Endomorphisms of Fano 3-folds and log Bott vanishing

Burt Totaro

A basic problem of algebraic dynamics is to determine which projective varieties admit an endomorphism of degree greater than 1. To avoid degenerate cases, we focus on *int-amplified* endomorphisms $f\colon X \to X$, meaning that there is an ample divisor H such that f^*H-H is ample [18, 19]. Fakhruddin, Meng, Zhang, and Zhong conjectured that a smooth complex projective rationally connected variety that admits an int-amplified endomorphism must be a toric variety [7, Question 4.4], [20, Question 1.1]. Kawakami and I introduced a new approach to this problem, showing that a variety with an int-amplified endomorphism must satisfy *Bott vanishing* [14, Theorem C].

In this paper, we extend that result to a logarithmic version of Bott vanishing for an endomorphism with a totally invariant divisor (Theorem 3.1). We also connect log Bott vanishing with some related problems: which varieties are images of toric varieties (Theorem 5.1)? Which varieties admit morphisms of unbounded degree from some other variety (Theorem 4.1)?

We apply these results to Fano 3-folds. Meng, Zhang, and Zhong showed that the only smooth complex Fano 3-folds that admit an int-amplified endomorphism are the toric ones [20, Theorem 1.4]. Also, Achinger, Witaszek, and Zdanowicz showed that the only smooth complex Fano 3-folds that are images of toric varieties are the toric ones [1, proof of Theorem 4.4.1], [2, Theorems 6.9 and 7.7]. Using log Bott vanishing, we reprove both results and extend them to characteristic p, for morphisms of degree prime to p (Theorems 6.1 and 7.2). This resolves [14, Question 1.6]. For Fano 3-folds with Picard number 1, these extensions already appeared in [14, Theorem A and Proposition 3.10].

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1 Notation

A Weil divisor (with integer coefficients) on a normal projective variety is called *ample* if some positive multiple is an ample Cartier divisor. A *contraction* of a normal variety X over a field k is a proper morphism $\pi\colon X \to Y$ with $\pi_*O_X = O_Y$. For a projective variety X over k, $N_1(X)$ is the vector space of 1-cycles with real coefficients modulo numerical equivalence, which has finite dimension. The *Néron-Severi* space $N^1(X)$ is the space of $\mathbf{R}$-Cartier divisors modulo numerical equivalence, and so $N^1(X) = N_1(X)^*$.

A morphism of varieties $f\colon Y \to X$ over a field k is *separable* if it is dominant and the function field $k(Y)$ is a separable field extension of $k(X)$. For k algebraically closed, f is separable if and only if the derivative of f is surjective at some smooth point of Y. For an endomorphism f of a variety X, a closed subset S of X is *totally invariant* under f if $f^{-1}(S) = S$.

For a normal variety X over a field k and $i \geq 0$, we write Ω^i_X for $\Omega^i_{X/k}$. The sheaf of *reflexive differentials* $\Omega^{[i]}_X$ is the double dual $(\Omega^i_X)^{**}$. For a reduced divisor D on X, $\Omega^{[i]}_X(\log D)$ (reflexive differentials with *log poles* along D) denotes the sheaf of i-forms α on the smooth locus of $X - D$ such that both α and $d\alpha$ have at most a simple pole along each component of D. For a reflexive sheaf M and a Weil divisor E on X, we write $M(E)$ for the reflexive sheaf $(M \otimes O_X(E))^{**}$. If X is smooth over k, then $O_X(E)$ is a line bundle, and $M(E)$ is just the tensor product $M \otimes O_X(E)$.

2 The trace of a differential form with log poles

Here is the key lemma for this paper, extending the proof of [14, Theorem C] to allow log poles.

Lemma 2.1. *Let $f\colon Y \to X$ be a finite surjective morphism of normal varieties over a perfect field k. Let A be a Weil divisor on X. Let E_X and D_X be reduced divisors on X with $0 \leq E_X \leq D_X$. Let D_Y be the sum of the components of $f^{-1}(D_X)$ along which the ramification degree e of f is invertible in k, and let E_Y be the corresponding divisor inside $f^{-1}(E_X)$. Then, for every $i \geq 0$, the pullback and pushforward of differential forms give maps of reflexive sheaves*

$$\Omega^{[i]}_X(\log D_X)(A - E_X) \to f_*(\Omega^{[i]}_Y(\log D_Y)(f^*A - E_Y)) \to \Omega^{[i]}_X(\log D_X)(A - E_X),$$

with composition equal to multiplication by $\deg(f)$.

In characteristic p, even if the morphism f in Lemma 2.1 has degree prime to p, it may be wildly ramified along some divisors. That requires the careful choice of D_Y and E_Y in the statement, in order to get something true.

Proof. Since k is perfect, the normal varieties X and Y over k are geometrically normal [25, Tag 038O], hence smooth over k in codimension 1.

As in the statement, let D_Y be the sum of the components of $f^{-1}(D_X)$ along which the ramification degree e of f is invertible in k. (The *ramification degree* e along a component D_2 of $f^{-1}(D_X)$ is the coefficient of D_2 in the pullback Weil divisor f^*D_X, which is defined since f is finite and D_X is generically Cartier. The condition that e is invertible in k is not the same as f being tamely ramified along D_2; tame ramification would mean that e is invertible in k *and* that $f\colon D_2 \to f(D_2)$ is separable.) Likewise, let E_Y be the sum of the components of $f^{-1}(E_X)$ along which the ramification degree of f is invertible in k.

We want to construct pullback and pushforward maps of reflexive sheaves:

$$\Omega^{[i]}_X(\log D_X)(-E_X) \to f_*\Omega^{[i]}_Y(\log D_Y)(-E_Y) \to \Omega^{[i]}_X(\log D_X)(-E_X).$$

First consider the pullback map. Since we are mapping to a reflexive sheaf, it suffices to define this map outside the subset $X^{\text{sing}} \cup f(Y^{\text{sing}})$, which has codimension at least 2 in X. The map is the usual pullback of differential forms outside D_X. So it suffices to show that the pullback sends $\Omega^i_X(\log D_X)(-E_X)$ into $\Omega^i_Y(\log D_Y)(-E_Y)$ near the generic point of each component D_2 of $f^{-1}(D_X)$. Let $D_1 = f(D_2)$.

There are four cases.

Case 1 D_1 not in E_X, $e \in k^*$. Then we need to show that $\Omega^i_X(\log D_1)$ pulls back into $\Omega^i_Y(\log D_2)$ near the generic point of D_2. Let g be a local defining function of D_1 in X and h a local defining function of D_2 in Y. We have $f^*(g) = h^e u$ for some unit $u \in O^*_{Y,D_2}$. Then $f^*(\frac{dg}{g}) = e\frac{dh}{h} + \frac{du}{u}$, and so it is clear that forms in $\Omega^i_X(\log D_1) = \Omega^i_X + \frac{dg}{g}\Omega^{i-1}_X$ pull back to $\Omega^i_Y(\log D_2) = \Omega^i_Y + \frac{dh}{h}\Omega^{i-1}_Y$ near the generic point of D_2.

Case 2 D_1 not in E_X, $e = 0 \in k$. Then we need to show that forms in $\Omega^i_X(\log D_1)$ pull back to Ω^i_Y near the generic point of D_2. This is clear by the formulas in Case 1, using that $e = 0 \in k$.

Case 3 D_1 in E_X, $e \in k^*$. Then we need to show that $\Omega^i_X(\log D_1)(-D_1)$ pulls back into $\Omega^i_Y(\log D_2)(-D_2)$ near the generic point of D_2. This follows from Case 1, using that g pulls back to a unit times a positive power of h.

