Jumping of the nef cone for Fano varieties

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Among all projective algebraic varieties, Fano varieties (those with ample anticanonical bundle) can be considered the simplest. Birkar, Cascini, Hacon, and McKernan showed that the Cox ring of a Fano variety, the ring of all sections of all line bundles, is finitely generated [4]. This implies a fundamental fact about the birational geometry of a Fano variety: there are only finitely many small **Q**-factorial modifications of the variety, parametrized by a chamber decomposition of the movable cone into rational polyhedral cones (the nef cones of the modifications). See Kollár-Mori [21] or Hu-Keel [14] for definitions.

There are also strong results about deformations of Fano varieties. For any deformation of a **Q**-factorial terminal Fano variety X_0, de Fernex and Hacon showed that the Cox ring deforms in a flat family [5, Theorem 1.1, Proposition 6.4]. This had been proved for smooth Fanos by Siu [29, Corollary 1.2]. In other words, all line bundles have the "same number" of sections on X_0 as on deformations of X_0. It follows that the movable cone remains constant when X_0 is deformed [5, Theorem 6.8]. To be clear, we use "deformation" to mean a nearby deformation; for example, the statement on the movable cone means that the movable cone is constant for t in some open neighborhood of 0, for any flat family $X \to T$ with fiber X_0 over a point $0 \in T$.

When a **Q**-factorial terminal Fano variety X_0 is deformed, de Fernex and Hacon asked whether the chamber decomposition of the movable cone also remains constant [5, Remark 6.2]. This would say in particular that the nef cone, or dually the cone of curves, remains constant under deformations of X_0. The answer is positive in dimension at most 3, and also in dimension 4 when X_0 is Gorenstein [5, Theorem 6.9]. In any dimension, Wiśniewski showed that the nef cone of a *smooth* Fano variety remains constant under deformations [30, 31].

In this paper, we show that the blow-up of $\mathbf{P}^4$ along a line degenerates to a **Q**-factorial terminal Fano 4-fold X_0 with a strictly smaller nef cone (Theorem 2.1). Therefore the results by de Fernex and Hacon on deformations of 3-dimensional Fanos are best possible. The example is based on the existence of high-dimensional flips which deform to isomorphisms, generalizing the Mukai flop. This phenomenon will be common, and we give a family of examples in various dimensions, including a Gorenstein example in dimension 5 (Theorem 1.1). The examples also disprove the "volume criterion for ampleness" on **Q**-factorial terminal Fano varieties [5, Question 5.5].

In view of Wiśniewski's theorem, it would be interesting to describe the largest class of Fano varieties for which the nef cone remains constant under deformations. It will not be enough to assume that the variety is **Q**-factorial and terminal, but the optimal assumption should also apply to many Fanos which are not **Q**-factorial or terminal.

For example, de Fernex and Hacon proved that $\mathbf{Q}$-factorial terminal toric Fano varieties are rigid, which implies that the nef cone is constant under deformations in that case [5, Theorem 7.1]. In this paper, we show more generally that a toric Fano variety which is smooth in codimension 2 and $\mathbf{Q}$-factorial in codimension 3 is rigid (Theorem 5.1). (This terminology means that the variety is smooth outside some closed subset of codimension at least 3, and $\mathbf{Q}$-factorial outside some closed subset of codimension at least 4.)

We show in Theorem 4.1 that the divisor class group is constant under deformations of klt Fano varieties which are smooth in codimension 2 and $\mathbf{Q}$-factorial in codimension 3. This extends results of Kollár-Mori [20] and de Fernex-Hacon [5].

We show that for any deformation of a terminal Fano variety which is $\mathbf{Q}$-factorial in codimension 3, the Cox ring (of all Weil divisors) deforms in a flat family (Theorem 6.1). The proof uses de Fernex-Hacon's extension theorem on the $\mathbf{Q}$-factorial case. On the other hand, section 7 shows that the Cox ring need not vary in a flat family for Fano varieties with slightly worse singularities, thus answering a question in the first version of this paper. In a sense, section 7 shows that, for Fano varieties, de Fernex-Hacon's extension theorem is optimal.

A side result which seems to be new is that for a complex projective n-fold X with rational singularities such that $H^1(X, O) = H^2(X, O) = 0$, the divisor class group maps isomorphically to the homology group $H_{2n-2}(X, \mathbf{Z})$ (Theorem 3.1). For 3-folds with isolated singularities, this was proved by Namikawa-Steenbrink [26, Theorem 3.2].

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1 Jumping of the nef cone

Theorem 1.1 *For any positive integers $a \geq b$, there is a smooth variety X of dimension $a + b + 1$ and a flat projective morphism $t : X \to A^1$ with the following properties. The fibers X_t for $t \neq 0$ are isomorphic to $\mathbf{P}^a \times \mathbf{P}^b$, and the nef cone of X_0 is a proper subset of the nef cone of $X_t = \mathbf{P}^a \times \mathbf{P}^b$. The fiber X_0 is a terminal Gorenstein Fano variety for $a > b$ and is $\mathbf{Q}$-factorial if $a > b > 1$. For $a = b$, X_0 is a smooth weak Fano ($-K_{X_0}$ is nef and big).*

In particular, for $a = 3$ and $b = 2$, X_0 is a $\mathbf{Q}$-factorial terminal Gorenstein Fano 5-fold whose nef cone changes when X_0 is deformed to $X_t = \mathbf{P}^3 \times \mathbf{P}^2$. In fact, the only singular point of X_0 is a node. This shows the optimality of de Fernex and Hacon's theorem that the nef cone is constant under deformation for $\mathbf{Q}$-factorial terminal Gorenstein Fano 4-folds [5, Theorem 6.9]. Theorem 1.1 works over any field.

Proof. We first describe why the nef cone changes in these examples. The smooth variety X has a projective morphism $X \to Z$ over A^1 which contracts a copy of $\mathbf{P}^a$ contained in the special fiber X_0. This is a flipping contraction of X for $a > b$ and a flopping contraction for $a = b$ (in particular, the canonical bundle K_X has degree negative or zero, respectively, on any curve in $\mathbf{P}^a$). Let $Y \to Z$ be the corresponding flip or flop; in our example, Y is also smooth, with the $\mathbf{P}^a$ in X replaced by a $\mathbf{P}^b$ in Y. Then the birational map $X \dashrightarrow Y$ restricts to a birational map $X_0 \dashrightarrow Y_0$ which is itself a flip or flop, since $K_{X_0} = (K_X + X_0)|_{X_0} = (K_X)|_{X_0}$. (For $b = 1$, $X_0 \to Z_0$ contracts a divisor $\mathbf{P}^a$ in X_0, but this divisor is not $\mathbf{Q}$-Cartier and $X_0 \dashrightarrow Y_0$ must be regarded as a flip; see Kollár et al. [17, 2.26] on the minimal model program for non-$\mathbf{Q}$-factorial varieties. For $b > 1$, we are in the more familiar case where X_0 is $\mathbf{Q}$-factorial and the flipping contraction $X_0 \to Z_0$ contracts a subvariety of codimension greater than 1 in X_0.)

Thus, for $a > b$, we have a flip $X_0 \dashrightarrow Y_0$ which deforms to an isomorphism $X_t \cong Y_t$ for $t \neq 0$. This implies that the nef cone of X_0 is not the whole nef cone of X_t, as we want. Indeed, an ample divisor on Y restricts to an ample divisor on Y_0 and also on $Y_t \cong X_t$ for $t \neq 0$, but the corresponding divisor on X_0 is not ample because $X_0 \dashrightarrow Y_0$ is not an isomorphism. For $a = b$, the flop $X_0 \dashrightarrow Y_0$ is the Mukai flop, which replaces $\mathbf{P}^a \subset X_0$ with normal bundle $T^*\mathbf{P}^a$ by another copy of $\mathbf{P}^a$ [23]. The Mukai flop is well known to have the property here: it deforms into an isomorphism [15, Theorem 4.6]. The flip $X_0 \dashrightarrow Y_0$ for $a > b$ was considered by Kawamata [16, Example 4.2].

