The elliptic genus of a singular variety

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1 Rigidity of the Ochanine genus and the complex elliptic genus

The key property of elliptic cohomology that remains interesting even after tensoring with the rationals is its *rigidity.* Here is one way to describe what rigidity means.

We can begin by asking which bordism invariants for oriented manifolds are multiplicative under fiber bundles. Equivalently, what is the quotient ring of the bordism ring MSO_* of oriented manifolds by the ideal I of differences $E - F \cdot B$, for all fiber bundles $F \to E \to B$ of closed oriented manifolds? In order to have an interesting answer, we will always assume that our fiber bundles have structure group a compact connected Lie group G. That is, we start with a principal G-bundle over B and an action of G on F, and E is the associated F-bundle over B. (Examples are sphere bundles, or projective bundles.) The answer has been known since the 1960s. I will just state the answer rationally.

Theorem 1.1

$$MSO_*/I \otimes \mathbf{Q} \cong \mathbf{Q}[x],$$

where $|x| = 4$*. The homomorphism* $MSO_* \to \mathbf{Q}[x]$ *is given by the signature of an oriented manifold.*

In other words, the signature behaves multiplicatively on fiber bundles of the above type (by Chern-Hirzebruch-Serre), and it is the universal genus with this property. There is a similar answer for complex manifolds. As above, let I denote the ideal in MU_* of differences $E - F \cdot B$, for all fiber bundles of stably complex manifolds with structure group a compact connected Lie group.

Theorem 1.2

$$MU_*/I \otimes \mathbf{Q} \cong \mathbf{Q}[x_1, x_2],$$

where $|x_i| = 2i$*. The homomorphism* $MU_* \to \mathbf{Q}[x_1, x_2]$ *is given by the Hirzebruch* χ_y*-genus of a complex manifold,*

$$\chi_y(X) = \sum_i \chi(X, \Omega^i_X) y^i.$$

Both the signature and the χ_y-genus are related to K-theory, rather than to any deeper cohomology theory.

If one changes the question, however, one finds much more interesting answers. Namely, consider the same question for spin manifolds.

Theorem 1.3
$$MSpin_*/I \otimes \mathbf{Q} \cong \mathbf{Q}[\delta, \epsilon],$$
where $|\delta| = 4$ *and* $\epsilon = 8$. *The homomorphism* $MSpin_* \to \mathbf{Q}[\delta, \epsilon]$ *is the Ochanine-Landweber-Ravenel-Stong elliptic genus.*

This is remarkable when one considers that the bordism ring of spin manifolds is the same, rationally, as the bordism ring of oriented manifolds. This theorem says that actions of compact connected Lie groups on spin manifolds are much more special, in an unexpected way, than actions on mere oriented manifolds. This is the "rigidity" property of the elliptic genus which has fascinated topologists since the discovery of the elliptic genus in the mid-1980s. The conference volume [4] is an excellent reference.

One can also consider the same question for complex manifolds with $c_1 = 0$, "Calabi-Yau manifolds", which are roughly a complex analogue of spin manifolds. The result is ([3], [2], [5]):

Theorem 1.4
$$MSU_*/I \otimes \mathbf{Q} \cong \mathbf{Q}[x, y, g_2],$$
where $|x| = 4$, $|y| = 6$, *and* $|g_2| = 8$. *The homomorphism* $MSU_* \to \mathbf{Q}[x, y, g_2]$ *is the Krichever-Höhn complex elliptic genus.*

The coefficient rings of the Ochanine genus and the complex elliptic genus are naturally described in terms of elliptic curves. For the Ochanine genus, $\mathbf{Q}[\delta, \epsilon]$ is the ring of modular forms for the group $\Gamma_0(2) \subset SL(2, \mathbf{Z})$. In other words, it is the ring of "functions" on the moduli space of elliptic curves together with a point of order 2. (Because one compactifies this space to get a projective variety, one actually considers sections of powers of a natural line bundle, rather than functions.)