Case 4 D_1 in E_X, $e = 0 \in k$. Then we need to show that $\Omega^i_X(\log D_1)(-D_1)$ pulls back into Ω^i_Y near the generic point of D_2. This is clear, since $\Omega^i_X(\log D_1)(-D_1)$ is contained in Ω^i_X.

Next, we want to define the *pushforward* (or *trace*) map

$$f_*\Omega^{[i]}_Y(\log D_Y)(-E_Y) \to \Omega^{[i]}_X(\log D_X)(-E_X).$$

Outside $D_X \cup X^{\text{sing}} \cup f(Y^{\text{sing}})$, this trace map was defined by Garel and Kunz [9], [16, section 16], [25, Tag 0FLC]. (They assume that the finite morphism f is flat with lci fibers; that holds on the open set mentioned, since both X and Y are smooth there [25, Tags 00R4 and 09Q7].) Since we are mapping into a reflexive sheaf, it remains to check that the trace map above is regular near the generic point of each component D_1 of D_X. It suffices to check this after replacing X by an elementary étale neighborhood [25, Tag 02LE] of the generic point of D_1, in such a way that Y becomes a disjoint union of varieties containing the different components D_2 of $f^{-1}(D_1)$. (Then the trace from Y to X is the sum of the traces for these different varieties.) We can work on one of these varieties; that is, we can assume that $f^{-1}(D_1)$ is an irreducible divisor D_2.

There are four cases.

Case 1 D_1 not in E_X, $e \in k^*$. Then we need to show that the pushforward of a form in $\Omega^i_Y(\log D_2)$ lies in $\Omega^i_X(\log D_1)$ near the generic point of D_1. We have $\Omega^i_Y(\log D_2) = \Omega^i_Y + \frac{dh}{h}\Omega^{i-1}_Y$. Since $f^*(g) = h^e u$ in the notation above, we have $e\frac{dh}{h} = f^*(\frac{dg}{g}) - \frac{du}{u}$. Since $e \in k^*$, it follows that $\Omega^i_Y(\log D_2) = \Omega^i_Y + f^*(\frac{dg}{g})\Omega^{i-1}_Y$. By the projection formula [25, Tag 0FLC], the trace of these forms lies in $\Omega^i_X + \frac{dg}{g}\Omega^{i-1}_X = \Omega^i_X(\log D_1)$.

Case 2 D_1 not in E_X, $e = 0 \in k$. Then we need to show that forms in Ω^i_Y push forward to $\Omega^i_X(\log D_1)$ near the generic point of D_1. This is easy, since the pushforward is contained in $\Omega^i_X \subset \Omega^i_X(\log D_1)$.

For cases 3 and 4, we use the following property of the trace map on differential forms [11, Remark 5.7]:

Proposition 2.2. *Let $f\colon Y \to X$ be a finite flat morphism with lci fibers over a field k. Suppose that D_1 is an irreducible divisor in X such that $D_2 := f^{-1}(D_1)$ is also irreducible. Let e be the ramification degree of f along D_2. Then the following diagram commutes near the generic point of D_1:*

$$\begin{array}{ccc} \Omega^*_Y & \longrightarrow & \Omega^*_{D_2} \\ \big\downarrow{\scriptstyle \mathrm{tr}_f} & & \big\downarrow{\scriptstyle e\,\mathrm{tr}_f} \\ \Omega^*_X & \longrightarrow & \Omega^*_{D_1}. \end{array}$$

In our situation (above), the flatness and lci assumptions of Proposition 2.2 hold near the generic points of D_1 and D_2, because X and Y are smooth there.

Case 3 D_1 in E_X, $e \in k^*$. Then we need to show that forms in $\Omega^i_Y(\log D_2)(-D_2)$ push forward into $\Omega^i_X(\log D_1)(-D_1)$ near the generic point of D_1. That is, we want to show that the trace of $h\,\Omega^i_Y + h\frac{dh}{h}\Omega^{i-1}_Y$ is contained in $g\,\Omega^i_X + g\frac{dg}{g}\Omega^{i-1}_X$. Proposition 2.2 gives that the trace of $(h, dh)\Omega^*_Y$ is contained in $(g, dg)\Omega^*_X$, which proves the claim.

Case 4 D_1 in E_X, $e = 0 \in k$. Then we need to show that forms in Ω^i_Y push forward to $\Omega^i_X(\log D_1)(-D_1)$ near the generic point of D_1. That is, we want $\mathrm{tr}_f(\Omega^i_Y)$ to be contained in $(g, dg)\Omega^*_X$. This follows from Proposition 2.2, using that $e = 0 \in k$. Thus we have constructed the pushforward map in all cases.

We now return to the original morphism $f\colon Y \to X$ (before we restricted to an étale neighborhood of X). The composition of pullback and pushforward is multiplication by $\deg(f)$, as one can check on an open subset where X and Y are smooth, by the projection formula: for a form α on Y, $\mathrm{tr}_f(f^*\alpha) = \mathrm{tr}_f(1)\alpha = \deg(f)\alpha$ [25, Tag 0FLC].

Tensoring the maps above with $O_X(A)$ and taking double duals, we have pullback and pushforward maps

$$\Omega^{[i]}_X(\log D_X)(A - E_X) \to f_*(\Omega^{[i]}_Y(\log D_Y)(f^*A - E_Y)) \to \Omega^{[i]}_X(\log D_X)(A - E_X),$$

with composition equal to multiplication by $\deg(f)$, as we want. (Note that f^*A is a Weil divisor (with integer coefficients), because f is a finite surjective morphism between normal varieties.) □

3 Endomorphisms and log Bott vanishing

Kawakami and I showed that a projective variety with a suitable endomorphism must satisfy Bott vanishing [14, Theorem C]. (The endomorphism is assumed to be int-amplified and of degree invertible in the base field.) We now prove a logarithmic generalization, when the endomorphism has a totally invariant divisor. This generalizes Fujino's work on the case of toric varieties [8, Theorem 1.3]. Indeed, every toric variety has an action of the multiplicative monoid of positive integers, and every toric divisor is totally invariant under those endomorphisms.

Theorem 3.1. *Let X be a normal projective variety over a perfect field k. Suppose that X admits an int-amplified endomorphism f whose degree is invertible in k. Let*

D be a reduced divisor on X such that $f^{-1}(D) \subset D$. Then (X, D) satisfies log Bott vanishing. That is,

$$H^j(X, \Omega_X^{[i]}(\log D)(A - E)) = 0$$

for every reduced divisor E with $0 \le E \le D$, $i \ge 0$, $j > 0$, and A an ample Weil divisor.

It may seem artificially strong to assume that an endomorphism has a totally invariant divisor. But in fact, this property comes up naturally in classifying varieties with endomorphisms. See [20] or the proof of Theorem 6.1.

Theorem 3.1 is the most general vanishing property I know how to prove, for a variety with an endomorphism. The idea of allowing any divisor $0 \le E \le D$ was suggested by Fujino's result for toric varieties [8, Theorem 1.3].

Remark 3.2. Log Bott vanishing can fail if we only assume that f is separable (rather than of degree invertible in k). (It is an open question whether the original form of Bott vanishing holds for separable endomorphisms. Bott vanishing clearly fails for inseparable endomorphisms, since *every* projective variety over a finite field has the Frobenius endomorphism, which is int-amplified.) Namely, let p be a prime number, and let X be the blow-up of $\mathbf{P}^2$ in characteristic p at the $p+1$ $\mathbf{F}_p$-points on a line defined over $\mathbf{F}_p$. Write H for the pullback to X of the line bundle $O(1)$. Nakayama showed that X has a separable int-amplified endomorphism f of degree p^2 [22, Example 4.5]. The sum of the exceptional divisors, $D := \sum_{i=0}^{p} E_i$, is totally invariant under f. But log Bott vanishing fails for the ample line bundle $A := (p+2)H - E_0 - \cdots - E_p$ and $E := D$.