For clarity, we first describe the example in a neighborhood of the subvariety being flipped. The flip $X \dashrightarrow Y$ will be the simplest flip between smooth varieties, with X the total space of the vector bundle $O(-1)^{\oplus b+1}$ over $\mathbf{P}^a$ and Y the total space of the vector bundle $O(-1)^{\oplus a+1}$ over $\mathbf{P}^b$. These spaces can also be described in terms of linear algebra. Let $V = A^{b+1}$ and $W = A^{a-b}$, viewed as vector spaces, so that $V \oplus W$ has dimension $a + 1$. In this local picture, the image of the flipping contraction $X \to Z$ is the affine variety

$$Z = \{f = (f_1, f_2) : V \to V \oplus W \text{ linear, } \operatorname{rank}(f) \leq 1\}.$$

And X and Y are the two obvious resolutions of Z, depending on whether we look at the image or the kernel of the linear map f:

$$X = \{(f, L) : L \subset V \oplus W \text{ a line, } f : V \to L \text{ linear}\}$$

and

$$Y = \{(f, S) : S \subset V \text{ of codimension 1, } f : V/S \to V \oplus W \text{ linear}\}.$$

We map Z to A^1 by the trace $t = \operatorname{tr}(f_1)$; composing gives morphisms $X \to A^1$ and $Y \to A^1$. (For this local picture, t could be replaced by any sufficiently general function on Z vanishing at the origin.) The fiber Y_0 over the origin is smooth, since we can describe it as $\{(f, S) : S \subset V \text{ of codimension 1, } f : V/S \to S \oplus W \text{ linear}\}$, which exhibits Y_0 as the total space of the vector bundle $T^*\mathbf{P}^b \oplus O(-1)^{\oplus a-b}$ over $\mathbf{P}^b$. By contrast, X_0 has singular set of codimension $2b+1$ (here X_0 has dimension $a+b$, so X_0 is actually smooth in the flop case, $a = b$). Explicitly, if $x_0, \ldots, x_a$ are homogeneous coordinates on $\mathbf{P}^a$ and $\xi_0, \ldots, \xi_b$ are fiber coordinates on $O(-1)$, then the morphism $X \to A^1$ is given by $t = \sum_{i=0}^{b} x_i \xi_i$. We see that the singularity of X_0 is locally a node on a $(2b+1)$-fold times a smooth variety. In particular, X_0 is terminal and Gorenstein for any $b \geq 1$, and it is $\mathbf{Q}$-factorial for $b \geq 2$, as we want.

Now let us compactify X, Y and Z over A^1. We will use the same names for the compactified varieties. Namely, define

$$Z = \{([f_1, f_2, g], t) : (f_1, f_2) : V \to V \oplus W \text{ of rank} \leq 1, g \in A^1, t \in A^1, \operatorname{tr}(f_1) = tg\}.$$

(The brackets $[f_1, f_2, g]$ mean that f_1, f_2, g are not all zero and we only consider them up to a common scalar factor.) Taking $g = 1$ gives the open subset of Z we considered before. The number t gives the projective morphism $Z \to A^1$ we want. The varieties $X \to Z$ and $Y \to Z$ are defined by

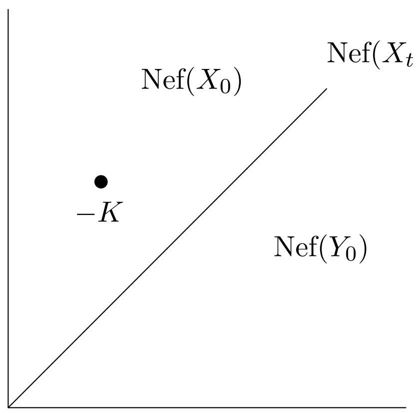
$$X = \{([f_1, f_2, g], L, t) : L \subset V \oplus W \text{ a line}, \\ (f_1, f_2) : V \to L \subset V \oplus W, g \in A^1, t \in A^1, \text{tr}(f_1) = tg\}$$

and

$$Y = \{([f_1, f_2, g], S, t) : S \subset V \text{ of codimension } 1, \\ (f_1, f_2) : V/S \to V \oplus W, g \in A^1, t \in A^1, \text{tr}(f_1) = tg\}.$$

Write X_0, Y_0, Z_0 for the fibers of X, Y, Z over 0 in A^1. The varieties X and Y are isomorphic outside $\mathbf{P}^a \subset X_0$ and $\mathbf{P}^b \subset Y_0$, which are contracted to a point in Z_0. In fact, the projective morphisms $X \to A^1$ and $Y \to A^1$ are products over $A^1 - 0$, with all fibers isomorphic to $\mathbf{P}^a \times \mathbf{P}^b$, by mapping to $(L, S) \in \mathbf{P}^a \times \mathbf{P}^b$. Here X and Y are smooth varieties, but X_0 is singular in codimension $2b + 1$ (with its singular set contained in $\mathbf{P}^a \subset X_0$). By the local calculation above, X_0 is terminal and Gorenstein for any $b \geq 1$, and it is $\mathbf{Q}$-factorial for $b \geq 2$. The flipped variety Y_0 is smooth; in fact, it is the projective bundle $P(TP^b \oplus O(1)^{\oplus a-b} \oplus O)$ over $\mathbf{P}^b$ (the space of hyperplanes in the given vector bundle). In particular, Y_0 has Picard number 2.

Finally, let us show that X_0 is Fano for $a > b$. (For $a = b$, X_0 and Y_0 are weak Fano, since we compute that the line bundles $-K_{X_0}$ and $-K_{Y_0}$ are semi-ample and give the birational contractions $X_0 \to Z_0$ and $Y_0 \to Z_0$.) The Picard number of X_0 is 2 (when $b > 1$, this is immediate from the small $\mathbf{Q}$-factorial modification $X_0 \dashrightarrow Y_0$). Therefore the nef cone of X_0 has exactly two extremal rays, which we can compute by exhibiting two nontrivial contractions of X_0. By construction, we have $X_0 \to \mathbf{P}^a$ and $X_0 \to Z_0$. Both contractions are K-negative; in the first case, this uses that $X_0 \to \mathbf{P}^a$ is a projective bundle over the generic point of $\mathbf{P}^a$. Therefore X_0 is Fano, as we want. QED



2 Jumping of the nef cone in dimension 4

Theorem 2.1 *There is a variety X and a flat projective morphism $t : X \to A^1$ with the following properties. The fibers X_t for $t \neq 0$ are isomorphic to the blow-up* *of $\mathbf{P}^4$ along a line, and the nef cone of X_0 is a proper subset of the nef cone of X_t. The fiber X_0 is a $\mathbf{Q}$-factorial terminal Fano 4-fold.*

Again, this shows the optimality of de Fernex-Hacon's results, as discussed in the introduction. Note that X_0 cannot be Gorenstein, and so these examples are unavoidably more complicated than those in Theorem 1.1. Nonetheless, the 5-fold flip $X \dashrightarrow Y$ that comes up is closely analogous to the simplest 3-fold flip [21, Example 2.7].

Proof. The plan of the example is the same as for Theorem 1.1. We start with a 5-fold flip $X \dashrightarrow Y$ that transforms a weighted projective plane $P(1,1,2)$ into a $\mathbf{P}^2$, with flipping contraction $X \to Z$. The flip we use can be defined as a quotient of the 5-fold flop $(\mathbf{P}^2, O(-1)^{\oplus 3}) \dashrightarrow (\mathbf{P}^2, O(-1)^{\oplus 3})$ by the group $\mathbf{Z}/2$ (as in the simplest example of a 3-fold flip [21, Example 2.7]). In this case (unlike in Theorem 1.1), X will be singular; it has one singular point in the subvariety $P(1,1,2)$, and X is not Gorenstein at that point. We consider a general function $t : Z \to A^1$ vanishing at the singular point of Z. We have to check that the 4-fold $X_0 := t^{-1}(0) \subset X$ is $\mathbf{Q}$-factorial and terminal. With that done, we have a "Mukai-type" flip of 4-folds $X_0 \dashrightarrow Y_0$, meaning one that deforms into an isomorphism $X_t \cong Y_t$. Finally, we compactify X over A^1, and check that we can arrange for X_0 to be a Fano variety.

For brevity, we will define a compactified family $X \to A^1$ without describing the local picture in detail. First define

$$W = \{([x_1, x_2, x_3, y_2, y_3, g], t) \in P(1^5, 2) \times A^1 : x_1^2 + x_2 y_2 + x_3 y_3 = tg\}.$$

(A reference on weighted projective spaces is Dolgachev [8].) Then $t : W \to A^1$ is a flat projective morphism with fibers isomorphic to $\mathbf{P}^4$ outside 0 in A^1 (since we can solve the equation for g when t is not zero). It is a smooth morphism outside the one singular point of W, $([0,0,0,0,0,1],0)$, where W has a quotient singularity $A^5/\{\pm 1\}$. The fiber W_0 over 0 in A^1 is the quotient by $\mathbf{Z}/2$ of a quadric 4-fold in $\mathbf{P}^5$ with a node, and so W_0 has Picard number 1.