For the complex elliptic genus, the coefficient ring $\mathbf{Q}[x, y, g_2]$ is the ring of Jacobi forms. This is the ring of "functions" on the universal elliptic curve $\overline{M_{1,2}}$. Here there is a natural fibration
$$E \to \overline{M_{1,2}} \to \overline{M_{1,1}},$$
where $\overline{M_{1,1}}$ is the moduli space of elliptic curves. Corresponding to the surjection $\overline{M_{1,2}} \to \overline{M_{1,1}}$ of projective varieties, there is an inclusion of rings:
$$\mathbf{Q}[g_2, g_3] \subset \mathbf{Q}[x, y, g_2].$$
Here $\mathbf{Q}[g_2, g_3]$ (sometimes called $\mathbf{Q}[E_4, E_6]$), with $|g_2| = 8$ and $|g_3| = 12$, is the ring of "functions" on $\overline{M_{1,1}}$, that is, the ring of modular forms for $SL(2, \mathbf{Z})$. The above inclusion of rings is described by the Weierstrass equation:
$$y^2 = 4x^3 - g_2 x - g_3.$$

2 The elliptic genus of a singular variety

I found in [5] that the complex elliptic genus arises naturally in the context of the following questions.

Question (Goresky-MacPherson) Which characteristic numbers can be defined for compact complex algebraic varieties with singularities?

Question (Morava) Does a singular complex variety have a fundamental class in complex bordism?

Morava's question is suggested by the fact that any singular complex variety Y (which I will always assume to be compact) has a resolution of singularities $f : X \to Y$. (That is, X is smooth, f is proper, and f is an isomorphism over a dense open subset of Y.) For Y of complex dimension n, any resolution of Y determines an element of $MU_{2n}Y$, which is a sort of fundamental class of Y. The problem is that resolutions of Y are not unique, and different resolutions typically define different elements of $MU_{2n}Y$. For example, if X_1 and X_2 are resolutions of Y which define the same element of $MU_{2n}Y$, then in particular they have the same Chern numbers; but this is false in general. In particular, blowing up a point on one resolution of Y gives a "bigger" resolution of Y, with different Chern numbers.

A natural way to address this problem is to consider only resolutions which are *minimal* in some sense. However, even with this improvement, I found that Morava's question has a negative answer, in the following sense: There is a singular variety Y with two different and "equally good" minimal resolutions X_1 and X_2 which have different Chern numbers. Thus a singular variety does not, in general, have a well-defined fundamental class in complex bordism. (This can be formulated precisely by requiring that the class is compatible with minimal resolutions.)

In the examples I constructed, X_1 and X_2 are related in an explicit way, by what I called a *classical flop.* This made it natural to ask what is the quotient ring of the bordism ring defined by identifying any pair of manifolds related by a classical flop. This leads to a new interpretation of the complex elliptic genus [5]:

Theorem 2.1 *The quotient ring of MSU_* by the ideal of differences $X_1 - X_2$, with X_1 and X_2 related by a classical flop, is, after inverting 2, a polynomial ring $\mathbf{Z}[1/2][x_2, x_3, x_4]$. The corresponding homomorphism*

$$MSU_* \to \mathbf{Z}[1/2][x_2, x_3, x_4]$$

is the complex elliptic genus.

Thus we find that the complex elliptic genus arises naturally from a geometric problem. However, classical flops are a very special class of pairs of manifolds which are both minimal resolutions of the same singular variety. I asked whether this theorem could extend to all pairs of manifolds which are minimal resolutions of the same singular variety. This problem was solved in a spectacular paper by Borisov and Libgober [1]:

Theorem 2.2 *The quotient ring of MSU_* by the ideal of differences $X_1 - X_2$, for any pair of complex manifolds X_1 and X_2 with $c_1 = 0$ which are both minimal resolutions of a singular variety Y, is, after inverting 2, a polynomial ring $\mathbf{Z}[1/2][x_2, x_3, x_4]$. The corresponding homomorphism*

$$MSU_* \to \mathbf{Z}[1/2][x_2, x_3, x_4]$$

is the complex elliptic genus.