Indeed, $H^1(X, A - E) \ne 0$ in this case, since the Euler characteristic $\chi(X, A - E) = (p^2 + p + 6)/2$ is less than $h^0(X, A - E) = (p^2 + 3p + 4)/2$. To compute the Euler characteristic, use that for the blow-up of a smooth surface Z at a k-point P, $\pi\colon Y \to Z$ with exceptional curve E, we have $R\pi_* O_Y(-2E) = (I_{P/Z})^2 \subset O_Z$ and $\dim_k(O_Z/(I_{P/Z})^2) = 3$. So $\chi(X, A - E) = \chi(X, (p+2)H - 2E_0 - \cdots - 2E_p) = h^0(\mathbf{P}^2, (p+2)H) - 3(p+1) = \binom{p+4}{2} - 3(p+1) = (p^2+p+6)/2$. To compute h^0, use that $A - E$ has degree $-p < 0$ on the strict transform of the line, $L \sim H - E_0 - \cdots - E_p$. So $h^0(X, A - E) = h^0(X, A - E - L) = h^0(X, (p+1)H - E_0 - \cdots - E_p) = \binom{p+3}{2} - (p+1) = (p^2 + 3p + 4)/2$.

Proof. (Theorem 3.1) Since f is int-amplified, there is an ample Cartier divisor H on X such that $f^*H - H$ is ample. In particular, f^*H is ample, and so f does not contract any curves. So $f\colon X \to X$ is finite. Since $f^{-1}(D) \subset D$ and f is surjective, we have $f^{-1}(D) = D$.

Since $f^{-1}(D) = D$, f permutes the (finitely many) irreducible components of D. After replacing f by a positive iterate, we can assume that f maps each component of D to itself. So $f^{-1}(D_1) = D_1$, for each component D_1 of D. In particular, since E is a reduced divisor with $0 \le E \le D$, we now have that $f^{-1}(E) = E$. (In what follows, we use only that $f^{-1}(E) = E$, not that f maps each component of D to itself. This makes the proof clearer, in terms of notation.)

We are given that the degree of f is invertible in k. The inverse image of each irreducible component D_1 of D is a single irreducible component D_2 of D. Therefore, the degree of $f\colon X \to X$ is the product of the degree of $f\colon D_2 \to D_1$ and the ramification degree of f along D_2. It follows that this ramification degree is invertible in k and that $k(D_2)$ is a finite separable extension of $k(D_1)$ via f. That is, f is tamely ramified over each component of D.

Let A be an ample Weil divisor on X. By Lemma 2.1, we have pullback and pushforward maps

$$\Omega_X^{[i]}(\log D_X)(A-E) \to f_*(\Omega_X^{[i]}(\log D)(f^*A-E)) \to \Omega_X^{[i]}(\log D_X)(A-E),$$

with composition equal to multiplication by $\deg(f)$. (Note that f^*A is a Weil divisor (with integer coefficients), because f is a finite surjective morphism between normal varieties.)

Given this, the proof of [14, Theorem C] works. Namely, since $\deg(f)$ is invertible in k, it follows that the pullback map is split injective as a map of O_X-modules. Let $j > 0$. Taking cohomology (and using that f is finite), it follows that $H^j(X, \Omega_X^{[i]}(\log D)(A-E)) \to H^j(X, \Omega_X^{[i]}(\log D)(f^*A-E))$ is split injective. The same argument works for any iterate f^e with $e \geq 1$.

Using that f is int-amplified, we showed that the iterates $(f^e)^*(A)$ become arbitrarily large in the ample cone of X, as e goes to infinity [14, proof of Theorem C]. By Fujita vanishing for Weil divisors [14, Lemma 2.6], it follows that there is an $e \geq 1$ such that $H^j(X, \Omega_X^{[i]}(\log D)((f^e)^*A-E)) = 0$. By the previous paragraph, we have $H^j(X, \Omega_X^{[i]}(\log D)(A-E)) = 0$, as we want. $\square$

4 Morphisms

Let X and Y be projective varieties of the same dimension with Picard number 1. If X does not satisfy Bott vanishing, then there is an upper bound on the degrees of all morphisms $Y \to X$ with degree invertible in k [14, Proposition 2.7]. We now prove a log version of that result.

Theorem 4.1. *Let X and Y be normal projective varieties of the same dimension over a perfect field k. Assume that X and Y have Picard number 1. Let $D_X \subset X$ and $B_Y \subset Y$ be reduced divisors. Suppose that there are morphisms $f: Y \to X$ of arbitrarily high degree such that the degree is invertible in k and $f^{-1}(D_X) \subset B_Y$. Then (X, D_X) satisfies log Bott vanishing. That is, we have*

$$H^j(X, \Omega_X^{[i]}(\log D_X)(A-E)) = 0$$

for every reduced divisor $0 \leq E \leq D_X$, $i \geq 0$, $j > 0$, and A an ample Weil divisor on X.

The proof of Theorem 4.1 shows that one can drop the assumption that X and Y have Picard number 1 if one replaces "f of arbitrarily high degree" by "f^*H arbitrarily large in the ample cone of Y", for a fixed ample Cartier divisor H on X.

Proof. Since each morphism $f: Y \to X$ has degree invertible in k, the degree is positive. Since Y has Picard number 1, the pullback of an ample Cartier divisor on X is ample on Y, and so f is finite.

For each morphism $f: Y \to X$, let D_Y be the sum of the components of $f^{-1}(D_X)$ along which the ramification degree of f is invertible in k, and likewise define $E_Y \subset f^{-1}(E_Y)$. Since D_Y and E_Y are contained inside the fixed divisor B_Y, we can assume (after passing to a subsequence of the morphisms f) that D_Y and E_Y are the same for all the morphisms $f\colon Y \to X$ that we consider.

Let A be an ample Weil divisor on X. By Lemma 2.1, we have pullback and pushforward maps

$$\Omega_X^{[i]}(\log D_X)(A-E) \to f_*(\Omega_Y^{[i]}(\log D_Y)(f^*A - E_Y)) \to \Omega_X^{[i]}(\log D_X)(A-E).$$

The composition is multiplication by $\deg(f)$, and so the pullback map is split injective. It follows that $H^j(X, \Omega_X^{[i]}(\log D_X)(A-E))$ injects into $H^j(Y, \Omega_Y^{[i]}(\log D_Y)(f^*A - E_Y))$. Since we have morphisms f of arbitrarily large degree, f^*A becomes arbitrarily large in the ample cone of Y (here just one ray). Therefore, for f of sufficiently large degree, Fujita vanishing for Weil divisors gives that $H^j(Y, \Omega_Y^{[i]}(\log D_Y)(f^*A - E_Y))$ is equal to zero [14, Lemma 2.6]. It follows that $H^j(X, \Omega_X^{[i]}(\log D_X)(A-E)) = 0$, as we want. □

5 Images of toric varieties: log Bott vanishing

A projective variety that is an image of a toric variety (by a morphism of degree invertible in k) must satisfy Bott vanishing [14, Proposition 3.10]. We now prove a log version of that result.