The singularity of W_0 (the quotient of a 4-fold node by ± 1) is obtained by contracting a smooth quadric 3-fold D with normal bundle $O(-2)$ in a smooth 4-fold M. Since D has Picard number 1, W_0 is $\mathbf{Q}$-factorial. We compute that $\pi : M \to W_0$ has $K_M = \pi^* K_{W_0} + (1/2)D$. Since the discrepancy 1/2 is positive, W_0 is terminal.

Let X be the blow-up of W along the subvariety $\mathbf{P}^1 \times A^1 = \{([0,0,0,y_2,y_3,0],t)\}$ of W. Since this is contained in the smooth locus of $W \to A^1$, the morphism $X \to A^1$ is smooth outside one point. Clearly the fibers of $X \to A^1$ outside 0 in A^1 are isomorphic to the blow-up of $\mathbf{P}^4$ along a line. Since the fiber X_0 is obtained by blowing up W_0 at a $\mathbf{P}^1$ in the smooth locus of W_0, X_0 is $\mathbf{Q}$-factorial and terminal, and it has Picard number 2.

We now define a flipping contraction $X \to Z$ over A^1 which contracts a surface $P(1,1,2)$ in the special fiber X_0 to a point. Let

$$Z = \{([u_{11}, u_{12}, u_{13}, u_{21}, \ldots, u_{33}, g], t) \in P(1^3, 2^7) \times A^1 : \operatorname{rank}(u_{ij}) \leq 1, u_{11}^2 + u_{22} + u_{33} = tg\}.$$

Here we view (u_{ij}) as a 3×3 matrix to define its rank. We define the contraction $X \to Z$ as the following rational map $W \dashrightarrow Z$, which becomes a morphism after blowing up the subvariety $\mathbf{P}^1 \times A^1$ of W:

$$([x_1, x_2, x_3, y_2, y_3, g], t) \mapsto ([x_1, x_2, x_3, x_1 y_2, x_2 y_2, x_3 y_2, x_1 y_3, x_2 y_3, x_3 y_3, g], t).$$

The resulting contraction $X \to Z$ contracts only a surface $S \cong P(1,1,2)$, the birational transform [21, Notation 0.4] of the surface $\{([0,0,0,y_2,y_3,g],0)\}$ in W.

It is straightforward to check that the contraction $X \to Z$ is K_X-negative; locally, it is a terminal toric flipping contraction, corresponding to a relation $e_1 + e_2 + 2e_3 = e_4 + e_5 + e_6$ in $N \cong \mathbf{Z}^5$ in Reid's notation [27, Proposition 4.3]. The toric picture shows that the flip $Y \to Z$ of $X \to Z$ replaces the surface $P(1,1,2)$ in the 5-fold X with a copy of $\mathbf{P}^2$.

Since X_0 has Picard number 2 and has two nontrivial K-negative contractions (to W_0 and Z_0), X_0 is Fano. Because $X \to Z$ contracts only a surface in X_0, X and Y are the same except over 0 in A^1. That completes the proof: X_0 is a $\mathbf{Q}$-factorial terminal Fano 4-fold with a flip $X_0 \dashrightarrow Y_0$ that deforms into an isomorphism $X_t \cong Y_t$. It follows that the nef cone of X_0 is a proper subset of that of X_t, because an ample divisor on Y is ample on $X_t \cong Y_t$ but not on X_0. QED

3 The divisor class group of a Fano variety

Theorem 3.1 *Let X be a complex projective n-fold with rational singularities such that $H^1(X,O) = H^2(X,O) = 0$. Then Pic$(X) \to H^2(X,\mathbf{Z})$ and Cl$(X) \to H_{2n-2}(X,\mathbf{Z})$ are isomorphisms.*

The theorem applies, for example, if there is an effective $\mathbf{R}$-divisor Δ with (X,Δ) a dlt Fano pair. The statement on the Picard group is well known, but it is harder for the divisor class group, even for terminal Fano varieties. When X has dimension 3 and the singularities are isolated, the statement on the divisor class group was proved by Namikawa-Steenbrink [26, Theorem 3.2].

Proof. To say that (X,Δ) is a dlt Fano pair means that the pair (X,Δ) is dlt [21, Definition 2.37] and that $-(K_X+\Delta)$ is ample. In that case, X has rational singularities [21, Theorem 5.22], and $H^i(X,O) = 0$ for $i > 0$ by Kodaira vanishing [10, Theorem 2.42].

Let X be a projective variety with rational singularities such that $H^1(X,O) = H^2(X,O) = 0$. By the exponential sequence, $\text{Pic}(X) \to H^2(X,\mathbf{Z})$ is an isomorphism. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities, which we can assume to be an isomorphism over the smooth locus X^{sm} and with the inverse image of the singular set a divisor $E = \cup E_i$ having simple normal crossings. By definition of rational singularities, we have $\pi_* O_{\widetilde{X}} = O_X$ and $R^j\pi_* O_{\widetilde{X}} = 0$ for $j > 0$. We deduce that $H^1(\widetilde{X},O) = H^2(\widetilde{X},O) = 0$, and so $\text{Pic}(\widetilde{X})$ maps isomorphically to $H^2(\widetilde{X},\mathbf{Z})$. Equivalently, $\text{Cl}(\widetilde{X})$ maps isomorphically to $H_{2n-2}(\widetilde{X},\mathbf{Z})$.

We have localization sequences in Chow groups and Borel-Moore homology, using that $\widetilde{X} - E \cong X^{\text{sm}}$:

$$\begin{array}{ccccccc}
\oplus \mathbf{Z}E_i & \longrightarrow & \text{Cl}(\widetilde{X}) & \longrightarrow & \text{Cl}(X^{\text{sm}}) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \\
\oplus \mathbf{Z}E_i & \longrightarrow & H_{2n-2}(\widetilde{X},\mathbf{Z}) & \longrightarrow & H^{BM}_{2n-2}(X^{\text{sm}},\mathbf{Z}) & \longrightarrow & H_{2n-3}(E,\mathbf{Z}).
\end{array}$$

The diagram implies that $\text{Cl}(X^{\text{sm}}) \to H^{BM}_{2n-2}(X^{\text{sm}},\mathbf{Z})$ is injective. Equivalently, $\text{Cl}(X) \to H_{2n-2}(X,\mathbf{Z})$ is injective.

Namikawa showed that for any variety X with rational singularities, all 2-forms on the smooth locus extend to 2-forms on a resolution $\widetilde{X} \to X$ [25, Theorem A.1]. In our situation, we deduce that every 2-form on $\widetilde{X}$ with log poles on the divisor E extends to a 2-form on $\widetilde{X}$, which must be zero by the Hodge symmetry $h^{pq} = h^{qp}$ from $H^2(\widetilde{X}, O) = 0$.

By definition of Deligne's Hodge filtration on $H^2(X^{\text{sm}}, \mathbf{C})$, the vanishing of 2-forms on $\widetilde{X}$ with log poles on E means precisely that $F^2H^2(X^{\text{sm}}, \mathbf{C}) = 0$ [7, Theorem 3.2.5]. The mixed Hodge structure on H^2 of any smooth variety has Hodge numbers h^{pq} equal to zero unless $p \leq 2$, $q \leq 2$, and $p + q \geq 2$ [7, Corollary 3.2.15]. We have shown that $h^{20} = h^{21} = h^{22} = 0$; using Hodge symmetry, it follows that the only nonzero Hodge number for $H^2(X^{\text{sm}}, \mathbf{C})$ is h^{11}. In particular, $H^2(X^{\text{sm}}, \mathbf{C})$ is all in weight 2, which means that $H^2(\widetilde{X}, \mathbf{Q}) \to H^2(X^{\text{sm}}, \mathbf{Q})$ is surjective [7, Corollary 3.2.17]. Here $\text{Cl}(\widetilde{X})$ is isomorphic to $H^2(\widetilde{X}, \mathbf{Z})$ and therefore maps onto $H^2(X^{\text{sm}}, \mathbf{Z}) = H_{2n-2}(X, \mathbf{Z})$ after tensoring with the rationals. It follows that $\text{Cl}(X)$ maps onto $H_{2n-2}(X, \mathbf{Z})$ after tensoring with the rationals.

