector bundles on B.

For a vector bundle $V \to B$, the corresponding principal bundle is

$$E = \bigcup_{x\in B} \mathrm{Iso}(\mathbb{R}^n, V_x)$$

with G-action $g(\varphi) = \varphi \circ g$.

Theorem. *Let G be a topological group. Then there is a bijection*

$$\{\textit{isomorphism classes of principal } G\textit{-bundles on } X\} \longleftrightarrow [X, BG]$$

Sketch proof. Given a map $f\colon X \to BG$ the pull-back of $EG \to BG$ by f is a principal bundle on X.

The isomorphism class of f^*EG depends only on the homotopy type of f.

For a principal G-bundle $E \to X$, construct a map $X \to BG$ by the following diagram

$$\begin{tikzcd} E \arrow[r] & (E\times EG)/G \arrow[r] \arrow[d, shift left] & BG \\ X \arrow[r, equal] & E/G \arrow[u, shift left, "\simeq"] \arrow[ur] & \end{tikzcd}$$

□

The construction shows that for any principal G-bundle $G \to E \to X$, there is a fibration $E \to X \to BG$. Looping this gives $G \to E \to X$ again.

Definition. A ***characteristic class*** (with values in $H^i(-;R)$) for principal G-bundles is an assignment to every principal G-bundle $E \to X$ of an element $\alpha(E) \in H^i(X;R)$ such that for every $f\colon Y \to X$

$$\alpha(f^*E) = f^*(\alpha(E))$$

Lemma. *There is a one-one correspondence between $H^i(BG;R)$ and the set of characteristic classes for principal G-bundles with values in $H^i(X;R)$.*

Sketch proof. Given any characteristic class α apply α to the universal G-bundle $G \to EG \to BG$ to get $\alpha(EG) \in H^i(BG;R)$. Conversely, given an element $\alpha \in H^i(BG;R)$ and any principal G-bundle f^*EG define $\alpha(f^*EG) = f^*\alpha$. □

Example.

$$H^*(BO(n);\mathbb{Z}/2) \cong (\mathbb{Z}/2)[w_1E,\ldots,w_nE]$$

where E is the universal rank-n vector bundle on $BO(n)$ (same proof as for $U(n)$, see below).

What is $H^*(BO(n);\mathbb{Z})$? It is easier to first look at $H^*(BU(n);\mathbb{Z})$.

Principal $GL(n,\mathbb{C})$-bundles $\longleftrightarrow$ complex vector bundles

Principal $U(n)$-bundles $\longleftrightarrow$ complex vector bundles with Hermitian product

Since $U(n) \simeq GL(n,\mathbb{C})$ we have $BU(n) \simeq BGL(n,\mathbb{C})$ and

$$H^*(BU(n);\mathbb{Z}) = H^*(BGL(n,\mathbb{C});\mathbb{Z}).$$

First, what is $H^*(BU(1);\mathbb{Z})$? $U(1) \cong \mathbb{S}^1$, $BU(1) = B\mathbb{S}^1 = \mathbb{C}P^\infty$. There is a universal (tautological) line bundle $L = \mathscr{O}(-1)$ on $\mathbb{C}P^\infty$.

$$H^*(BU(1);\mathbb{Z}) = H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[u], \quad u \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$$

So for any complex line bundle $f^*(\mathscr{O}(1))$, we define the ***first Chern class***

$$c_1(f^*(\mathscr{O}(1))) = f^*u \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$$

Lemma. *For any CW-complex X, discrete Abelian group A and $n \geq 1$,*

$$[X, K(A,n)] = H^n(X;A)$$

Sketch proof. Construct maps $X \to K(A,n)$ cell-by-cell. □

Corollary. *For any space X,*

$$c_1 \colon \{\textit{isomorphism classes of } \mathbb{C}\textit{-line bundle on } X\} \xrightarrow{\cong} H^2(X;\mathbb{Z})$$

Proof. $\mathbb{C}P^\infty = K(\mathbb{Z},2)$, so both sets correspond to $[X,\mathbb{C}P^\infty]$. □

Theorem. *For any complex vector bundle $E \to X$, we can define $c_i(E) \in H^2(X;\mathbb{Z})$ for all $i \geq 0$, called the **Chern classes** of E. They satisfy*

(1) $c_0(E) = 1$, $c_i(E) = 0$ for $i > \operatorname{rank}_{\mathbb{C}} E$.

