The topology of smooth divisors and the arithmetic of abelian varieties

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We have three main results. First, we show that a smooth complex projective variety which contains three disjoint codimension-one subvarieties in the same homology class must be the union of a whole one-parameter family of disjoint codimension-one subsets. More precisely, the variety maps onto a smooth curve with the three given divisors as fibers, possibly multiple fibers (Theorem 2.1). The beauty of this statement is that it fails completely if we have only two disjoint divisors in a homology class, as we will explain. The result seems to be new already for curves on a surface. The key to the proof is the Albanese map.

We need Theorem 2.1 for our investigation of a question proposed by Fulton, as part of the study of degeneracy loci. Suppose we have a line bundle on a smooth projective variety which has a holomorphic section whose divisor of zeros is smooth. Can we compute the Betti numbers of this divisor in terms of the given variety and the first Chern class of the line bundle? Equivalently, can we compute the Betti numbers of any smooth divisor in a smooth projective variety X in terms of its cohomology class in $H^2(X, \mathbf{Z})$?

The point is that the Betti numbers (and Hodge numbers) of a smooth divisor are determined by its cohomology class if the divisor is ample or if the first Betti number of X is zero (see section 4). We want to know if the Betti numbers of a smooth divisor are determined by its cohomology class without these restrictions. The answer is no. In fact, there is a variety which contains two homologous smooth divisors, one of which is connected while the other is not connected. Fortunately, we can show that this is a rare phenomenon: if a variety contains a connected smooth divisor which is homologous to a non-connected smooth divisor, then it has a surjective morphism to a curve with some multiple fibers, and the two divisors are both unions of fibers. This is our second main result, Theorem 5.1.

We also give an example of two connected smooth divisors which are homologous but have different Betti numbers. Conjecture 6.1, suggested by this example, asserts that two connected smooth divisors in a smooth complex projective variety X which are homologous should have cyclic etale coverings which are deformation equivalent to each other. The third main result of this paper, Theorem 6.3, is that this conjecture holds, in a slightly weaker form (allowing deformations into positive characteristic), under the strange assumption that the Picard variety of X is isogenous to a product of elliptic curves. The statement in general would follow from a well-known open problem in the arithmetic theory of abelian varieties, Conjecture 6.2: for any abelian variety A over a number field F, there are infinitely many primes $\mathfrak{p}$ of the ring of integers o_F such that the finite group $A(o_F/\mathfrak{p})$ has order prime to the characteristic of the field $o_F/\mathfrak{p}$.

I am grateful to Bill Fulton for asking the right question. Brendan Hassett found an elegant example related to these questions, a version of which is included in section 2.

1 Notation

If X is a smooth algebraic variety, a *divisor* is an element of the free abelian group on the set of codimension-one subvarieties of X. (Varieties are irreducible by definition.) In other words, a divisor is a finite sum $\sum a_i D_i$ where a_i are integers and D_i are codimension-one subvarieties of X. An *effective* divisor is such a sum with every integer a_i nonnegative. A *smooth* divisor is a sum $\sum D_i$ with each subvariety D_i smooth and D_i disjoint from D_j for $i \neq j$. The *support* of a divisor $\sum a_i D_i$ is the union of the subvarieties D_i with $a_i \neq 0$. An effective divisor is *connected* if it is not 0 and its support is connected.

Let X be a smooth complex projective variety of dimension n, and let H be a fixed ample divisor on X. Throughout this paper, we use the intersection pairing on divisors defined by

$$(D, E) = D \cdot E \cdot H^{n-2} \in \mathbf{Z}.$$

We often use the easy fact that if D and E are effective divisors with no irreducible component in common, then $(D, E) \geq 0$, with equality if and only if D and E are disjoint. We also use the Hodge index theorem for divisors: the symmetric bilinear form (D, E) on the group of divisors modulo homological equivalence, tensored with the real numbers, is nondegenerate with signature $(1, N-1)$. This follows from the Hodge-Riemann bilinear relations [4], p. 123.

Alternatively, using the Lefschetz hyperplane theorem, the above Hodge index theorem for divisors follows from the Hodge index theorem for divisors on a surface, applied to a surface in X which is the intersection of $n-2$ divisors linearly equivalent to multiples of H. This proof has the advantage that it works for varieties over fields of any characteristic, using etale cohomology.