To show that $\text{Cl}(X)$ maps onto $H_{2n-2}(X, \mathbf{Z})$, we observe that the cokernel of $\text{Cl}(X) \to H_{2n-2}(X, \mathbf{Z})$ is torsion-free for every normal algebraic variety X. Indeed, it is equivalent to show that $\text{Pic}(X^{\text{sm}}) \to H^2(X^{\text{sm}}, \mathbf{Z})$ has torsion-free cokernel. That follows from the commutative diagram, for any positive integer m, where the top row is the Kummer sequence for the etale topology:

$$\begin{array}{ccccccc}
H^1(X^{\text{sm}}, \mathbf{Z}/m) & \longrightarrow & \text{Pic}(X^{\text{sm}}) & \xrightarrow[\cdot m]{} & \text{Pic}(X^{\text{sm}}) & \longrightarrow & H^2(X^{\text{sm}}, \mathbf{Z}/m) \\
\downarrow \cong & & \downarrow & & \downarrow & & \downarrow \cong \\
H^1(X^{\text{sm}}, \mathbf{Z}/m) & \longrightarrow & H^2(X^{\text{sm}}, \mathbf{Z}) & \xrightarrow[\cdot m]{} & H^2(X^{\text{sm}}, \mathbf{Z}) & \longrightarrow & H^2(X^{\text{sm}}, \mathbf{Z}/m).
\end{array}$$

We have shown that $\text{Cl}(X) \to H_{2n-2}(X, \mathbf{Z})$ is an isomorphism. QED

4 Deforming Weil divisors

Theorem 4.1 *Let X_0 be a complex projective variety with rational singularities such that $H^1(X_0, O) = H^2(X_0, O) = 0$. Suppose that X_0 is smooth in codimension 2 and $\mathbf{Q}$-factorial in codimension 3. Then the divisor class group is unchanged under nearby deformations of X_0. More precisely, given a deformation $X \to (T, 0)$ of X_0 over a smooth curve T, there is an etale morphism of curves $(T', 0) \to (T, 0)$ such that the class group of $X_{T'}$ maps split surjectively to the class group of $Cl(X_t)$ for all $t \in T'$, and all these surjections have the same kernel.*

The assumption that X_0 has rational singularities and $H^1(X_0, O) = H^2(X_0, 0) = 0$ holds, for example, if there is an effective $\mathbf{R}$-divisor Δ with (X_0, Δ) a dlt Fano pair.

In Theorem 4.1, the divisor class group means the group of Weil divisors (with integer coefficients) modulo linear equivalence. For $\mathbf{Q}$-factorial terminal Fanos, deformation invariance of the class group was proved by de Fernex and Hacon [5, Proposition 6.4, Lemma 7.2], building on the work of Kollár and Mori [20, Proposition 12.2.5].

The assumptions on singularities are explained by some simple examples. A quadric surface X_0 with a node is $\mathbf{Q}$-factorial but singular in codimension 2, and it deforms to a smooth quadric surface X_t; then $\mathrm{Cl}(X_0) \cong \mathbf{Z}$ and $\mathrm{Cl}(X_t) \cong \mathbf{Z}^2$. A quadric 3-fold X_0 with a node is smooth in codimension 2 but not $\mathbf{Q}$-factorial in codimension 3, and it deforms to a smooth quadric 3-fold X_t; then $\mathrm{Cl}(X_0) \cong \mathbf{Z}^2$ and $\mathrm{Cl}(X_t) \cong \mathbf{Z}$. Generalizing the last case, Sturmfels, Batyrev, and others found many non-toric Fanos such as Grassmannians which degenerate to terminal Gorenstein toric Fano varieties X_0 [2]. In their examples, $\mathrm{Pic}(X_0) \cong \mathrm{Pic}(X_t) \cong \mathbf{Z}$, but X_0 is not $\mathbf{Q}$-factorial because of nodes in codimension 3, whereas X_t is smooth. So $\mathrm{Cl}(X_0)$ is bigger than $\mathrm{Cl}(X_t) \cong \mathbf{Z}$. These examples suggest the problem of finding "minimal" assumptions which imply that the Picard group of a Fano variety, rather than the divisor class group, is constant under deformation.

Finally, Theorem 4.1 does not hold if the assumption that X_0 has rational singularities is weakened to X_0 Cohen-Macaulay. An example is provided by taking X_0 to be the projective cone over a smooth quartic surface Y_0 with Picard number 1, and deforming it to projective cones X_t over other smooth quartic surfaces Y_t. Then X_0 is Cohen-Macaulay, smooth in codimension 2, $\mathbf{Q}$-factorial, and $H^1(X_0, O) = H^2(X_0, O) = 0$. (It is also a log-canonical Fano 3-fold.) But $\mathrm{Cl}(X_t) \cong \mathrm{Pic}(Y_t)$ can be non-constant even on an analytic neighborhood of $0 \in T$. On the good side, if the assumption that X_0 has rational singularities in Theorem 4.1 is weakened to X_0 Cohen-Macaulay, then the proof at least shows that there is an etale morphism of curves $(T', 0) \to (T, 0)$ such that the class group of $X_{T'}$ maps split surjectively to the class group of X_0.

Proof. We know that $\mathrm{Cl}(X_0)$ is finitely generated by the more precise Theorem 3.1, $\mathrm{Cl}(X_0) = H_{2n-2}(X_0, \mathbf{Z})$. Let us prove that $\mathrm{Cl}(X_R) \to \mathrm{Cl}(X_0)$ is injective, where R is the henselization of the local ring of T at 0 [22, section I.4] and X_R denotes the pullback of $X \to T$ via $\mathrm{Spec}(R) \to T$. (We will briefly replace T by a ramified covering later in this proof, but the same argument will apply.) Namely, by Griffith's local Lefschetz theorem for divisor class groups, since X_0 is Cohen-Macaulay and smooth in codimension 2, the restriction map $\mathrm{Cl}(O_{X_R,x}) \to \mathrm{Cl}(O_{X_0,x})$ is injective for every point $x \in X_0$ [11, Proposition 3.5]. Therefore a Weil divisor class D on X_R which restricts to 0 in $\mathrm{Cl}(X_0)$ is trivial near each point of X_0. This means that D is a Cartier divisor, and its restriction to X_0 is linearly equivalent to zero. But $\mathrm{Pic}(X_R) \to \mathrm{Pic}(X_0)$ is injective because $H^1(X_0, O) = 0$ [12, Corollary 4.6.4]. So D is linearly equivalent to zero on X_R. We have shown that $\mathrm{Cl}(X_R) \to \mathrm{Cl}(X_0)$ is injective.

Let K be the quotient field of the henselian local ring R. Geometrically, X_K is obtained from X_R by removing the special fiber X_0, which is linearly equivalent to zero as a divisor in X_R, and so the restriction map $\mathrm{Cl}(X_R) \to \mathrm{Cl}(X_K)$ is an isomorphism. The algebraic closure $\overline{K}$ of K has Galois group $\mathrm{Gal}(\overline{K}/K) \cong \widehat{\mathbf{Z}}$ (the completed fundamental group of a ball in T minus the point 0). I claim that $\mathrm{Cl}(X_R) = \mathrm{Cl}(X_K)$ maps isomorphically to the Galois invariants $\mathrm{Cl}(X_{\overline{K}})^{\widehat{\mathbf{Z}}}$. To prove this, let U be the smooth locus of X_K; since X_K is normal, it suffices to show that $\mathrm{Pic}(U) \to \mathrm{Pic}(U_{\overline{K}})^{\widehat{\mathbf{Z}}}$ is an isomorphism. Consider the Hochschild-Serre spectral sequence for etale cohomology,

$$H^p(K, H^q_{\mathrm{et}}(U_{\overline{K}}, O^*)) \Rightarrow H^{p+q}_{\mathrm{et}}(U, O^*).$$

Here $H^0_{\mathrm{et}}(U_{\overline{K}}, O^*) \cong \overline{K}^*$ since X_K is normal, connected, and proper over K. So $H^1(K, H^0_{\mathrm{et}}(U_{\overline{K}}, O^*)) = H^1(K, \overline{K}^*) = 0$ by Hilbert's theorem 90. Also, $H^2(K, H^0_{\mathrm{et}}(U_{\overline{K}}, O^*))$

is equal to the Brauer group $H^2(K, \overline{K}^*)$, which is zero because K is the quotient field of a henselian discrete valuation ring with algebraically closed residue field [22, Example III.2.2]. Therefore the spectral sequence gives that $\text{Pic}(U) = H^1_{\text{et}}(U, O^*)$ maps isomorphically to $\text{Pic}(U_{\overline{K}})^{\widehat{\mathbf{Z}}} = H^0(K, H^1_{\text{et}}(U_{\overline{K}}, O^*))$. Thus we have shown that $\text{Cl}(X_R) = \text{Cl}(X_K)$ maps isomorphically to $\text{Cl}(X_{\overline{K}})^{\widehat{\mathbf{Z}}}$.