(2) $c_i(f^*E) = f^*(c_i(E))$.

(3) The total Chern class $c(E) = \sum_i c_i(E) \in H^*(X;\mathbb{Z})$ has

$$c(E \oplus F) = c(E)c(F)$$

(4) The complex line bundle $\mathscr{O}(1)$ on $\mathbb{C}P^1$ has $c_1(\mathscr{O}(1)) = u$, the standard generator for $H^2(\mathbb{C}P^1;\mathbb{Z})$.[3]

Remark. The Chern classes are determined by these axioms.

$$H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1(E),\ldots,c_n(E)]$$

where E is the universal rank-n vector bundle over $BU(n)$.

Lecture 24

Let $p\colon E \to X$ be complex rank-n vector bundle and $\pi\colon P(E) \to X$ the corresponding complex projective bundle. There is an obvious universal $\mathbb{C}$-line bundle L on $P(E)$.

Let $u = -c_1(L) \in H^2(P(E);\mathbb{Z})$. Clearly u restricted to fibre $\mathbb{C}P^{n-1}$ is the standard generator for $H^2(\mathbb{C}P^{n-1};\mathbb{Z}) \cong \mathbb{Z}$. The elements $1, u, \ldots, u^{n-1} \in H^*(P(E);\mathbb{Z})$ restrict to a basis for $H^*(\mathbb{C}P^{n-1};\mathbb{Z}) \cong \mathbb{Z}[u]/(u^n)$ as a free $\mathbb{Z}$-module. So by Leray-Hirsch,

$$H^*(P(E);\mathbb{Z}) = \bigoplus_{i=0}^{n-1} u^i H^*(X;\mathbb{Z})\,.$$

[3] i.e. $\int_{\mathbb{C}P^1} c_1(\mathscr{O}(1)) = 1$. Note that $\mathscr{O}(1)$ is the dual of the tautological line bundle $\mathscr{O}(-1)$.

Note that L is a subbundle of $\pi^* E$. Using an arbitrary Hermitian metric on $\pi^* E$ we get a splitting

$$\pi^* E = L \oplus E' ,$$

where E' has rank $n-1$. Iterating gives

Theorem (Complex Splitting Principle)**.** *For any complex vector bundle E on a paracompact space X there is a map $f\colon Y \to X$ such that*

(1) $f^* E$ is a direct sum of line bundles on Y, and

(2) $f^*\colon H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ is injective (in fact split injective, i.e. the image is a direct summand of $H^*(Y;\mathbb{Z})$).

To construct the Chern classes we first compute $H^*(BU(n);\mathbb{Z})$: Let $E \to BU(n)$ be the universal vector bundle on $BU(n)$, and let Y be the corresponding space given by the splitting principle. We can describe Y explicitly: The fibre of $Y \to BU(n)$ over X is

$$\{(L_1,\dots,L_n)\colon L_i \text{ a } \mathbb{C}\text{-line in } E_x, E = L_1 \oplus \cdots \oplus L_n, \\ L_i \text{ pairwise orthogonal w.r.t. some fixed metric on } E\}$$

This fibre is $U(n)/T$, where $T \subseteq U(n)$ is the torus of diagonal unitary matrices.

We can identify the “splitting map” for $BU(n)$ as

$$U(n)/T \to BT \to BU(n)$$

(for any $H < G$ topological groups there is a fibration $G/H \to BH \to BG$), so

$$H^*(BU(n);\mathbb{Z}) \subseteq H^*(B((\mathbb{S}^1)^n);\mathbb{Z}) = H^*((\mathbb{C}P^\infty)^n;\mathbb{Z}) = \mathbb{Z}[t_1,\dots,t_n] ,$$

where each t_i has degree 2.