2 Characterization of varieties which fiber over a curve

We prove a little more than was stated in the introduction.

Theorem 2.1 *Let X be a smooth complex projective variety. Let $D_1, \ldots, D_r$, $r \geq 3$, be connected effective divisors (not 0) which are pairwise disjoint and whose rational cohomology classes lie in a line in $H^2(X, \mathbf{Q})$. Then there is a map $f : X \to C$ with connected fibers to a smooth curve C such that $D_1, \ldots, D_r$ are all positive rational multiples of fibers of f. In fact, there is only one map f with these properties.*

In this statement, and in the rest of the paper, if $f : X \to C$ is a map from a smooth variety onto a smooth curve, then a "fiber" of f is defined to be the divisor $f^{-1}(p)$ for a point p in C, that is, the sum of the irreducible components of the set $f^{-1}(p)$ with multiplicities. To compute the multiplicity of a given irreducible component D in the divisor $f^{-1}(p)$, let z be a local coordinate function on the curve which vanishes at p, and compute the order of vanishing of the composed function

$z(f)$ along the divisor D. In particular, if the divisor $f^{-1}(p)$ equals aD for some smooth irreducible divisor D and some integer $a \geq 2$, we call D a smooth multiple fiber of f.

Interestingly, the theorem becomes false if we have only two disjoint homologous divisors D_1 and D_2, as shown by the following example.

Example. Let D be any curve of genus at least one, and let L be a line bundle of degree 0 on D which is not torsion in Pic D. Let X be the ruled surface $P(O \oplus L)$ over D. Then X contains two copies of D, call them D_1 and D_2, at 0 and infinity: they are disjoint smooth curves and are homologous to each other. But the conclusion of the above theorem fails: D_1 and D_2 are not fibers or multiple fibers of any map of X to a curve. Indeed, if $f : X \to C$ is a map with $f^{-1}(\text{point}) = aD_1$, for some positive integer a, then the normal bundle of D_1 must be a-torsion in Pic D_1. But in this example, D_1 has normal bundle L, which we assumed is not torsion. Thus Theorem 2.1 would be false for $r = 2$. This example was used for essentially the same purpose by Kollár [8].

Brendan Hassett found that the failure of Theorem 2.1 if we have only two disjoint homologous divisors is not at all restricted to ruled varieties. The following example is a variant of his.

Example. Let Y be any smooth projective variety of dimension at least 2 with $H^1(Y, \mathbf{Q}) \neq 0$. Thus $\text{Pic}_0(Y)$ is a nontrivial abelian variety. If a line bundle is ample, it remains ample upon adding an element of $\text{Pic}_0(Y)$, so Y contains two smooth ample divisors D_1 and D_2 which are homologous, but differ in Pic Y by a non-torsion element of $\text{Pic}_0(Y)$. We can also arrange that D_1 intersects D_2 transversely.

Let X be the blow-up of Y along the smooth codimension-two subscheme $D_1 \cap D_2$. Then D_1 and D_2 become disjoint in X, and they are still homologous. The normal bundle of D_1 in X is the restriction of $D_1 - D_2 \in \text{Pic}_0(Y)$ to D_1. By the Lefschetz hyperplane theorem, since Y has dimension at least 2, the restriction map $H^1(Y, \mathbf{Q}) \to H^1(D_1, \mathbf{Q})$ is injective, and so the restriction map $\text{Pic}_0(Y) \to \text{Pic}_0(D_1)$ has finite kernel. Since $D_1 - D_2$ is non-torsion in $\text{Pic}_0(Y)$, the normal bundle of D_1 in X is non-torsion in $\text{Pic}_0(D_1)$. So Theorem 2.1 again fails here if we have only two divisors D_1, D_2.

Proof of Theorem 2.1. Write $\tilde{D}_1$ for a resolution of singularities of the reduced divisor underlying D_1, so that $\tilde{D}_1$ is a disjoint union of smooth varieties.

The proof is in two cases, depending on whether the map

$$H^1(X, \mathbf{Q}) \to H^1(\tilde{D}_1, \mathbf{Q})$$

is injective. The amazing thing is that if this map is injective, then we can construct a map from X to $\mathbf{P}^1$, and if it is not injective, then we can construct a map from X to a curve of genus at least 1. This dichotomy was used in a special case by Neeman [11], pp. 109-110.