Notice that the class group $\text{Cl}(X_{\overline{K}})$ is isomorphic to the class group of X_t for very general points $t \in T$ (that is, for all but countably many points in the curve T). Indeed, the morphism $X \to T$ is defined over some finitely generated subfield F of $\mathbf{C}$, $X_F \to T_F$. For very general points $t \in T(\mathbf{C})$, the morphism $t : \text{Spec}(\mathbf{C}) \to T \to T_F$ factors through a geometric generic point, $\overline{F(T)}$ (that is, it is transcendental over F). So both X_t and $X_{\overline{K}}$ are extensions of $X_{\overline{F(T)}}$ from one algebraically closed field to another. Since the class group of X_t is finitely generated, this implies that the class group of X_t (for t very general) is isomorphic to that of $X_{\overline{K}}$.

I claim that the Galois group $\widehat{\mathbf{Z}}$ acts trivially on $\text{Cl}(X_{\overline{K}})$. (In geometric terms, this means that the monodromy $\mathbf{Z}$ around 0 in T acts trivially on $\text{Cl}(X_t) = H_{2n-2}(X_t, \mathbf{Z})$ for $t \neq 0$ near 0.) Our earlier proof that $\text{Cl}(X_R) = \text{Cl}(X_K)$ maps injectively to $\text{Cl}(X_0)$ applies to any finite extension field E in place of K, to show that the specialization map $\text{Cl}(X_E) \to \text{Cl}(X_0)$ is injective. By taking a direct limit, it follows that the specialization map $\text{Cl}(X_{\overline{K}}) \to \text{Cl}(X_0)$ is injective. But this specialization map factors through the coinvariants of the Galois group $\text{Gal}(\overline{K}/K) = \widehat{\mathbf{Z}}$ on $\text{Cl}(X_{\overline{K}})$. Therefore, $\text{Gal}(\overline{K}/K) = \widehat{\mathbf{Z}}$ acts trivially on $\text{Cl}(X_{\overline{K}})$, as we wanted. Together with the argument two paragraphs back, this shows that $\text{Cl}(X_R) = \text{Cl}(X_K)$ maps isomorphically to $\text{Cl}(X_{\overline{K}})$.

Since R is the henselian local ring of T at 0, this implies that after replacing T by some algebraic curve T' with an etale morphism $(T', 0) \to (T, 0)$, the class group $\text{Cl}(X)$ maps onto $\text{Cl}(X_t)$ for very general $t \in T$. We can assume (after replacing T by a Zariski open neighborhood of 0) that $X \to T$ is a topological fibration over $T - 0$. Since we have a topological interpretation of the divisor class group as $\text{Cl}(X_t) = H_{2n-2}(X_t, \mathbf{Z})$, the class group $\text{Cl}(X)$ maps onto $\text{Cl}(X_t)$ for all $t \neq 0$ in T.

It remains to show that $\text{Cl}(X_R)$ maps onto $\text{Cl}(X_0)$. Let D be a Weil divisor on X_0. The sheaf $O(D)$ restricts to a line bundle L on the smooth locus U_0 of X_0. Write jU_0 for the subscheme of X defined locally by the jth power of a function defining U_0. We have an exact sequence

$$H^1(U_0, O) \to \text{Pic}((j+1)U_0) \to \text{Pic}(jU_0) \to H^2(U_0, O),$$

because the normal bundle of X_0 in X is trivial. Because $X_0 - U_0$ has codimension at least 3 and X_0 is Cohen-Macaulay, the restriction map $H^j(X_0, O) \to H^j(U_0, O)$ is bijective for $i < 2$ and injective for $i = 2$ [13, Proposition III.3.3]. Since $H^1(X_0, O) = 0$, that implies that $H^1(U_0, O) = 0$. Moreover, U_0 is contained in some $\mathbf{Q}$-factorial open subset V_0 whose complement has codimension at least 4 in X_0. By the same arguments, $H^1(V_0, O) = H^2(V_0, O) = 0$.

Because V_0 is $\mathbf{Q}$-factorial, there is a positive integer m such that the line bundle $L^{\otimes m}$ on U_0 extends to a line bundle M on V_0. By the exact sequence we wrote for U_0, M extends uniquely to a line bundle M on the formal scheme $V_0^\wedge$ (the completion of V_0 in X). It follows that the obstruction in $H^2(U_0, O)$ to extending L from U_0 to $2U_0$ is killed by m and hence is zero, because $H^2(U_0, O)$ is a complex vector space. Because $H^1(U_0, O) = 0$, the extension of L to $2U_0$ is unique, and repeating the argument shows that L extends to a line bundle on the formal scheme $U_0^\wedge$, the completion of U_0 in X.

Let $j : U_0^\wedge \to X_0^\wedge$ be the inclusion of formal schemes, where $X_0^\wedge$ is the completion of X_0 in X. Because X_0 is Cohen-Macaulay and the codimension of U_0 in X_0 has codimension at least 3, Grothendieck's existence theorem shows that $E := j_*L$ is a coherent sheaf on the formal scheme $X_0^\wedge$ [13, Theorem IX.2.2]. Because $X \to T$ is a projective morphism, E is algebraizable [12, Corollary III.5.1.6]; that is, E can be viewed as a coherent sheaf on the scheme $X_{R^\wedge}$, where $R^\wedge$ is the completed local ring of the curve T at the point 0. Because the restriction of E to $U_0 \subset X_0$ is a line bundle, E is a line bundle on $X_{R^\wedge} - S$ for some closed subset $S \subset X_{R^\wedge}$ of codimension at least 3. So, after replacing the Weil divisor D on X_0 we started with by a linearly equivalent divisor, D extends to a Weil divisor D on $X_{R^\wedge}$.

The relative Hilbert scheme of codimension-1 subschemes of fibers of $X \to T$ with given Hilbert polynomial p, $\mathrm{Hilb}^p(X/T)$, is projective over T. Write $D = D_1 - D_2$ where D_1 and D_2 are effective divisors on $X_{R^\wedge}$. A projective morphism which has a section over the completed local ring of T at 0 also has a section over the henselian local ring R of T at 0. Therefore, D extends to a Weil divisor on X_R. We have shown that $\mathrm{Cl}(X_R) \to \mathrm{Cl}(X_0)$ is surjective.

We showed at the start of the proof that $\mathrm{Cl}(X_R) \to \mathrm{Cl}(X_0)$ is injective, and so this restriction map is an isomorphism. (We also know that $\mathrm{Cl}(X_R) \to \mathrm{Cl}(X_{\overline{K}})$ is an isomorphism.) But $\mathrm{Cl}(X_R)$ is the direct limit of the class groups of all pullbacks of X by etale morphisms $(T', 0) \to (T, 0)$. Therefore, after some such pullback, the homomorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(X_0)$ becomes split surjective. By what we have shown, the kernels of the restriction maps $\mathrm{Cl}(X) \to \mathrm{Cl}(X_0)$ and $\mathrm{Cl}(X) \to \mathrm{Cl}(X_t)$ for $t \neq 0$ in T are both equal to the kernel of $\mathrm{Cl}(X) \to \mathrm{Cl}(X_R)$. QED

5 Deformations of toric Fano varieties

Theorem 5.1 *A toric Fano variety which is smooth in codimension 2 and* **Q***-factorial in codimension 3 is rigid.*

This strengthens Bien-Brion's theorem that smooth toric Fano varieties are rigid [3] and de Fernex-Hacon's theorem that **Q**-factorial terminal toric Fano varieties are rigid [5]. (Note that a toric variety is **Q**-factorial if and only if the corresponding fan is simplicial.) The assumptions on singularities cannot be omitted. First, a quadric surface with a node is a toric Fano variety that is not rigid. Likewise, many toric Fano varieties with codimension-3 node singularities are not rigid, by the examples after Theorem 4.1. The varieties in Theorem 5.1 are klt (like every **Q**-Gorenstein toric variety) but need not be terminal or canonical.