Thus any characteristic class α for a rank-n $\mathbb{C}$-vector bundle is determined by $\alpha(L_1 \oplus \cdots \oplus L_n)$, which must be a polynomial with $\mathbb{Z}$-coefficients in $t_1,\dots,t_n$ $(t_i = c_1(L_i))$

For any $\sigma \in S_n$, we must have

$$\alpha(L_1 \oplus \cdots \oplus L_n) = \alpha(L_{\sigma(1)} \oplus \cdots \oplus L_{\sigma(n)}) ,$$

so

$$H^*(BU(n);\mathbb{Z}) \subseteq \mathbb{Z}[t_1,\dots,t_n]^{S_n} = \mathbb{Z}[e_1,\dots,e_n] ,$$

where $e_1,\dots,e_n$ are the elementary symmetric functions of $t_1,\dots,t_n$.

In fact equality holds. This follows by counting dimensions: For any space X, define the ***Poincaré series*** to be

$$p(X) = \sum_{i\geq 0} b_i(X) t^i .$$

The Leray-Hirsch theorem gives

$$\begin{aligned} p(B((\mathbb{S}^1)^n)) &= p(U(n)/T)p(BU(n)) \\ &= p(\mathbb{C}P^1)p(\mathbb{C}P^2)\cdots p(\mathbb{C}P^{n-1})p(BU(n)) \\ &= (1+t^2)(1+t^2+t^4)\cdots(1+t^2+\cdots+t^{2n-2})p(BU(n)) \end{aligned}$$

Also

$$p(B((\mathbb{S}^1)^n)) = \frac{1}{(1-t^2)^n}$$

Thus

$$p(BU(n)) = \frac{1}{(1-t^2)(1-t^4)\cdots(1-t^{2n})} = p(\mathbb{Z}[e_1,\ldots,e_n])$$

where $\deg e_i = 2i$. So

$$H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[e_1,\ldots,e_n]$$

Definition. We define $c_i(E_{univ}) = e_i$, $c_i(f^*E_{univ}) = f^*e_i$.

Example Sheet 1

(1) Show that the 2-torus minus a point deformation retracts onto the wedge $\mathbb{S}^1 \vee \mathbb{S}^1$. (In general, for pointed spaces (X, x_0) and (Y, y_0), $X \vee Y$ denote the pointed space obtained from the disjoint union $X \amalg Y$ by identifying x_0 and y_0.)

(2) (a) For a triple (X, A, x_0) (X is a space, A is a subspace, x_0 is a point in A), prove the exactness of the relative homotopy sequence

$$\cdots \to \pi_n(A, x_0) \to \pi_n(X, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0) \to \dots$$

(For example, write out the proof of exactness at $\pi_n A$ in detail, and make sure you can work out the remaining cases as well.)

(b) The end of the above sequence is slightly different, because it involves pointed sets, but is also easier to understand geometrically. Thus, check by hand that the sequence

$$\pi_1(X, x_0) \to \pi_1(X, A, x_0) \to \pi_0(A, x_0) \to \pi_0(X, x_0)$$

is exact.

(3) Show that any compact subspace of a CW-complex is contained in a finite subcomplex.

(4) In the context of homotopy theory, $\mathbb{S}^\infty$ denotes the CW complex $\mathbb{S}^\infty = \bigcup_{n \geq 0} \mathbb{S}^n$, where $\mathbb{S}^0 \subset \mathbb{S}^1 \subset \mathbb{S}^2 \subset \dots$ in the natural way. Show that $\mathbb{S}^\infty$ is contractible (a) using Whitehead's theorem plus question (3), and (b) directly.

(5) By definition, an ***H*-space** is a pointed space (X, e) together with a "multiplication" map $\mu \colon X \times X \to X$ such that the two maps $X \to X$ given by $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are both homotopic to the identity through maps $(X, e) \to (X, e)$. For example, a topological group X with identity element e is an H-space; in that case, the two maps $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are actually equal to the identity map.

(a) For an H-space X, show that the group operation in $\pi_n(X, e)$ can also be defined by the rule $(fg)(x) = \mu(f(x), g(x))$.