First, suppose the above map is injective. Then the map of abelian varieties

$$\text{Pic}_0(X) \to \text{Pic}_0(\tilde{D}_1)$$

has finite kernel. (Here $\text{Pic}_0(\tilde{D}_1)$ means the product of the Picard varieties of the connected components of $\tilde{D}_1$.) Since the divisors D_2 and D_3 on X are in multiples of the same rational cohomology class, there are positive integers a_2, a_3 such that $a_2 D_2 - a_3 D_3$ is 0 in $H^2(X, \mathbf{Z})$; equivalently, the divisor $a_2 D_2 - a_3 D_3$

defines an element of $\mathrm{Pic}_0(X)$. Since D_1 is disjoint from D_2 and D_3, the divisor class $a_2D_2 - a_3D_3$ restricts to 0 in Pic D_1 and hence in Pic $\tilde{D}_1$. Since the above map has finite kernel, there are larger positive integers b_2, b_3 such that

$$b_2D_2 - b_3D_3 = 0 \in \mathrm{Pic}_0(X).$$

That is, the effective divisors b_2D_2 and b_3D_3 are linearly equivalent. Since these divisors are disjoint, there is a map

$$g : X \to \mathbf{P}^1$$

with $g^{-1}(0) = b_2D_2$ and $g^{-1}(\infty) = b_3D_3$. This essentially solves the problem. The full conclusion of Theorem 2.1 in this case (that is, for $H^1(X,\mathbf{Q}) \to H^1(\tilde{D}_1,\mathbf{Q})$ injective) follows from the following lemma, whose proof we put off until section 3.

Lemma 2.2 *Let X be a smooth complex projective variety of dimension n, and let H be an ample divisor on X. Let D_1 be a connected effective divisor (not 0) on X, and suppose that $(D_1, D_1) = D_1^2 \cdot H^{n-2}$ is 0. Suppose there is a map from X onto some possibly singular curve which maps D_1 to a point. Then there is a map $f : X \to C$ onto a smooth curve C such that f has connected fibers and D_1 is a positive rational multiple of a fiber of f. Moreover, f is unique with these properties.*

Also, any connected effective divisor which is homologous to a rational multiple of D_1 is a positive rational multiple of a fiber of f.

Now we prove Theorem 2.1 in the other case, for $H^1(X,\mathbf{Q}) \to H^1(\tilde{D}_1,\mathbf{Q})$ not injective. We use this in the form: the dual map of abelian varieties

$$\mathrm{Alb}\ (\tilde{D}_1) \to \mathrm{Alb}\ (X)$$

is not surjective. (Of course, Alb $(\tilde{D}_1)$ means the product of the Albanese varieties of the connected components of $\tilde{D}_1$.) There is a natural map from zero cycles of degree 0 on X to Alb (X). Consider the map g from X to the quotient abelian variety Alb (X)/Alb $(\tilde{D}_1)$ given by $x \mapsto x - p$, for a chosen point p in D_1. For any point $x \in D_1$ (inside X), $x - p \in$ Alb (X) is a sum of differences $x_1 - x_2$ where x_1, x_2 are two points in the image of the same component of $\tilde{D}_1$, since D_1 is connected. So for $x \in D_1$, the element $x - p \in$ Alb (X) lies in the image of Alb $(\tilde{D}_1)$. Thus the map

$$g : X \to \mathrm{Alb}\ (X)/\mathrm{Alb}\ (\tilde{D}_1)$$

sends D_1 to the point 0.

Also, the image of g generates the abelian variety Alb (X)/Alb $(\tilde{D}_1)$, and this abelian variety is nonzero by our assumption. So $g(X)$ has dimension at least 1. In fact, it has dimension exactly 1, by the following argument. Let L be the pullback of a hyperplane section on $g(X)$ to X. (Since $g(X)$ is a subvariety of an abelian variety, it is projective.) If $g(X)$ has dimension at least 2, then, in the notation of section 1, $(L, L) = L^2 \cdot H^{n-2} \in \mathbf{Z}$ is positive, since L^2 is represented by a nonzero effective codimension-two cycle on X. Also, $(L, D_1) = 0$ since D_1 maps to a point in $g(X)$. So the Hodge index theorem (section 1) implies that $(D_1, D_1) < 0$. But in fact we know that D_1 is homologous to a disjoint divisor D_2, so that $(D_1, D_1) = 0$, a contradiction. It follows that the variety $g(X)$ has dimension 1. Now we can apply Lemma 2.2 (to be proved in section 3), and Theorem 2.1 is proved. QED

3 Proof of Lemma 2.2

This is straightforward.