Proof. For any normal variety X and $j \geq 0$, the double dual $(\Omega^j_X)^{**}$ is the reflexive sheaf that extends the vector bundle Ω^j_U on the smooth locus U of X; that is, $(\Omega^j_X)^{**} = f_*\Omega^j_U$ where $f : U \to X$ is the inclusion. Danilov's vanishing theorem says that for any ample line bundle $O(D)$ on a projective toric variety X, $H^i(X, (\Omega^j \otimes O(D))^{**}) = 0$ for all $i > 0$ and $j \geq 0$ [6, Theorem 7.5.2]. More generally, for any ample Weil divisor D on a projective toric variety X, Mustață showed that $H^i(X, (\Omega^j \otimes O(D))^{**}) = 0$ for all $i > 0$ and $j \geq 0$ [24, Proposition 2.3]. Fujino gave a simple proof of this vanishing using the action of the multiplicative monoid of natural numbers on a toric variety [9, Proposition 3.2].

Define the tangent sheaf of a normal variety X of dimension n as $\mathcal{T}_X^0 = \mathrm{Hom}(\Omega_X^1, O_X)$. This is the reflexive sheaf that agrees with $\Omega_X^{n-1} \otimes (K_X)^*$ on the smooth locus of X, and so we have $\mathcal{T}_X^0 = (\Omega^{n-1} \otimes (K_X)^*)^{**}$. By the Danilov-Mustață vanishing theorem, for any toric Fano variety X, we have $H^i(X, \mathcal{T}_X^0) = 0$ for all $i > 0$. Since $H^1(X, \mathcal{T}_X^0) = 0$, all locally trivial deformations of X are trivial [28, Proposition 1.2.9]. (Roughly speaking, a deformation is locally trivial if it does not change the singularities of X.)

Finally, Altmann showed that any $\mathbf{Q}$-Gorenstein affine toric variety which is smooth in codimension 2 and $\mathbf{Q}$-factorial in codimension 3 is rigid [1, Corollary 6.5(1)]. Thus, for X a toric Fano variety which is smooth in codimension 2 and $\mathbf{Q}$-factorial in codimension 3, all deformations are locally trivial. By the previous paragraph, all deformations of X are in fact trivial. QED

6 Flatness of Cox rings

Theorem 6.1 *Let X_0 be a terminal Fano variety over the complex numbers which is $\mathbf{Q}$-factorial in codimension 3. Then, for any deformation $X \to (T, 0)$ of X_0 over a smooth curve, the Cox rings Cox(X_t) form a flat family over an etale neighborhood of 0 in T. That is, after replacing X by its pullback via some etale morphism of curves $(T', 0) \to (T, 0)$, every divisor class on each X_t extends to X, and $h^0(X_t, O(D))$ is independent of t for every Weil divisor D on X.*

Here by the Cox ring of X_t we mean the sum over all classes of Weil divisors, $\oplus_{D \in \mathrm{Cl}(X_t)} H^0(X_t, O(D))$. For X_0 $\mathbf{Q}$-factorial rather than $\mathbf{Q}$-factorial in codimension 3, the theorem was proved by de Fernex and Hacon [5, Theorem 1.1, Proposition 6.4]. The proof of Theorem 6.1 combines their results with Theorem 4.1 on deformation invariance of the divisor class group.

The statement about flatness in Theorem 6.1 can be explained in more detail. Let $X \to (T, 0)$ be a deformation over an affine curve, as in the theorem. Theorem 4.1 shows that after replacing X by its pullback by some etale morphism of curves $(T', 0) \to (T, 0)$, we can assume that $\mathrm{Cl}(X) \to \mathrm{Cl}(X_t)$ is split surjective for all t in the new curve T, and all these surjections have the same kernel. Let $A \subset \mathrm{Cl}(X)$ be a subgroup which maps isomorphically to $\mathrm{Cl}(X_t)$ for all $t \in T$. By choosing divisors whose classes generate the abelian group A, we can define a ring structure on $\oplus_{D \in A} H^0(X, O(D))$, which depends on choices because the group A may have torsion. Then Theorem 6.1 shows that this ring is a flat $O(T)$-algebra. Tensoring it with the homomorphism $O(T) \to \mathbf{C}$ associated to any point $t \in T$ gives the Cox ring of X_t.

Proof. By Kawamata and Nakayama, X is terminal, after shrinking T to a Zariski open neighborhood of $0 \in T$ [16, Theorem 1.5]. Let $\pi : W \to X$ be a $\mathbf{Q}$-factorialization, which exists by Birkar-Cascini-Hacon-McKernan [4, Corollary 1.4.3]. That is, W is $\mathbf{Q}$-factorial and terminal and $W \to X$ is a small projective birational morphism. It follows that $K_W = \pi^*(K_X)$.

We will show that $W_0 \to X_0$ is a $\mathbf{Q}$-factorialization. Since $W \to X$ is a small birational morphism, $W_0 \to X_0$ is birational. Since X is terminal (in particular, K_X is $\mathbf{Q}$-Cartier) and X_0 is terminal, the pair (X, X_0) is plt [21, Theorem 5.50]. Therefore the pair (W, W_0) is plt, since $K_W = \pi^*(K_X)$ and $W_0 = \pi^*(X_0)$ as a Cartier divisor. It follows that W_0 is normal [21, Proposition 5.51].

Since $K_{W_0} = K_W|_{W_0}$ and $K_{X_0} = K_X|_{X_0}$, we have $K_{W_0} = \pi^* K_{X_0}$. Since X_0 is terminal, it follows that W_0 is terminal and the birational morphism $W_0 \to X_0$ is small.

By Theorem 4.1, after replacing T by a curve T' with an etale morphism $(T', 0) \to (T, 0)$, our assumptions imply that $\mathrm{Cl}(X) \to \mathrm{Cl}(X_0)$ is onto. That is, every Weil divisor on X_0 is the restriction of some Weil divisor D on X. Let D_W be the birational transform of D on W. Since W is $\mathbf{Q}$-factorial, D_W is $\mathbf{Q}$-Cartier, and hence so is $D_W|_{W_0}$. Clearly this divisor pushes forward to $D|_{X_0}$, and since $W_0 \to X_0$ is small, $D_W|_{W_0}$ must be the birational transform of $D|_{X_0}$. Every divisor on W_0 is the birational transform of a divisor on X_0, and so we have shown that W_0 is $\mathbf{Q}$-factorial. That is, the morphism $W_0 \to X_0$ is a $\mathbf{Q}$-factorialization. In other words, our assumptions imply that "$\mathbf{Q}$-factorialization works in families."

Because $W_0 \to X_0$ is small, so is $W_t \to X_t$ for all $t \in T$, after shrinking T around 0. Therefore the Cox rings of these two varieties are the same: $\mathrm{Cox}(W_t) \cong \mathrm{Cox}(X_t)$. So, to prove flatness of the Cox rings for the family $X \to T$, it suffices to prove flatness of the Cox rings for the family $W \to T$. Here W_0 is no longer Fano, but it is a $\mathbf{Q}$-factorial terminal weak Fano (meaning that $-K_{W_0}$ is nef and big), and X_0 is the anticanonical model of W_0.

We know that the class groups of W_t are independent of t by Theorem 4.1. Our goal is to show that for every integral Weil divisor L on X, the restriction map $H^0(W, O(L)) \to H^0(W_0, O(L|_{W_0}))$ is surjective. We can assume that $H^0(W_0, O(L|_{W_0}))$ is not zero.