Theorem 5.1. *Let X be a normal projective variety over a perfect field k. Suppose that there is a morphism f from a proper toric variety Y onto X. If $Y \to Y_1 \to X$ is the Stein factorization of f, assume that the degree of $Y_1 \to X$ is invertible in k. Let $D_X \subset X$ be a reduced divisor such that $f^{-1}(D_X)$ is a union of toric divisors. Then (X, D_X) satisfies log Bott vanishing. That is, we have*

$$H^j(X, \Omega_X^{[i]}(\log D_X)(A-E)) = 0$$

for every reduced divisor $0 \leq E \leq D_X$, $i \geq 0$, $j > 0$, and A an ample Weil divisor on X.

It may seem unrealistically strong to assume that $f^{-1}(D_X)$ is a union of toric divisors. Nonetheless, the proof of Theorem 7.2 shows how Theorem 5.1 can be used for classifying images of toric varieties.

Proof. Every contraction of a toric variety is toric [26, Proposition 2.7]. (This was known earlier for projective toric varieties [6, Theorem 6.28 and exercise 7.2.3].) So, replacing Y by Y_1, we can assume that the surjection $f\colon Y \to X$ is finite. By our assumptions, the degree of f is invertible in k. Let D_Y be the sum of the components of $f^{-1}(D_X)$ along which the ramification degree e of f is invertible in k. Likewise, let E_Y be the sum of the components of $f^{-1}(E)$ along which the ramification degree of f is invertible in k. By our assumptions, D_Y and E_Y are sums of toric divisors.

Let A be an ample Weil divisor on X. By Lemma 2.1, we have pullback and pushforward maps

$$\Omega_X^{[i]}(\log D_X)(A-E) \to f_*(\Omega_Y^{[i]}(\log D_Y)(f^*A - E_Y)) \to \Omega_X^{[i]}(\log D_X)(A-E),$$

with composition equal to multiplication by $\deg(f)$. Since $\deg(f)$ is invertible in k, it follows that the pullback map is split injective as a map of O_X-modules. Let $j > 0$.

Taking cohomology (and using that f is finite), it follows that $H^j(X, \Omega_X^{[i]}(\log D_X)(A-E)) \to H^j(Y, \Omega_Y^{[i]}(\log D_Y)(f^*A - E_Y))$ is split injective. Since f^*A is ample on Y, the latter cohomology group is zero by log Bott vanishing for toric varieties [8, Theorem 1.3]. (Alternatively, this follows from Theorem 3.1, using the multiplication maps on a toric variety.) It follows that $H^j(X, \Omega_X^{[i]}(\log D_X)(A - E)) = 0$, as we want. □

6 Endomorphisms of del Pezzo surfaces and Fano 3-folds

Theorem 6.1. *Let X be a smooth Fano 3-fold over an algebraically closed field k. If X has an int-amplified endomorphism of degree invertible in k, then X is toric.*

The proof uses Tanaka's theorem that the classification of smooth Fano 3-folds has essentially the same form in every characteristic [27, Theorem 1.1]. Without that, our proof applies to Fano 3-folds in any characteristic that are given by the same construction as one of the Fano 3-folds over $\mathbf{C}$ (classified by Iskovskikh and Mori-Mukai). (Tanaka leaves one question open: whether there is a Fano 3-fold with Picard number 1, Fano index 1, and genus $g = 11$ (that is, $(-K_X)^3 = 2g - 2 = 20$) in some positive characteristic. However, if such a variety exists, it does not satisfy Bott vanishing [14, Proposition 3.8], and so it is irrelevant for the results in this paper such as Theorem 6.1.)

In characteristic zero, Theorem 6.1 was proved by Meng, Zhang, and Zhong [20, Theorem 1.4]. Here we give a new proof which works in any characteristic. For Fano 3-folds with Picard number 1, this is [14, Theorem A]. We also prove an analogous result for del Pezzo surfaces (Proposition 6.4).

To prove Theorem 6.1, our basic idea is to use that X satisfies Bott vanishing, but that is not enough by itself: among the smooth complex Fano 3-folds, 18 are toric while 19 others also satisfy Bott vanishing [28]. In proving Theorem 6.1, one key point is that when X has an endomorphism as above, not only X but also every contraction of X satisfies Bott vanishing (Lemma 6.2).

Proof. (Theorem 6.1) Assume that X has an int-amplified endomorphism of degree invertible in k. Then X satisfies Bott vanishing [14, Theorem C]. Among all smooth Fano 3-folds, exactly 19 non-toric Fano 3-folds (up to isomorphism) satisfy Bott vanishing [28, Theorem 0.1]. In Mori-Mukai's numbering [21, 12, 3], these are (2.26), (2.30), (3.15)–(3.16), (3.18)–(3.24), (4.3)–(4.8), (5.1), and (6.1). (To be precise, the answer is a subset of this in characteristic 2, where only 9 non-toric Fano 3-folds on the known list satisfy Bott vanishing [28, section 2].) We need to show that none of these 19 varieties has an int-amplified endomorphism f of degree invertible in k. Table 1 describes these 19 varieties, using information from Mori-Mukai or the web site Fanography [21, 3]. In the table, V_5 is the smooth quintic del Pezzo 3-fold $\mathrm{Gr}(2, 5) \cap \mathbf{P}^6 \subset \mathbf{P}^9$, Q is the smooth quadric 3-fold, W is the flag manifold $GL(3)/B$ (or equivalently, a smooth divisor of degree $(1, 1)$ in $\mathbf{P}^2 \times \mathbf{P}^2$), (3.17) is a smooth divisor of degree $(1, 1, 1)$ in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$, and S_5 is the quintic del Pezzo surface.

For each of these 19 Fano 3-folds, the nef cone of X is the same as the (known) nef cone in characteristic zero. In particular, it is rational polyhedral. (The nef

Fano 3-fold	Description
(2.26)	the blow-up of $V_5 \subset \mathbf{P}^6$ along a general line in it
(2.30)	the blow-up of $Q \subset \mathbf{P}^4$ at a point
(3.15)	the blow-up of Q along a disjoint line and conic
(3.18)	the blow-up of Q along a conic and then along a fiber in the exceptional divisor
(3.19)	the blow-up of Q at 2 non-collinear points
(3.20)	the blow-up of Q along 2 disjoint lines
(3.23)	the blow-up of Q along a line and then along a fiber of the exceptional divisor
(4.4)	the blow-up of Q along a conic and then along 2 fibers of the exceptional divisor
(5.1)	the blow-up of Q along a conic and then along 3 fibers of the exceptional divisor
(3.16)	the blow-up of $W \subset \mathbf{P}^2 \times \mathbf{P}^2$ along a curve of degree $(2, 1)$
(3.24)	the blow-up of W along a curve of degree $(1, 0)$
(4.7)	the blow-up of W along disjoint $(1, 0)$ and $(0, 1)$ curves
(4.3)	the blow-up of (3.17) in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ along a curve of degree $(1, 1, 0)$
(3.21)	the blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ along a curve of degree $(2, 1)$
(3.22)	the blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ along a curve of degree $(0, 2)$
(4.5)	the blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ along disjoint $(2, 1)$ and $(1, 0)$ curves
(4.6)	the blow-up of $\mathbf{P}^3$ along 3 disjoint lines
(4.8)	the blow-up of $(\mathbf{P}^1)^3$ along a curve of degree $(0, 1, 1)$
(6.1)	$\mathbf{P}^1 \times S_5$

Table 1: The 19 non-toric Fano 3-folds that satisfy Bott vanishing

cone for every smooth complex Fano 3-fold is given in [5]. The nef cone was recomputed for the 19 varieties above in any characteristic, yielding the same result [28]. The argument also shows that all nef divisors are semi-ample, and hence the contractions of these varieties have the same description in every characteristic.)