Proof of Lemma 2.2. We start with the given map $g : X \to g(X)$ from a smooth projective variety X onto a singular curve. Form the Stein factorization $X \to C \to g(X)$: $f : X \to C$ has connected fibers, C is normal and $C \to g(X)$ is finite [6], p. 280. Since C is normal, it is a smooth curve. The connected divisor D_1 in X maps to a point in C since it maps to a point in $g(X)$.

Consider the intersection pairing on divisors discussed in section 1,

$$(D, E) = D \cdot E \cdot H^{n-2} \in \mathbf{Z}$$

for a fixed ample divisor H on X. For effective divisors D and E with no irreducible components in common, we have $(D, E) \geq 0$ with equality if and only if D and E are disjoint. So $(D_1, f^{-1}(p)) = 0$ for a general point p in C. We can now start to check the last statement of the lemma: for any connected effective divisor D which is homologous to a rational multiple of D_1, we have $(D, f^{-1}(p)) = 0$. So D is disjoint from $f^{-1}(p)$ for general points p in C. Equivalently, $f(D)$ is a finite subset of C. Since D is connected, f maps the divisor D to a point.

We now strengthen this statement to say that any connected effective divisor D which is homologous to a rational multiple of D_1 must be a positive rational multiple of a fiber $f^{-1}(p)$. (This is the last statement of Lemma 2.2.) The statement will apply in particular to D_1 itself. It is a consequence of the following lemma. For curves on a surface (a case to which one can easily reduce via hyperplane sections), Beauville gives an elementary proof of this lemma in [2], pp. 122-123. It goes back to Enriques's classification of surfaces with Kodaira dimension zero. For completeness we give a proof here, using the Hodge index theorem.

Lemma 3.1 *Let X be a smooth projective variety which has a map $f : X \to C$ with connected fibers onto a smooth curve. Then any nonzero effective divisor D on X such that $(D, D) = 0$ which maps to a point p in C must be a positive rational multiple of the divisor $f^{-1}(p)$.*

Proof. We will prove a bit more, namely that any divisor D (not necessarily effective) with $(D, D) = 0$ which is supported in $f^{-1}(p)$ is a rational multiple of $f^{-1}(p)$. The idea is that, if $f^{-1}(p) = \sum_i a_i E_i$, then the intersection form (D, E) on

$$\mathbf{R} \cdot E_1 \oplus \mathbf{R} \cdot E_2 \oplus \cdots$$

is negative definite except for one 0 eigenvalue. Indeed, $f^{-1}(p)$ is homologous on X to any other fiber of f, so it has 0 intersection number with each E_i. By the Hodge index theorem (section 1), the intersection form (D, E) on the subspace of ($\mathbf{R}$-divisors on X/homological equivalence) which is orthogonal to a nonzero element A (here $= f^{-1}(p)$) with $(A, A) = 0$ is negative definite except for one 0 eigenvalue, corresponding to A itself. So any divisor D with $(D, D) = 0$ and $f(D) = p$ must be rationally homologous on X to a rational multiple of $f^{-1}(p)$.

To show that D is actually a rational multiple of $f^{-1}(p)$ as a divisor, it suffices to check that the irreducible components $E_1, E_2, \ldots$ of $f^{-1}(p)$ are linearly independent in $H^2(X, \mathbf{Q})$. If they are not, then some positive linear combination of some of the

E_i's is homologous to a positive linear combination of a disjoint subset of the E_i's. But then this homology class would have nonnegative intersection number with each E_i, as one sees immediately. Also, it has positive intersection number with each E_i which intersects the first set of E_i's without being contained in it; such an E_i exists, because the union of the E_i's ($= f^{-1}(p)$) is connected. Therefore this homology class has positive intersection number with $f^{-1}(p) = \sum a_i E_i$, since all the a_i's are positive. This is a contradiction, since $f^{-1}(p)$ has 0 intersection number with every E_i. This proves that the irreducible components E_i of $f^{-1}(p)$ are linearly independent in $H^2(X, \mathbf{Q})$. QED (Lemma 3.1).