Because $\pi : W \to X$ is birational, there is an ample divisor A on X such that $L + \pi^* A$ is $\mathbf{Q}$-linearly equivalent to an effective $\mathbf{Q}$-divisor B on W. Choose $e > 0$ small enough that the pair (W, eB) is terminal. Here $K_W + eB$ is big over X (just because $W \to X$ is birational), and so Birkar-Cascini-Hacon-M$^{\mathrm{c}}$Kernan showed that (W, eB) has a minimal model W' over X [4, Theorem 1.2]. Because the morphism $W \to X$ is small, so is $W' \to X$, and we end up with another small $\mathbf{Q}$-factorialization $\pi : W' \to X$ such that $K_{W'} + eB$ is nef over X. Equivalently, since $K_{W'} = \pi^* K_X$ and $B \sim_{\mathbf{Q}} L + \pi^*(A)$, L becomes nef over X on W'. Let us write W instead of W'; our earlier arguments still apply to this new $\mathbf{Q}$-factorialization of X. In particular, $W_0 \to X_0$ is a small $\mathbf{Q}$-factorialization, and we have now arranged that L is nef over X on W. Making a small $\mathbf{Q}$-factorial modification does not change the space of global sections of a reflexive sheaf of rank one, and so our goal is still to show that $H^0(W, O(L)) \to H^0(W_0, O(L|_{W_0}))$ is surjective.

Let D_1 be an effective $\mathbf{Q}$-divisor $\sim_{\mathbf{Q}} -K_W + bL$ for some small $b > 0$; this is possible because $-K_W$ is big over T. Let B be $1/m$ times a general divisor in $|-mK_W|$ for some $m > 1$ such that $-mK_W$ is basepoint-free. Let $D = (1 - c/b)B + (c/b)D_1$ for a small $c \in (0, 1)$; clearly $D \sim_{\mathbf{Q}} -K_W + cL$, and the pair (W, D) is klt for c small enough [18, Theorem 4.8]. Since we arranged that L is nef on W over X and $-K_W = \pi^*(-K_X)$, D is nef on W over X.

Since $-K_{X_0}$ is ample, $-K_X$ is ample over T after shrinking the curve T around the point 0. So $-K_W = \pi^*(-K_X)$ is nef and big over T. By the cone theorem, the cone of curves of W over T is rational polyhedral [21, Theorem 3.7]. (Since $-K_W$ is nef and big over T, and W is $\mathbf{Q}$-factorial and klt, it is straightforward to construct an effective $\mathbf{Q}$-divisor E on W such that the pair (W, E) is klt and Fano over T. That is the assumption we need for the cone theorem.) Since D is nef over X and $-K_W$ is the pullback of $-K_X$ which is ample over T, $D - aK_W$ is nef over T for a

at least some positive real number a_0, since it suffices to check that $D - aK_W$ has nonnegative intersection with each extremal ray of the cone of curves of W over T. Choose an $a > a_0$.

Following the idea of de Fernex-Hacon [5, Theorem 4.5(b)], we run a (W, D)-minimal model program over T with scaling of $D - aK_W$:

$$W = W^0 \dashrightarrow W^1 \dashrightarrow W^2 \dashrightarrow \cdots .$$

(In order to run this minimal model program, we need to know that $(K_W + D) + t(D - aK_W)$ is nef over T for some $t > 0$, which is true by our choice of a.) Here $D - aK_W$ is only nef over T, not ample over T as in [5, Theorem 4.5(b)], but because W is weak Fano over T, we still know that this minimal model program terminates. In fact, every minimal model program on W over T terminates, by Birkar-Cascini-Hacon-M$^{\text{c}}$Kernan [4, Corollary 1.3.1].

By de Fernex-Hacon [5, Theorem 4.1], which applies to our minimal model program with scaling by the nef divisor $D - aK_W$, each fiber type (resp. divisorial, resp. flipping) contraction of W restricts to a fiber type (resp. divisorial, resp. flipping) contraction of W_0. Therefore, the given (W, D)-minimal model program over T induces an $(W_0, D|_{W_0})$-minimal model program on W_0. We assumed that the **Q**-divisor class $K_{W_0} + D|_{W_0} \sim_{\mathbf{Q}} cL|_{W_0}$ is effective, and so we never have a fiber type contraction of W_0. Therefore, we never have a fiber type contraction of W. Thus the minimal model program ends with an (W, D)-minimal model $W \dashrightarrow W'$ which induces a minimal model $W_0 \dashrightarrow W_0'$ for $(W_0, D|_{W_0})$. That is, writing D and L for the birational transforms on W' of these divisors, the pair (W', D) is **Q**-factorial and klt, W' is terminal, and $K_{W'} + D \sim_{\mathbf{Q}} cL$ is nef over T. (Although we could arrange for the pair (W, D) to be terminal, that property may be lost in the course of the minimal model program, because some components of D may be contracted.)

Because $K_W + D \sim_{\mathbf{Q}} cL$ where L is an integral Weil divisor and $c > 0$, it is a standard property of minimal models that $H^0(W, O(L)) = H^0(W', O(L))$ and $H^0(W_0, O(L)) = H^0(W_0', O(L))$ [5, proof of Theorem 4.5]. It remains to show that $H^0(W', O(L)) \to H^0(W_0', O(L))$ is surjective.

Since D is big on W, it is big on W', and so we can write $D = G + H$ for some ample **Q**-divisor H and effective **Q**-divisor G on W' such that the pair (W', G) is klt. Then $H^1(W', O_{W'}(L)) = 0$ follows from Kodaira vanishing, after shrinking the curve T around 0. (In more detail: we have $L \sim_{\mathbf{Q}} (1/c)(K_{W'} + D)$ where $1/c > 1$. So $L \sim_{\mathbf{Q}} (K_{W'} + G) + (1/c - 1)(K_{W'} + D) + H$, where $K_{W'} + D$ is nef over T and H is ample over T, which lets us apply Kodaira vanishing on W' over T [10, Theorem 2.42].) After shrinking the curve T around 0, the divisor W_0' is linearly equivalent to zero on W', and so we have $H^1(W', O_{W'}(L - W_0')) = 0$. By de Fernex and Hacon, we have a short exact sequence

$$0 \to O_{W'}(L - W_0') \to O_{W'}(L) \to O_{W_0'}(L) \to 0$$

of sheaves on W' [5, Lemma 4.6]. Therefore $H^0(W', O(L)) \to H^0(W_0', O(L))$ is surjective. Equivalently, $H^0(X, O(L)) \to H^0(X_0, O(L))$ is surjective. QED

7 Counterexamples to extension theorems

Theorem 4.1 shows that the divisor class group remains constant under deformations of a klt Fano variety X_0 which is smooth in codimension 2 and **Q**-factorial in codimension 3. The first version of this paper asked whether the Cox ring (of all Weil divisors) deforms in a flat family under these assumptions. Equivalently, we are asking whether $H^0(X, O(L)) \to H^0(X_0, O(L))$ is surjective for every Weil divisor L on the total space X. In this section, we give two negative answers to that question. The answer is positive by de Fernex and Hacon for **Q**-factorial terminal Fano varieties [5, Theorem 1.1, Proposition 6.4], and by Theorem 6.1 for terminal Fanos which are **Q**-factorial in codimension 3. The argument by de Fernex and Hacon actually shows flatness of Cox rings when we deform **Q**-factorial canonical Fano varieties which are smooth in codimension 2. But the following examples show that the results mentioned are essentially optimal.

Theorem 7.1 *There is a canonical Fano 4-fold X_0 which is smooth in codimension 3, a deformation $X \to A^1$ of X_0, and a Weil divisor L on X such that $H^0(X, O(L)) \to H^0(X_0, O(L))$ is not surjective. The fibers X_t for $t \neq 0$ are isomorphic to $\mathbf{P}^1 \times \mathbf{P}^3$.*

Proof. Embed $\mathbf{P}^1 \times \mathbf{P}^3$ in projective space $\mathbf{P}^{19}$ by the line bundle $H = O(1, 2)$, and let M be a smooth hyperplane section of $\mathbf{P}^1 \times \mathbf{P}^3$. Let X_0 be the projective cone over M. Taking X to be a suitable open subset of the blow-up of the projective cone over $\mathbf{P}^1 \times \mathbf{P}^3$ along M, we see that X_0 deforms over A^1 with fibers X_t isomorphic to $\mathbf{P}^1 \times \mathbf{P}^3$ for $t \neq 0$.

Clearly X_0 is smooth in codimension 3. We have $-K_{\mathbf{P}^1 \times \mathbf{P}^3} = O(2, 4) = 2H$, and so $-K_M = H$. Therefore the singularity of X_0 (the affine cone over M embedded by $-K_M$) is canonical. Also, X_0 is Fano, being embedded in projective space by an ample line bundle H with $-K_{X_0} = 2H$.