Lemma 6.2. *Let X be a normal projective variety over a perfect field k. Suppose that X has only finitely many contractions (for example, this holds if X is a Mori dream space, or if the nef cone is rational polyhedral). Assume that X has an int-amplified endomorphism of degree invertible in k. Then:*

(i) Every contraction of X admits an int-amplified endomorphism of degree invertible in k.

(ii) Every contraction of X satisfies Bott vanishing for ample Weil divisors.

If we only assume that X has a separable int-amplified endomorphism, then every contraction of X also has a separable int-amplified endomorphism.

One can get around the assumption that X has only finitely many contractions by considering only contractions that reduce the Picard number by 1 [19, Definition 5.2 and Theorem 7.9].

Proof. Let $f\colon X \to X$ be a separable int-amplified endomorphism, and let $\pi\colon X \to Z$ be a contraction. Every curve on X is the image of a curve under f. It follows that the pullback linear map $f^*\colon N^1(X) \to N^1(X)$ is injective, hence an isomorphism. It also follows that $f^*(\mathrm{Nef}(X)) = \mathrm{Nef}(X)$. A contraction is determined by a face of the cone of curves, or equivalently by a face of the nef cone in $N^1(X)$; so f permutes the set of contractions of X. Since X has only finitely many contractions, after replacing f by a positive iterate, we can assume that f preserves the contraction π. That is, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{g} & Z, \end{array}$$

for an endomorphism g of Z. Here g is separable since f is.

An endomorphism f of a normal projective variety over a field k is int-amplified if and only if all eigenvalues of f^* on $N^1(X)_{\mathbf{C}}$ have absolute value greater than 1 [18, Theorem 1.1]. (Meng assumes that k has characteristic zero, but his proof works in any characteristic.) In our situation, the pullback $\pi^*\colon N^1(Z) \to N^1(X)$ is injective (because every curve on Z is the image of some curve on X). Since all eigenvalues of f^* on $N^1(X)_{\mathbf{C}}$ have absolute value greater than 1, the same holds for the eigenvalues of g^* on $N^1(Z)_{\mathbf{C}}$. That is, g is an int-amplified endomorphism of Z. That completes the proof for a separable int-amplified endomorphism f.

Suppose in addition that f has degree invertible in k. The degree of g divides the degree of f and hence is invertible in k. It follows that Z satisfies Bott vanishing by [14, Theorem C] (generalized as Theorem 3.1 above). $\square$

This immediately rules out 13 of the 19 Fano 3-folds above, since they contract to other varieties where Bott vanishing fails, by Table 1. Namely, by [28, section 2] (or earlier work), Bott vanishing fails for the quintic del Pezzo 3-fold V_5, the quadric

3-fold Q, the flag manifold $W = GL(3)/B$, and the 3-fold (3.17), a smooth divisor in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ of degree $(1,1,1)$.

That leaves: (3.21), (3.22), (4.5), (4.6), (4.8), and (6.1). To analyze these, we will use:

Lemma 6.3. *Let $f\colon Y \to X$ be a finite surjective morphism of normal varieties over an algebraically closed field k. Let $\pi_X\colon X \to X_1$ and $\pi_Y\colon Y \to Y_1$ be contractions, and suppose that there is a morphism $g\colon Y_1 \to X_1$ making the diagram commute:*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_X} \\ Y_1 & \xrightarrow{g} & X_1. \end{array}$$

Let Δ_X be the set of points $P \in X_1(k)$ with $\pi_X^{-1}(P)$ reducible, and likewise define $\Delta_Y \subset Y_1(k)$. Then $g^{-1}(\Delta_X) \subset \Delta_Y$.

Proof. For each k-point P in Y_1, I claim that the fiber of Y over P maps onto the fiber of X over $g(P)$. This is fairly clear geometrically; to be precise, we can imitate the proof of [4, Lemma 7.3]. Namely, suppose that $Y_P \to X_{g(P)}$ is not surjective. Let $S = g^{-1}(g(P)) - \{P\}$, a finite set. Then $S \neq \emptyset$ and $U := Y - \pi_Y^{-1}(S)$ is an open dense subset of Y not equal to Y, using that Y is irreducible. Since X is normal, the finite surjection $f\colon Y \to X$ is an open map [25, Tag 0F32], and so $f(U)$ is an open dense subset of X. In particular, $f(U) \cap X_{g(P)}$ is open in $X_{g(P)}$. Note that $f(U) = (X - X_{g(P)}) \cup f(Y_P)$. So $f(Y_P)$ is open in $X_{g(P)}$. It is not empty, since Y_P is not empty. Since f is proper, $f(Y_P)$ is also closed in $X_{g(P)}$. Since $\pi_X\colon X \to X_1$ is a contraction, $X_{g(P)}$ is connected, and so $f(Y_P) = X_{g(P)}$, as we want.

An irreducible scheme cannot map onto a reducible scheme. Therefore, $g^{-1}(\Delta_X)$ is contained in Δ_Y. ☐

Let us continue the proof of Theorem 6.1, that a Fano 3-fold with an int-amplified endomorphism of degree invertible in k is toric. Of the 6 remaining cases, we start with (6.1), which is $\mathbf{P}^1$ times the quintic del Pezzo surface X. By Lemma 6.2, it suffices to show that X has no int-amplified endomorphism of degree invertible in k. This was shown by Nakayama [22, Proposition 4.4]. We prove a bit more, namely that X has no separable int-amplified endomorphism.

Proposition 6.4. *A smooth del Pezzo surface over an algebraically closed field k that admits a separable int-amplified endomorphism must be toric.*

Proof. The classification of smooth del Pezzo surfaces has the same form in any characteristic [15, section III.3]. In particular, a smooth del Pezzo surface X is toric if and only if its degree $(-K_X)^2$ is at least 6. (Then X is $\mathbf{P}^1 \times \mathbf{P}^1$ or else the blow-up of $\mathbf{P}^2$ at 3 or fewer points in general position.) And every smooth del Pezzo surface of degree at most 5 is a blow-up of the quintic del Pezzo surface (the blow-up of $\mathbf{P}^2$ at 4 points in general position). By Lemma 6.2, it suffices to show that the quintic del Pezzo surface X does not have a separable int-amplified endomorphism.

Suppose that the quintic del Pezzo surface X has a separable int-amplified endomorphism f. Let π_X be one of the contractions of X to $\mathbf{P}^1$. By the proof of Lemma 6.2, after replacing f by a positive iterate, the contraction π_X is f-equivariant, giving a separable int-amplified endomorphism g of $\mathbf{P}^1$.

The point is that the contraction π_X from the quintic del Pezzo surface to $\mathbf{P}^1$ has 3 singular fibers, and these fibers are reducible (namely, the union of two copies of $\mathbf{P}^1$). Let $\Delta_X \subset \mathbf{P}^1$ be the discriminant locus of π_X, consisting of 3 points. By Lemma 6.3, $g^{-1}(\Delta_X)$ is contained in Δ_X. Then pulling back differential forms gives a map

$$\Omega^1_{\mathbf{P}^1}(\log \Delta_X) \to g_* \Omega^1_{\mathbf{P}^1}(\log \Delta_X).$$

Equivalently, we can view this as a map of line bundles from $g^*(K_{\mathbf{P}^1} + \Delta_X)$ to $K_{\mathbf{P}^1} + \Delta_X$, hence as a section of $K_{\mathbf{P}^1} + \Delta_X - g^*(K_{\mathbf{P}^1} + \Delta_X)$. Since g is separable, this section is not identically zero. But $K_{\mathbf{P}^1} + \Delta_X$ has degree 1, so $K_{\mathbf{P}^1} + \Delta_X - g^*(K_{\mathbf{P}^1} + \Delta_X)$ has degree $1 - \deg(g) < 0$, a contradiction. Thus the quintic del Pezzo surface does not have a separable int-amplified endomorphism. $\square$

Since the 3-fold (6.1) is $\mathbf{P}^1$ times the quintic del Pezzo surface, Lemma 6.2 and Proposition 6.4 give that (6.1) does not admit a separable int-amplified endomorphism. A fortiori, it does not have an int-amplified endomorphism of degree invertible in k, as considered in Theorem 6.1.