Given my earlier result, what Borisov and Libgober had to show was that the complex elliptic genus of X_1 was equal to the complex elliptic genus of X_2, for any two minimal resolutions of a singular variety Y. In this context, we adopt the following definition of a *minimal* resolution, which fits with the notion of "minimal models" in algebraic geometry. Namely, we assume that the singular variety Y has rational Gorenstein singularities, so it has a canonical line bundle K_Y, and we assume that K_Y is trivial. Then we say that a resolution $X \to Y$ is minimal if K_X is also trivial.

This theorem of Borisov and Libgober is a new kind of rigidity theorem for the complex elliptic genus φ. Indeed, I was able to show that $\varphi(X_1) = \varphi(X_2)$ in the special case where X_1 and X_2 are related by a classical flop by identifying $X_1 - X_2$ in MU_* with the total space of a fiber bundle over the singular set of Y, and applying the usual rigidity theorem for the complex elliptic genus. However, if we know only that X_1 and X_2 are minimal resolutions of a singular variety Y, then there is no obvious geometric description of the relation between X_1 and X_2. Nonetheless, Borisov and Libgober were able to show that $\varphi(X_1) = \varphi(X_2)$.

The key to their proof is to define the elliptic genus of a singular variety Y with rational Gorenstein singularities using *any* resolution of singularities $f : X \to Y$, in such a way that if $f : X \to Y$ is a minimal resolution then the definition just gives the elliptic genus of X. The definition has the form

$$\varphi(Y) = \int_X \varphi(c_i TX, E_j, a_j)$$

for some explicit function φ, where E_j runs over the exceptional divisors of the map f, and the *discrepancies* a_j are defined by

$$K_X = f^* K_Y + \sum a_j E_j.$$

Thus, to prove their theorem, it suffices to show that this definition is the same for all resolutions of Y. For that, they use the "weak factorization theorem" of Abramovich, Matsuki, Karu, and Wlodarczyk, which says that (for any singular variety whatsoever) any two resolutions can be related by repeatedly blowing up and down *smooth* subvarieties. Thus, it suffices to show that the above definition of $\varphi(Y)$ does not change when the resolution X is replaced by a blow-up of X along a smooth subvariety. This turns out to be a straightforward calculation, using the properties of the Jacobi theta function (which goes into the definition of φ).

3 Possible characteristic numbers for real analytic spaces

It is now natural to ask: are there any further rigidity properties of the elliptic genus which remain to be discovered?

In particular: can one define the Ochanine elliptic genus φ for any singular oriented real analytic space Y, in such a way that

$$\varphi(Y) = \varphi(X)$$

for any minimal resolution $X \to Y$? (Every real analytic space has a resolution of singularities, and one can define a resolution to be minimal if the corresponding complexification, defined locally, is a minimal resolution in the sense defined above.) This would deserve to be called a "rigidity" property because it would imply that

$\varphi(X_1) = \varphi(X_2)$ whenever X_1 and X_2 are minimal resolutions of the same singular real analytic space Y.

This is what Borisov and Libgober do for complex analytic spaces. As evidence that the real analytic case may also have a positive answer, I can offer the following. There are other related calculations in my paper [6].

Theorem 3.1 *The quotient ring of MSO_* by the ideal generated by oriented real flops and complex flops is:*

$$\mathbf{Z}[\delta, 2\gamma, 2\gamma^2, 2\gamma^4, \ldots],$$

where $\mathbf{CP}^2$ maps to δ and $\mathbf{CP}^4$ maps to $2\gamma + \delta^2$. This quotient ring is exactly the image of MSO_* under the Ochanine elliptic genus ([4], p. 63).

References

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