By the Lefschetz hyperplane theorem, the divisor class group of X_0 is $\mathrm{Cl}(X_0) \cong \mathrm{Pic}(M) \cong \mathrm{Pic}(\mathbf{P}^1 \times \mathbf{P}^3) \cong \mathbf{Z}^2$. We have $H^0(X_t, O(-1, 2)) = 0$, but we will show that $H^0(X_0, O(-1, 2)) = \oplus_{j \geq 0} H^0(M, O(-1, 2) \otimes O(-jH))$ is not zero. It suffices to show that $H^0(M, O(-1, 2))$ is not zero. We have the exact sequence

$$\begin{aligned} 0 \quad &\to H^0(\mathbf{P}^1 \times \mathbf{P}^3, O(i-1, j-2)) \quad \to H^0(\mathbf{P}^1 \times \mathbf{P}^3, O(i, j)) \to H^0(M, O(i, j)) \\ &\to H^1(\mathbf{P}^1 \times \mathbf{P}^3, O(i-1, j-2)) \quad \to H^1(\mathbf{P}^1 \times \mathbf{P}^3, O(i, j)) \end{aligned}$$

for any $i, j \in \mathbf{Z}$. By the Künneth formula, $H^1(\mathbf{P}^1 \times \mathbf{P}^3, O(-2, 0)) \cong \mathbf{C}$ and $H^1(\mathbf{P}^1 \times \mathbf{P}^3, O(-1, 2)) = 0$. The exact sequence shows that $H^0(M, O(-1, 2)) \neq 0$, as we want. QED

Theorem 7.2 *There is a **Q**-factorial klt Fano 4-fold X_0 which is smooth in codimension 3, a deformation $X \to A^1$ of X_0, and a Weil divisor L on X such that $H^0(X, O(L)) \to H^0(X_0, O(L))$ is not surjective. The fibers X_t for $t \neq 0$ are isomorphic to the $\mathbf{P}^1$-bundle $P(O \oplus O(3))$ over $\mathbf{P}^3$.*

Proof. The example is similar to Theorem 2.1, although the conclusion is different; in that case, de Fernex-Hacon's extension theorem applies to show the surjectivity of restriction maps. First define

$$W = \{([x_1, x_2, x_3, x_4, y_2, g], t) \in P(1^4, 3, 4) \times A^1 : x_1^4 + x_2^4 + x_3^4 + x_4 y_2 = tg\}.$$

Then $t : W \to A^1$ is a flat projective morphism with fibers W_t isomorphic to the weighted projective space $P(1^4, 3)$ for $t \neq 0$, since we can solve the equation for

g when t is not zero. Write μ_r for the group of rth roots of unity. We compute that W_0 has two singular points, the point $[0,0,0,0,1,0]$, where the singularity is just A^4/μ_3, and the point $[0,0,0,0,0,1]$. Near the latter point, W_0 is isomorphic to $\{(x_1,x_2,x_3,x_4,y_2) \in A^5 : x_1^4+x_2^4+x_3^4+x_4y_2=0\}/\mu_4$, where μ_4 acts with weights $(1,1,1,1,3)$ on these variables. The hypersurface singularity $x_1^4+x_2^4+x_3^4+x_4y_2=0$ is terminal, for example by Kollár [19, Exercise 67], since it has the form $z_1z_2+f(z_3,\ldots,z_n)=0$ (called a "cA-type singularity") and is smooth in codimension 2. Since a quotient of a klt variety by a finite group is klt [21, Proposition 5.20(4)], W_0 is klt.

Also, W_0 is $\mathbf{Q}$-factorial by Grothendieck's theorem: a hypersurface that is smooth in codimension 3 is factorial [13, Corollaire XI.3.14]. The same theorem applied to the affine cone over W_0 shows that W_0 has Picard number 1.

Let X be the blow-up of W along the section $A^1 = \{([0,0,0,0,1,0],t)\}$ of $W \to A^1$. Since W is isomorphic to $(A^4/\mu_3)\times A^1$ near this section, blowing up gets rid of this singularity. As a result, the morphism $t : X \to A^1$ is smooth outside the single point $([0,0,0,0,0,1],0)$. Also, the contraction $X \to W$ is K_X-negative, because in each fiber of $X \to A^1$ we are contracting a $\mathbf{P}^3$ with normal bundle $O(-3)$. Since the fiber W_t is isomorphic to the weighted projective space $P(1^4,3)$ for $t \neq 0$, X_t is isomorphic to the $\mathbf{P}^1$-bundle $P(O\oplus O(3))$ over $\mathbf{P}^3$ for $t\neq 0$. Since the contraction $X_0 \to W_0$ is an isomorphism near the one singular point of X_0, X_0 is smooth in codimension 3, $\mathbf{Q}$-factorial, and klt. Finally, X_0 has Picard number 2, since it is a blow-up of W_0 at one point.

We now define a flipping contraction $X \to Z$ over A^1 which contracts a copy of $\mathbf{P}^1$ in the special fiber X_0 to a point. Let

$$Z = \{([u_{11},u_{12},u_{13},u_{14},u_{21},u_{22},u_{23},u_{24},g],t) \in P(1^4,4^5)\times A^1 : \\ \operatorname{rank}(u_{ij}) \leq 1,\ u_{11}^4+u_{12}^4+u_{13}^4+u_{24}=tg\}.$$

Here we view (u_{ij}) as a 2×4 matrix to define its rank. We define the contraction $X \to Z$ as the following rational map $W \dashrightarrow Z$, which becomes a morphism after blowing up the above section A^1 of $W \to A^1$:

$$([x_1,x_2,x_3,x_4,y_2,g],t) \mapsto ([x_1,x_2,x_3,x_4,x_1y_2,x_2y_2,x_3y_2,x_4y_2,g],t).$$

The resulting contraction $X \to Z$ contracts only a copy of $\mathbf{P}^1$, the birational transform of the curve $P(3,4) = \{([0,0,0,0,y_2,g],0)\}$ in W.

It is straightforward to check that the contraction $X \to Z$ is K_X-negative; locally, it is a terminal toric flipping contraction, corresponding to a relation $e_1+4e_2 = e_3+e_4+e_5+e_6$ in $N \cong \mathbf{Z}^5$ in Reid's notation [27, Proposition 4.3]. The toric picture shows that the flip $Y \to Z$ of $X \to Z$ replaces the curve $\mathbf{P}^1$ in the 5-fold X with a copy of $\mathbf{P}^3$.

Since X_0 has Picard number 2 and has two nontrivial K-negative contractions (to W_0 and Z_0), X_0 is Fano. Let us show that $H^0(X,O(L)) \to H^0(X_0,O(L))$ is not surjective for some Weil divisor L on X. The point is that the flip $X \dashrightarrow Y$ replaces a curve in the 4-fold X_0 by a divisor $\mathbf{P}^3$ in Y_0. Let A be an ample Cartier divisor on Y, and also write A for its birational transform on X. Then $H^0(X,O(mA)) = H^0(Y,O(mA))$ for each natural number m because X and Y are isomorphic in codimension one. Clearly, the images of these groups in $H^0(X_0,O(mA))$ and in

$H^0(Y_0, O(mA))$ can be identified. Define the section ring

$$R(X, A) = \oplus_{m\geq 0} H^0(X, O(mA)).$$

Since A is ample on Y, $H^0(Y, O(mA))$ maps onto $H^0(Y_0, O(mA))$ for m big enough, and $\operatorname{Proj}(\operatorname{im}(R(Y, A) \to R(Y_0, A))) = Y_0$. If the restrictions $H^0(X, O(mA)) \to H^0(X_0, O(mA))$ were all surjective, then the algebra $R(X_0, A)$ would be finitely generated and we would have $\operatorname{Proj} R(X_0, A) = Y_0$. But that is impossible, since (when a section ring is finitely generated) the map $X_0 \dashrightarrow \operatorname{Proj} R(X_0, A)$ is a rational contraction, by Hu and Keel [14, Lemma 1.6]. (A birational map $X_0 \dashrightarrow Y_0$ is called a rational contraction if a resolution $p : Z_0 \to X_0$ such that $q : Z_0 \to Y_0$ is a morphism has the property that every p-exceptional divisor is q-exceptional.) So in fact $H^0(X, O(mA)) \to H^0(X_0, O(mA))$ is not surjective for some $m \geq 0$. QED

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