The proof for (4.8), the blow-up X of $(\mathbf{P}^1)^3$ along a curve F of degree $(0,1,1)$, is somewhat similar. We can take F to be a point in $\mathbf{P}^1$ times the diagonal $\Delta^2_{\mathbf{P}^1}$ in $(\mathbf{P}^1)^2$. Suppose that X has an int-amplified endomorphism of degree invertible in k. The contraction π of X to $(\mathbf{P}^1)^2$ (corresponding to the last two $\mathbf{P}^1$ factors) has discriminant locus the diagonal $\Delta_{\mathbf{P}^1}$, and the fibers over $\Delta_{\mathbf{P}^1}$ are reducible. After replacing f by an iterate, π is f-equivariant, by Lemma 6.2. Write $g\colon (\mathbf{P}^1)^2 \to (\mathbf{P}^1)^2$ for the resulting endomorphism of $(\mathbf{P}^1)^2$, which is int-amplified and has degree invertible in k. By Lemma 6.3, $\Delta_{\mathbf{P}^1}$ is totally invariant under g.

By Theorem 3.1, it follows that $((\mathbf{P}^1)^2, \Delta_{\mathbf{P}^1})$ satisfies log Bott vanishing. In particular, taking $A = O(1,1)$ and $E = \Delta_{\mathbf{P}^1} \sim A$, we have $H^1(X, \Omega^1_{(\mathbf{P}^1)^2}(\log \Delta_{\mathbf{P}^1})) = 0$. But in fact, this cohomology group is not zero. Indeed, for any smooth divisor D in a smooth variety X over k, we have an exact sequence of coherent sheaves [25, Tag 0FMW]:

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \xrightarrow{\text{Res}} O_D \to 0.$$

So we have an exact sequence of cohomology groups $H^0(D, O_D) \to H^1(X, \Omega^1_X) \to H^1(X, \Omega^1_X(\log D))$. For $X = (\mathbf{P}^1)^2$ and $D = \Delta_{\mathbf{P}^1}$, we have $h^0(D, O_D) = 1$ and $h^1(X, \Omega^1_X) = 2$, and so $H^1(X, \Omega^1_X(\log D)) \neq 0$ as claimed. This contradiction shows that the Fano 3-fold (4.8) does not have an int-amplified endomorphism of degree invertible in k.

Next, we exclude the 3-fold (4.6), the blow-up of $\mathbf{P}^3$ along three pairwise disjoint lines L_1, L_2, L_3 over k. Here X has three contractions to $\mathbf{P}^1$, using that the blow-up of $\mathbf{P}^3$ along each line L_i is a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$. Let $\pi\colon X \to (\mathbf{P}^1)^2$ be the contraction given by the morphisms to $\mathbf{P}^1$ associated to L_1 and L_2. For clarity, first consider the blow-up Y of $\mathbf{P}^3$ along L_1 and L_2; then the contraction $Y \to (\mathbf{P}^1)^2$ is a $\mathbf{P}^1$-bundle. The line $L_3 \subset Y$ maps to a curve of degree $(1,1)$ in $(\mathbf{P}^1)^2$, which we can take to be the diagonal $\Delta_{\mathbf{P}^1}$. Since X is the blow-up of Y along L_3, the discriminant locus of $\pi\colon X \to (\mathbf{P}^1)^2$ is the curve $\Delta_{\mathbf{P}^1}$. The fibers of π over that curve are reducible (the union of two copies of $\mathbf{P}^1$).

If X has an int-amplified endomorphism of degree invertible in k, then the pair $((\mathbf{P}^1)^2, \Delta_{\mathbf{P}^1})$ satisfies log Bott vanishing. But this is false, as shown above. So the

Fano 3-fold (4.6) does not have an int-amplified endomorphism of degree invertible in k.

Next, we rule out (3.22), the blow-up X of $\mathbf{P}^1 \times \mathbf{P}^2$ along a conic in $p \times \mathbf{P}^2$, for a k-point p in $\mathbf{P}^1$. The contraction of X to $\mathbf{P}^2$ has discriminant locus a conic F in $\mathbf{P}^2$. As in the arguments above, if X has an int-amplified endomorphism of degree invertible in k, then so does $\mathbf{P}^2$, and F is totally invariant. Therefore, the pair $(\mathbf{P}^2, F)$ satisfies log Bott vanishing. But this is false. Namely, let $A = O(1)$ and $E = F \sim O(2)$; then we will show that $H^1(\mathbf{P}^2, \Omega^1_{\mathbf{P}^2}(\log F)(A - E)) = H^1(\mathbf{P}^2, \Omega^1_{\mathbf{P}^2}(\log F)(-1))$ is not zero. Use the exact sequence of coherent sheaves on $\mathbf{P}^2$:

$$0 \to \Omega^1_{\mathbf{P}^2} \to \Omega^1_{\mathbf{P}^2}(\log F) \xrightarrow{\mathrm{Res}} O_F \to 0.$$

By the exact sequence $0 \to \Omega^1_{\mathbf{P}^2}(-1) \to O(-2)^{\oplus 3} \to O(-1) \to 0$ on $\mathbf{P}^2$, $\Omega^1_{\mathbf{P}^2}(-1)$ has zero cohomology in all degrees. So we have an isomorphism

$$H^1(\mathbf{P}^2, \Omega^1_{\mathbf{P}^2}(\log F)(-1)) \cong H^1(F, O(-1)|_F).$$

Since the conic F is isomorphic to $\mathbf{P}^1$ and $O(-1)$ has degree -2 on F, $h^1(F, O(-1)|_F)$ is 1, not 0. So log Bott vanishing fails for $(\mathbf{P}^2, F)$. It follows that the 3-fold (3.22) does not have an int-amplified endomorphism of degree invertible in k.

The last cases are (3.21) and (4.5). These are handled by:

Lemma 6.5. *The Fano 3-folds (3.21) and (4.5) contract to the quintic del Pezzo 3-fold with one node, which does not satisfy Bott vanishing.*

Proof. Here (4.5) is a blow-up of the Fano 3-fold of type (3.21). Next, the Fano 3-fold X of type (3.21) is the blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ along a curve F of degree $(2, 1)$. There is a contraction from X to the quintic del Pezzo 3-fold Y with one node [24, section 5.4.2]. It remains to show that Y does not satisfy Bott vanishing. Indeed, I claim that the Euler characteristic $\chi(Y, \Omega^{[2]}_Y(1))$ is $-2 < 0$. This is harder than previous cases because Y is singular, but it is still manageable.