We can now finish the proof of Lemma 2.2. By Lemma 3.1 and the earlier part of this proof, we know that there is a map $f : X \to C$ with the properties we want: f is a map with connected fibers onto a smooth curve, and the given divisor D_1 is a positive rational multiple of a fiber.

It remains to check that there is only one map f with these properties. By Hironaka, later used by Mori, we know that maps with connected fibers from a given projective variety X onto normal projective varieties are uniquely characterized by which curves in X map to a point [9], p. 235. For a map f with the properties we want (a map from X onto a smooth curve C with connected fibers, such that the given divisor D_1 is a rational multiple of a fiber), the positive rational multiples of fibers of f are characterized as those connected effective divisors on X which are homologous to positive rational multiples of D_1. Thus f is determined by X and D_1. QED (Lemma 2.2).

4 Some general comments on the topology of smooth divisors

This section is not used in the rest of the paper. We will explain how the Betti numbers (and Hodge numbers) of a smooth *ample* divisor in a smooth projective variety are determined by its cohomology class, as mentioned in the introduction. (In fact, its rational cohomology class is enough.) Also, we will observe that the Betti numbers (and Hodge numbers) of any smooth divisor in a variety with first Betti number equal to 0 are determined by its integral cohomology class, although we will not try to compute these invariants explicitly.

Remark 1. Let X be a smooth projective variety of dimension n. To compute the Betti and Hodge numbers of an ample divisor $D \subset X$ in terms of its class in $H^2(X, \mathbf{Q})$, we first use the Lefschetz hyperplane theorem to deduce that $h^{ij}(D) = h^{ij}(X)$ for $i + j < n - 1$. The Hodge numbers $h^{ij}(D)$ for $i + j > n - 1$ follow from Poincaré duality. It remains to compute the Hodge numbers of D for $i + j = n - 1$.

The point is the natural exact sequence of vector bundles which describes the tangent bundle of D, for any smooth divisor $D \subset X$:

$$0 \to TD \to TX|_D \to O(D)|_D \to 0.$$

It follows that the Chern classes of D are the restriction to D of cohomology classes on X, $c(D) = c(X)(1 + [D])^{-1}$, where $[D] \in H^2(X, \mathbf{Q})$. As a result, all the Chern numbers of a smooth divisor D in a given variety X are determined by the rational cohomology class of D. By the Hirzebruch-Riemann-Roch theorem, then, the rational cohomology class of a smooth divisor D explicitly determines its Euler characteristic, and more generally certain linear combinations of Hodge numbers:

$$\chi(D, \Omega^i) = \sum_j (-1)^j h^{ij}(D)$$

[7]. (For example, we get the formula for the Euler characteristic of a smooth hypersurface D of degree d in projective space $\mathbf{P}^n$:

$$\chi(D) = d^{-1}[(1-d)^{n+1} + (n+1)d - 1].)$$

Combining Hirzebruch's results with the previous paragraph's observation, we see that if D is an ample smooth divisor, then the Betti numbers and Hodge numbers of D are determined by its rational cohomology class.

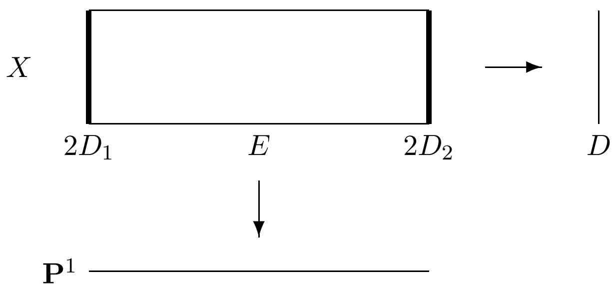
Many of these observations apply to more general degeneracy loci associated to a map of vector bundles. In particular, Harris and Tu gave formulas for the Chern numbers of any degeneracy locus which happens to be a smooth subvariety [5]. Also, for a map of vector bundles $\sigma : E \to F$ such that the vector bundle $\mathrm{Hom}(E, F)$ is ample, Fulton and Lazarsfeld proved nonemptiness and connectedness of the degeneracy loci under suitable dimension assumptions, in the spirit of the Lefschetz hyperplane theorem [3].