(b) Perhaps using (a), show that the fundamental group $\pi_1(X, e)$ of an H-space is Abelian.

(6) By definition, a pointed space X is n-connected if $\pi_i(X, x_0) = 0$ for $0 \leq i \leq n$. Show that an n-connected, n-dimensional CW complex is contractible.

Example Sheet 2

(1) Show that for any Abelian group G and any $n \geq 2$, there is a $K(G,n)$ CW-complex. (The construction was given in class for $G = \mathbb{Z}$. In general, it seems helpful to use the Hurewicz theorem.)

(2) You may use the following relative version of Hurewicz theorem: if a pair (X, A) of spaces is $(n-1)$-connected, $n \geq 2$, with A simply connected and nonempty, then $H_i(X, A) = 0$ for $i < n$ and the Hurewicz map $\pi_n(X, A) \to H_n(X, A; \mathbb{Z})$ is an isomorphism.

Show that a map $f\colon X \to Y$ between simply-connected CW-complexes X and Y such that $f_*\colon H_i(X;\mathbb{Z}) \to H_i(Y;\mathbb{Z})$ is an isomorphism for all $i \geq 0$ is a homotopy equivalence.

(3) Show that if X is a simply connected CW-complex with the same homology groups as $\mathbb{S}^n$, $n \geq 2$, then X is homotopy equivalent to $\mathbb{S}^n$. (Use problem (2))

(4) The unitary group $U(n)$ is defined as the group of $\mathbb{C}$-linear automorphism of $\mathbb{C}^n$ which preserves the length of all vectors, with length defined as $|(z_1, \dots, z_n)| := \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. You may use that the map from $U(n)$ to the sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ defined by $A \mapsto A(1,0,\dots,0)$ is a fibre bundle, with fibre $U(n-1)$.

   (a) Show that the group $GL(n,\mathbb{C})$ of $\mathbb{C}$-linear automorphisms of $\mathbb{C}^n$ is homotopy equivalent to $U(n)$.

   (b) Compute $\pi_i U(n)$ for $i \leq 3$ and any n.

(5) Check that the map $f\colon O(n) \to \mathbb{S}^{n-1}$ defined by $A \mapsto A(1,0,\dots,0)$ is a fibre bundle with fibre $O(n-1)$. (Hint: begin by finding sections of f over open subsets of $\mathbb{S}^{n-1}$ which cover $\mathbb{S}^{n-1}$.)

(6) We can obtain $\mathbb{C}P^\infty$ from $\mathbb{S}^2$ by adding cells of dimensions 4, 6, and so on. So there is a natural inclusion map from $\mathbb{S}^2$ to $\mathbb{C}P^\infty$. Show that the homotopy fibre of this inclusion is homotopy equivalent to $\mathbb{S}^3$. (Hint: first try to construct a map from $\mathbb{S}^3$ to this homotopy fibre. You may use the fact that the homotopy fibre of a map between CW-complexes is always homotopy equivalent to a CW-complex.)

What do you get by looping the resulting fibration? (Looping means passing from a fibration $F \to E \to B$ to the fibration $\Omega B \to F \to E$.)

Example Sheet 3

As in the course, you may assume that whatever spaces come up are paracompact, and homotopy equivalent to CW-complexes.

(1) Let L, M be complex line bundles on a space X. Then $L \otimes M$ is also a complex line bundle (with fibre at $x \in X$ equal to $L_x \otimes_{\mathbb{C}} M_x$). Show that

$$c_1(L \otimes M) = c_1(L) + c_1(M)$$

(Hint: reduce to a universal case, with $X = (\mathbb{C}P^\infty)^2$. Argue that there must be a formula $c_1(L \otimes M) = ac_1(L) + bc_1(M)$ for some integers a and b. Evaluate a and b by considering special cases.)

(2) Let E be a complex line bundle over a space X, and E^* be the dual bundle (with fibre at $x \in X$ equal to $(E_x)^* = \mathrm{Hom}_{\mathbb{C}}(E_x, \mathbb{C})$). Show that $c_i(E^*) = (-1)^i c_i(E)$. (Hint: use the splitting principle for complex vector bundles.)