To prove this, it is convenient to know how the sheaf Ω^2_Y is related to its reflexive hull $\Omega^{[2]}_Y$, for a 3-fold Y with a node. More generally, let Y be a normal hypersurface in a smooth variety W with $\dim(W) = n + 1$, so that locally Y is the zero locus of a regular function g. Define the *torsion* and *cotorsion* of a coherent sheaf M to be the kernel and cokernel of the natural map $M \to M^{**}$. Consider the complex K:

$$0 \to \Omega^0_W|_Y \to \Omega^1_W|_Y \to \cdots \to \Omega^{n+1}_W|_Y \to 0,$$

with differentials given by $\wedge dg$. Graf described the torsion and cotorsion of the sheaves Ω^j_Y in terms of this complex: we have $H^j(K) \cong \operatorname{tor} \Omega^j_Y$ and $H^j(K) \cong \operatorname{cotor} \Omega^{j-1}_Y$ [10, Theorem 1.11]. For Y the 3-fold node, we can take $W = A^4$ and $g = xy - zw$. The complex above is the Koszul complex for the regular sequence $\partial g/\partial x = y$, $\partial g/\partial y = x$, $\partial g/\partial z = -w$, $\partial g/\partial w = -z$, tensored over O_W with O_Y. These four functions generate the ideal of the origin P in $W = A^4$; so $\operatorname{tor} \Omega^i_Y \cong \operatorname{Tor}^{O_W}_{4-i}(O_Y, O_P)$ and $\operatorname{cotor} \Omega^i_Y \cong \operatorname{Tor}^{O_W}_{3-i}(O_Y, O_P)$. These Tor groups are easy to compute, using the free resolution $0 \to O_W \xrightarrow{g} O_W \to O_Y \to 0$ of O_Y as an O_W-module. Namely, it follows that $\operatorname{Tor}^{O_W}_*(O_Y, O_P)$ is the homology of the complex

$$0 \to O_P \xrightarrow{0} O_P \to 0,$$

which is a 1-dimensional k-vector space in degrees 1 and 0, and zero in other degrees. Therefore, Ω^1_Y is reflexive, while Ω^2_Y is torsion-free and its cotorsion is a 1-dimensional vector space, supported at the node P. That is, we have a short exact sequence of sheaves on a nodal 3-fold Y:

$$0 \to \Omega^2_Y \to \Omega^{[2]}_Y \to O_P \to 0.$$

It follows that for Y the quintic del Pezzo 3-fold with one node, we have $\chi(Y, \Omega^{[2]}_Y(1)) = 1 + \chi(Y, \Omega^2_Y(1))$. It remains to show that $\chi(Y, \Omega^2_Y(1)) = -3$.

For a nodal 3-fold Y in a smooth 4-fold W over k (as for any effective Cartier divisor), we have an exact sequence of coherent sheaves on Y [25, Tag 00RU]:

$$O(-Y)|_Y \to \Omega^1_W|_Y \to \Omega^1_Y \to 0.$$

Taking exterior powers, we have an exact sequence $\Omega^{j-1}_Y(-Y) \to \Omega^j_W|_Y \to \Omega^j_Y \to 0$ for any j. For $j = 2$, the map $\Omega^1_Y(-Y) \to \Omega^2_W|_Y$ is clearly injective outside the node. Since $\Omega^1_Y(-Y)$ is torsion-free (by the analysis above), this map is actually injective as a map of sheaves on all of Y. That is, we have an exact sequence $0 \to \Omega^1_Y(-Y) \to \Omega^2_W|_Y \to \Omega^2_Y \to 0$. Likewise, we have an exact sequence $0 \to O_Y(-Y) \to \Omega^1_W|_Y \to \Omega^1_Y \to 0$ on Y. Finally, we can tensor the exact sequence $0 \to O_W(-Y) \to O_W \to O_Y \to 0$ with any vector bundle on W, giving $0 \to \Omega^1_W(-Y) \to \Omega^1_W \to \Omega^1_W|_Y \to 0$ and $0 \to \Omega^2_W(-Y) \to \Omega^2_W \to \Omega^2_W|_Y \to 0$.

Apply this to the quintic del Pezzo 3-fold $Y \subset \mathbf{P}^6$ with a node, which is a codimension-3 linear section of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbf{P}^9$. The quintic del Pezzo 3-fold with one node is unique up to isomorphism, with explicit equations labelled $X_{5,2,4}$ in [17, section A.1.6]. By the equations, Y is a hyperplane section of a smooth codimension-2 linear section $W \subset \mathbf{P}^7$. By the previous paragraph, we can rewrite $\chi(Y, \Omega^2_Y(1))$ in terms of Euler characteristics on W; but this formula would be exactly the same with Y replaced by a smooth hyperplane section $V_5 \subset \mathbf{P}^6$. Therefore, $\chi(Y, \Omega^2_Y(1)) = \chi(V_5, \Omega^2_{V_5}(1)) = -3$ [28, section 2]. As discussed above, it follows that $\chi(Y, \Omega^{[2]}_Y(1)) = -2$, and so Y does not satisfy Bott vanishing. Lemma 6.5 is proved. $\square$

It follows that neither (3.21) nor (4.5) admits an int-amplified endomorphism of degree invertible in k. Theorem 6.1 is proved. $\square$

7 Images of toric varieties: del Pezzo surfaces and Fano 3-folds

Occhetta and Wiśniewski conjectured that a smooth complex projective variety X that admits a surjective morphism from a proper toric variety must be toric [23]. (This is known for contractions, and so it suffices to consider a finite surjective morphism.) Occhetta-Wiśniewski's conjecture was proved by Achinger, Witaszek, and Zdanowicz for X of dimension at most 2, and also for X a Fano 3-fold [1, proof of Theorem 4.4.1], [2, Theorems 6.9 and 7.7]. Using Bott vanishing, we will reprove this result for Fano 3-folds and extend it to positive characteristic (Theorem 7.2). Kawakami and I proved this extension earlier for Fano 3-folds with Picard number 1 [14, Proposition 3.10].

Another approach to Theorem 7.2 appeared after the first version of this paper. Namely, Kawakami and Takamatsu used Lemma 2.1 above to show that the image of an F-liftable variety in characteristic p by a morphism of degree prime to p is F-liftable [13, Corollary 3.13]. In particular, the image of a toric variety by a morphism of degree prime to p is F-liftable. But Achinger-Witaszek-Zdanowicz showed that a smooth Fano 3-fold that is F-liftable must be toric. So a Fano 3-fold that is an image of a toric variety by a morphism of degree prime to p must be toric.

We will prove Theorem 7.2 using log Bott vanishing rather than F-liftability. Thus our method for analyzing images of toric varieties closely parallels our approach to finding which varieties have nontrivial endomorphisms (Theorem 6.1).

We start with the analogous result in dimension 2.

Proposition 7.1. *A smooth del Pezzo surface over an algebraically closed field k that is the image of a proper toric variety must be toric.*

Surprisingly, Proposition 7.1 works in any characteristic with no assumption on the degree of the morphism f; f may even be inseparable. The proof does not use Bott vanishing.

Proof. A smooth del Pezzo surface X is toric if and only if its degree $(-K_X)^2$ is at least 6. And every smooth del Pezzo surface of degree at most 5 is a blow-up of the quintic del Pezzo surface (the blow-up of $\mathbf{P}^2$ at 4 points in general position). So it suffices to show that there is no surjection f from a proper toric variety Y to the quintic del Pezzo surface X.

Let $Y \to Z \to X$ be the Stein factorization of f; then Z is also toric, since every contraction of a toric variety is toric [26, Proposition 2.7]. Replacing Y by Z, we can assume that $f\colon Y \to X$ is finite and surjective.

Let π_X be one of the contractions of X to $\mathbf{P}^1$. Let $Y \to Y_1 \to \mathbf{P}^1$ be the Stein factorization of the composition $Y \to X \to \mathbf{P}^1$, where we write $\pi_Y\colon Y \to Y_1$ and $g\colon Y_1 \to \mathbf{P}^1$. Then Y_1 is a proper toric curve, hence isomorphic to $\mathbf{P}^1$, and $g\colon Y_1 \to \mathbf{P}^1$ is finite and surjective.