Remark 2. We now show that the Betti and Hodge numbers of any smooth divisor in a smooth projective variety X with $b_1(X) = 0$ are determined by its cohomology class. The point is that two linearly equivalent smooth divisors in a smooth projective variety always have the same Betti and Hodge numbers. This will imply that two homologous smooth divisors in a variety with first Betti number equal to 0 have the same Betti and Hodge numbers, since the assumption on the first Betti number implies that linear and homological equivalence of divisors are the same.

To see that two linearly equivalent smooth divisors have the same Betti and Hodge numbers, observe that the set of effective divisors in any linear equivalence class, if nonempty, is isomorphic to projective space $\mathbf{P}^N$ for some N. Moreover, the set of smooth effective divisors is a Zariski open subset. So the set of smooth effective divisors in a given linear equivalence class is always connected if it is nonempty. As a result, any two linearly equivalent smooth divisors belong to one connected family of smooth projective varieties. In particular, the two divisors have the same Betti and Hodge numbers.

5 Connectedness of smooth divisors

We turn to the second topic of this paper. First, we will give examples to show that a smooth connected divisor on a smooth projective variety can be homologous to a smooth non-connected divisor. Then we will show that the examples we give, which are on varieties which fiber over a curve with enough multiple fibers, are the only possible ones.



The simplest example of a smooth connected divisor which is homologous to a smooth non-connected divisor is the following (see the figure). Let D be a curve of genus at least 1, and let L be a nontrivial line bundle of degree 0 on D such that $L^{\otimes 2}$ is trivial. Let X be the ruled surface $P(O \oplus L)$ over D. Let D_1 and D_2 be the sections of this ruled surface at 0 and infinity. Then the divisor $2D_1$ is linearly equivalent to the disjoint divisor $2D_2$, so there is a morphism $f : X \to \mathbf{P}^1$ with $f^{-1}(0) = 2D_1$ and $f^{-1}(\infty) = 2D_2$. The inverse image of any other point in $\mathbf{P}^1$ is isomorphic to the double cover E of D which corresponds to the 2-torsion line bundle L. In this situation, the smooth connected curve $E \subset X$ is homologous to the non-connected smooth divisor $D_1 + D_2$.

This example can be generalized as follows. Let X be any smooth projective variety with a morphism $f : X \to C$ onto a smooth curve C, and suppose that all the fibers are connected. The general fibers of f are smooth connected divisors. There may be other fibers which are smooth “multiple fibers,” meaning that (as a divisor) $f^{-1}(p) = aD$, for some $a \geq 2$ and smooth divisor D in X. In this case, D is rationally homologous to $(1/a) \cdot$ (general fiber).

As a result, whenever there are enough smooth multiple fibers, we get examples of a smooth connected divisor (say, a general fiber) which is at least rationally homologous to a non-connected smooth divisor (say, a sum of multiple fibers). The surface constructed above has this form: it has a map $f : X \to \mathbf{P}^1$ with two double fibers, so a general fiber is rationally homologous to the sum of the two double fibers. (In that example, the general fiber happens to be integrally homologous to the sum of the two double fibers.)

The surprising thing is that these examples are the only thing that can go wrong, in the following sense.

Theorem 5.1 *Let X be a smooth projective variety. Let $A = \sum_i A_i$ and $B = \sum_i B_i$ be rationally homologous smooth divisors on X. (Thus $A_1, A_2, \ldots$ are disjoint smooth connected divisors, and so are $B_1, B_2, \ldots$.) Remove any components which occur in both A and B. Then at least one of the following statements holds.*

(1) $A = B = 0$.

(2) A and B are connected.

(3) There is a map $f : X \to C$ onto a smooth curve C such that all the fibers are connected and each of the divisors A_i and B_i is a fiber of f, possibly a multiple fiber. In fact there is a unique map f with these properties.

Proof. We have to show that if A or B has at least two components, then (3) holds.