(3) Let E be a complex vector bundle over a space X. Let $E_{\mathbb{R}}$ denote the same bundle, but now considered only as a real vector bundle. Show that the odd Stiefel-Whitney classes $w_{2i+1}(E_{\mathbb{R}})$ are zero, while $w_{2i}(E_{\mathbb{R}})$ is the image of the Chern class $c_i(E)$ under the natural map $H^{2i}(X;\mathbb{Z}) \to H^{2i}(X;\mathbb{Z}/2)$.

(4) I mentioned that for a CW-complex X, an Abelian group A and any $n \geq 1$, there is a natural isomorphism

$$[X, K(A,n)] \cong H^n(X;A).$$

(There is a natural “universal” element of $H^n(K(A,n);A)$, and pulling this back by a map $X \to K(A,n)$ gives an element of $H^n(X;A)$.)

Prove the following special case of the statement: given a CW-complex X, $n \geq 1$, and an element $\alpha \in H^n(X;\mathbb{Z})$, show that α is pulled back from the generator of $H^n(K(\mathbb{Z},n);\mathbb{Z}) = \mathbb{Z}$ by some map $X \to K(\mathbb{Z},n)$. (Hint: construct a cellular map, by induction on the skeleta of X. It helps to interpret $H^n(X;\mathbb{Z})$ as the cohomology of the cellular cochain complex

$$\cdots \to \prod_{(n-1)\text{-cells}} \mathbb{Z} \to \prod_{n\text{-cells}} \mathbb{Z} \to \prod_{(n+1)\text{-cells}} \mathbb{Z} \to \ldots.)$$

(5) Show that any involution of $\mathbb{R}^n$ (continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f^2 = \mathrm{id}$) must have a fixed point. (Hint: If not, we have a free action of $\mathbb{Z}/2$ on $\mathbb{R}^n$. What can you say about the cohomology of the quotient space?)

(6) Let p be a prime number. Apply the Leray-Hirsch theorem to the fibre bundle $\mathbb{S}^1 \to \mathbb{S}^\infty/(\mathbb{Z}/p) \to \mathbb{C}P^\infty$ to compute $H^*(B\mathbb{Z}/p;\mathbb{Z}/p)$. (The Leray-Hirsch theorem by itself does not give the ring structure completely; for that you will need different methods depending on whether $p = 2$ or not.)

(7) Let E be a real vector bundle of rank n. Show that $w_1(E) = w_1(\Lambda^n(E))$. Here $\Lambda^a(E)$ denotes the ath exterior power of E, which has rank $\binom{n}{a}$. Thus $\Lambda^n(E)$ is a line bundle, called the

determinant line bundle of E. (Depending on how you do this problem, you may need formulæ from linear algebra like

$$\Lambda^a(V \oplus W) = \bigoplus_{i=0}^{a} \Lambda^i(V) \otimes \Lambda^{a-i}(W).)$$

Application: one can define an ***orientation*** on a real vector bundle of rank n to be a trivialisation of its determinant line bundle. So this problem implies (why?) that a real vector bundle is orientable iff $w_1(E) = 0$. In particular, a manifold M is orientable if and only if $w_1(M) = 0$ in $H^1(M; \mathbb{Z}/2)$, where as usual the Stiefel-Whitney classes of a manifold are defined to be the Stiefel-Whitney classes of the tangent bundle.

(8) Show that $\mathbb{R}P^n$ is not a boundary for n even. Show directly (that is, without using Thom's theorem on cobordism) that the manifold $\mathbb{R}P^n$ is a boundary for n odd. (Hint: Start by showing that $\mathbb{R}P^n$, for n odd, is an $\mathbb{S}^1$-bundle over another manifold.

(9) A manifold M is said to admit a field of tangent k-planes if its tangent bundle admits a subbundle of dimension k. Show that $\mathbb{R}P^n$ admits a field of tangent 1-plane if and only if n is odd. Show that $\mathbb{R}P^4$ and $\mathbb{R}P^6$ do not admit fields of tangent 2-planes.