As we used in the proof of Proposition 6.4, the contraction π_X from the quintic del Pezzo surface to $\mathbf{P}^1$ has 3 singular fibers, and these fibers are reducible (namely, the union of two copies of $\mathbf{P}^1$). Let $\Delta_X \subset \mathbf{P}^1$ be the discriminant locus of π_X, consisting of 3 points. By Lemma 6.3, $g^{-1}(\Delta_X)$ is contained in the discriminant locus Δ_Y of $\pi_Y\colon Y \to Y_1$. But $Y \to Y_1$ is a toric morphism, so its discriminant locus is contained in the toric divisor of Y_1, which consists of 2 points. Since $g\colon Y_1 \to \mathbf{P}^1$ is surjective, this is a contradiction (2 points cannot map onto 3 points). We have shown that the quintic del Pezzo surface is not the image of a proper toric variety. □

Theorem 7.2. *Let X be a smooth Fano 3-fold over an algebraically closed field k. If X is the image of a proper toric 3-fold by a morphism of degree invertible in k, then X is toric.*

As in Theorem 6.1, the proof uses Tanaka's theorem that the classification of smooth Fano 3-folds has essentially the same form in every characteristic [27, Theorem 1.1]. Without that, our proof applies to Fano 3-folds in any characteristic that are given by the same construction as one of the Fano 3-folds over $\mathbf{C}$ (classified by Iskovskikh and Mori-Mukai).

Proof. Assume that X is the image of a proper toric 3-fold Y by a morphism f of degree invertible in k. After replacing f by its Stein factorization, we can also assume that $f: Y \to X$ is finite. Here Y is still toric, because every contraction of a toric variety is toric [26, Proposition 2.7]. Then X satisfies Bott vanishing, by Theorem 5.1. There are exactly 19 non-toric Fano 3-folds (up to isomorphism) that satisfy Bott vanishing [28, Theorem 0.1]. In Mori-Mukai's numbering, these are (2.26), (2.30), (3.15)–(3.16), (3.18)–(3.24), (4.3)–(4.8), (5.1), and (6.1). (To be precise, the answer is a subset of this in characteristic 2, where only 9 non-toric Fano 3-folds on the known list satisfy Bott vanishing [28, section 2].) We need to show that none of these 19 varieties can be the image of a toric variety Y by a finite morphism f of degree invertible in k.

Of these 19 varieties, 15 have contractions that do not satisfy Bott vanishing, by the proof of Theorem 6.1, including Lemma 6.5. This excludes all the cases except four: (3.22), (4.6), (4.8), and (6.1).

Here (6.1) is $\mathbf{P}^1$ times the quintic del Pezzo surface. That surface is not an image of a toric variety, by Proposition 7.1. So (6.1) is not the image of a toric variety.

Next, we rule out (3.22), the blow-up X of $\mathbf{P}^1 \times \mathbf{P}^2$ along a conic in $q \times \mathbf{P}^2$, for a k-point q in $\mathbf{P}^1$. The contraction of X to $\mathbf{P}^2$ has discriminant locus a conic $F \subset \mathbf{P}^2$. Let $X_1 = \mathbf{P}^1 \times \mathbf{P}^2$ and $X_2 = \mathbf{P}^2$, so we have contractions $X \to X_1 \to X_2$. Define toric varieties Y_1 and Y_2 as Stein factorizations of the morphisms from Y to X_1 and X_2; so we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ Y_1 \arrow[r] \arrow[d] & \mathbf{P}^1 \times \mathbf{P}^2 \arrow[d] \\ Y_2 \arrow[r] & \mathbf{P}^2, \end{tikzcd}$$

where the horizontal morphisms are finite and of degree invertible in k.

Since $X \to \mathbf{P}^1 \times \mathbf{P}^2$ contracts an irreducible divisor E to a curve, the toric contraction $Y \to Y_1$ must contract $f^{-1}(E)$ to a union of curves. It follows that $f^{-1}(E)$ is a toric divisor in Y. Since E maps onto a conic F in $\mathbf{P}^2$, the inverse image of F in Y_2 by the finite morphism $g: Y_2 \to \mathbf{P}^2$ has dimension 1 and is the image of $f^{-1}(E)$ in Y_2. So $g^{-1}(F)$ is a toric divisor in the toric surface Y_2. By Theorem 5.1, it follows that the pair $(\mathbf{P}^2, F)$ satisfies log Bott vanishing. But that is false, by the proof of Theorem 6.1. Thus (3.22) is not the image of a toric 3-fold by a morphism of degree invertible in k.

A similar argument rules out (4.8), the blow-up X of $(\mathbf{P}^1)^3$ along a curve F of degree $(0, 1, 1)$. The contraction of X to $(\mathbf{P}^1)^2$ (corresponding to the last two $\mathbf{P}^1$ factors) has discriminant locus a curve of degree $(1, 1)$, which we can take to be the diagonal $\Delta_{\mathbf{P}^1}$ in $(\mathbf{P}^1)^2$. Define toric varieties Y_1 and Y_2 by Stein factorization of the morphisms from Y to $(\mathbf{P}^1)^3$ and $(\mathbf{P}^1)^2$:

$$\begin{tikzcd} Y \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ Y_1 \arrow[r] \arrow[d] & (\mathbf{P}^1)^3 \arrow[d] \\ Y_2 \arrow[r] & (\mathbf{P}^1)^2, \end{tikzcd}$$

where the horizontal morphisms are finite and of degree invertible in k.

By the same argument as in the previous case, the inverse image in Y_2 of the diagonal $\Delta_{\mathbf{P}^1} \subset (\mathbf{P}^1)^2$ must be a toric divisor. By Theorem 5.1, it follows that the pair $((\mathbf{P}^1)^2, \Delta_{\mathbf{P}^1})$ satisfies log Bott vanishing. But that is false, by the proof of Theorem 6.1. Thus (4.8) is not the image of a toric 3-fold by a morphism of degree invertible in k.

Finally, let X be the Fano 3-fold (4.6), the blow-up of $\mathbf{P}^3$ along three pairwise disjoint lines L_1, L_2, L_3 over k. Suppose that there is a proper toric 3-fold Y with a morphism $f\colon Y \to X$ such that $\deg(f)$ is invertible in k. By the proof of Theorem 6.1, X has a contraction to $(\mathbf{P}^1)^2$ with discriminant locus the diagonal $\Delta_{\mathbf{P}^1}$ in $(\mathbf{P}^1)^2$. The fibers of π over that curve are reducible (the union of two copies of $\mathbf{P}^1$).

Let $Y \to Y_1 \to (\mathbf{P}^1)^2$ be the Stein factorization of the composition $Y \to X \to (\mathbf{P}^1)^2$. Then Y_1 is a proper toric surface over k, and $h\colon Y_1 \to X_1$ is finite, of degree invertible in k. The discriminant locus of $Y \to Y_1$ is contained in the union of the toric divisors. For a k-point p in Y_1, the fiber Y_p maps onto $X_{h(p)}$, as in part (1). Therefore, $\Delta_1 := h^{-1}(\Delta_{\mathbf{P}^1})$ is contained in the union of the toric divisors in Y_1.

Therefore, Theorem 5.1 applies to show that $((\mathbf{P}^1)^2, \Delta_{\mathbf{P}^1})$ satisfies log Bott vanishing. But that is false, by the proof of Theorem 6.1. This contradiction shows that the Fano 3-fold (4.6) is not the image of a toric 3-fold by a morphism of degree invertible in k. $\square$

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UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555
totaro@math.ucla.edu