As in section 1, we fix an ample divisor H on X and define an intersection pairing on divisors by

$$(D, E) = D \cdot E \cdot H^{n-2} \in \mathbf{Z}.$$

All the divisors A_i and B_i must have nonnegative self-intersection number, since, say for A_1:

$$\begin{aligned}(A_1, A_1) &= (A_1, A_1 + A_2 + \cdots) \\ &= (A_1, B_1 + B_2 + \cdots) \\ &\geq 0.\end{aligned}$$

The last inequality holds because A and B have no components in common. Since different components of A are disjoint, it follows that the components of A span a subspace of $H^2(X, \mathbf{Q})$ on which the intersection pairing (D, E) is nonnegative semidefinite. The Hodge index theorem (section 1) then implies that the components of A span only a 1-dimensional subspace of $H^2(X, \mathbf{Q})$. The same holds for B. As a result, all the components of A and B have rational cohomology classes in the same 1-dimensional subspace of $H^2(X, \mathbf{Q})$.

Since A or B has at least two components, say A, we have $(A_1, A_2) = 0$. Since all the components of A and B are homologous up to multiples, it follows that they all have self-intersection number 0, and they are all disjoint. Thus we have at least three disjoint smooth connected divisors on X (the components of A and B, together) whose rational cohomology classes lie in a line.

By Theorem 2.1, statement (3) holds. QED

6 Smooth connected divisors and the arithmetic of abelian varieties

We begin this section with an example of two disjoint homologous smooth divisors which are both connected but which have different Betti numbers. Conjecture 6.1, suggested by this example, says that any two homologous connected smooth divisors in a smooth complex projective variety X should have cyclic etale coverings which are deformation equivalent to each other. Theorem 6.3 proves a weaker form of this conjecture, allowing deformations into positive characteristic, under the assumption that the Picard variety of X is isogenous to a product of elliptic curves. This assumption could be omitted if we knew Conjecture 6.2, a well-known conjecture on the arithmetic of abelian varieties.

Example. We exhibit a smooth complex projective variety containing two disjoint homologous smooth divisors which are both connected but have different Betti numbers.

Let C_1 and C_2 be smooth curves, both of genus at least 1. Let $B_i \to C_i$ be a nontrivial double covering of C_i, for $i = 1, 2$. Then the group $(\mathbf{Z}/2)^2$ acts freely on $B_1 \times B_2$ with quotient $C_1 \times C_2$. Let $(\mathbf{Z}/2)^2$ also act on $\mathbf{P}^1$ with generators $x \mapsto -x$ and $x \mapsto 1/x$. The stabilizer of the point 0 in $\mathbf{P}^1$ is the subgroup $H_1 = \mathbf{Z}/2 \times 0$, and the stabilizer of 1 in $\mathbf{P}^1$ is the subgroup $H_2 = 0 \times \mathbf{Z}/2$. Let X be the quotient variety

$$X = (B_1 \times B_2 \times \mathbf{P}^1)/(\mathbf{Z}/2)^2.$$

Since $(\mathbf{Z}/2)^2$ acts freely on $B_1 \times B_2$, X is smooth. It is straightforward to check that $H^2(X, \mathbf{Z})$ is torsion-free.

The image of $B_1 \times B_2 \times 0$ in X is a smooth divisor D_1 isomorphic to $(B_1 \times B_2)/H_1 = C_1 \times B_2$, while the image of $B_1 \times B_2 \times 1$ in X is a smooth divisor D_2 isomorphic to $(B_1 \times B_2)/H_2 = B_1 \times C_2$. These two divisors are disjoint. They are rationally homologous, because $2D_1$ and $2D_2$ are both linearly equivalent to the image of $B_1 \times B_2 \times p$ for a general point p in $\mathbf{P}^1$. Since $H^2(X, \mathbf{Z})$ is torsion-free, D_1 and D_2 are integrally homologous. But they can have different Betti numbers. For example, we can assume that C_1 has genus 1 and C_2 has genus $g \geq 2$. Then the two divisors $B_1 \times C_2$ and $B_2 \times C_1$ have different Betti numbers, as shown in the following table. They must have the same Euler characteristic by Remark 1 in section 4.

i	0	1	2	3	4
$b_i(B_1 \times C_2)$	1	$2g+2$	$4g+2$	$2g+2$	1
$b_i(C_1 \times B_2)$	1	$4g$	$8g-2$	$4g$	1

In this example, D_1 and D_2 have isomorphic double coverings. More generally, for any variety X with a map to a curve such that D_1 and D_2 are smooth multiple fibers (as happens in the above example), a cyclic etale covering of D_1 will be deformation equivalent to a general fiber and hence to a cyclic etale covering of D_2. This leads to the following conjecture.

Conjecture 6.1 *Let D_1 and D_2 be smooth connected divisors in a smooth complex projective variety X which are rationally homologous. Then there is a positive integer n and an etale $\mathbf{Z}/n$-covering $\widetilde{D_1}$ of D_1 which is deformation equivalent to an etale $\mathbf{Z}/n$-covering $\widetilde{D_2}$ of D_2. Or we could ask only for $\widetilde{D_1}$ to be homotopy equivalent to $\widetilde{D_2}$.*

We can assume that D_1 and D_2 are disjoint in this conjecture. If they are not, let $f : X' \to X$ be the blow-up of X along the (possibly non-reduced) subscheme $D_1 \cap D_2$. An easy calculation shows that X' contains disjoint smooth divisors isomorphic to D_1 and D_2, and that X' is smooth in a neighborhood of these divisors. We have $f^*D_i = D_i + E$ where E is the exceptional divisor of f, so D_1 and D_2 are rationally homologous on X' if they were rationally homologous on X, and they are integrally homologous on X' if they were integrally homologous on X. Finally, we can resolve the singularities of X' by Hironaka without changing it in a neighborhood of D_1 and D_2. Thus, for any divisors D_1 and D_2 as in Conjecture 6.1, the same varieties D_1 and D_2 occur as *disjoint* homologous divisors in some other smooth projective variety. So we can and do assume that D_1 and D_2 are disjoint from now on.

The proof of Theorem 2.1 shows that Conjecture 6.1 is true in its stronger form if $D_1 - D_2$ is torsion in the Picard group of X, or, more generally (using that D_1 and D_2 are disjoint), if the normal bundle of D_1 in X is torsion in the Picard group of D_1. Indeed, under these assumptions, the proof of Theorem 2.1 gives a map from X to a curve in which D_1 and D_2 are smooth multiple fibers, say with multiplicity n (clearly the same for D_1 and D_2, since they are rationally homologous). Then there is an etale $\mathbf{Z}/n$-covering of D_1 which deforms to a general fiber of the map and hence to an etale $\mathbf{Z}/n$-covering of D_2.

But the normal bundle of D_1 in X need not be torsion in the Picard group of D_1, under the assumption of Conjecture 6.1 together with the assumption that D_1 and

D_2 are disjoint. Simple examples are given in section 2. The only way of attacking Conjecture 6.1 that comes to mind is to deform (X, D_1, D_2) in some way until the normal bundle of D_1 becomes torsion in the Picard group of D_1. Over the complex numbers, I do not see any way to do this.

We can instead consider a more general kind of deformation. Every smooth complex projective variety X can be deformed to one defined over a number field, and then reduced modulo prime ideals to get a smooth projective variety X_k over a finite field k. We can assume that D_1 and D_2 reduce to disjoint homologous divisors in X_k (using l-adic etale cohomology over the algebraic closure of k, for some prime number l invertible in k). The advantage of reducing to a finite field k, or its algebraic closure $\overline{k}$, is that a line bundle on $X_{\overline{k}}$ which is zero in $H^2(X_{\overline{k}}, \mathbf{Q}_l)$ is torsion in the Picard group of $X_{\overline{k}}$, because the group of points of an abelian variety over a finite field is finite. Therefore we can apply the proof of Theorem 2 to get a map f from $X_{\overline{k}}$ onto a smooth curve $C_{\overline{k}}$ such that $f_*O_X = O_C$ (that is, f has connected fibers), $f^{-1}(p_1) = nD_1$, and $f^{-1}(p_2) = nD_2$ for some points p_1 and p_2 in C and some positive integer n dividing the order of $D_1 - D_2$ in the Picard group of $X_{\overline{k}}$.

The problem is that the topological implications of such a map are not clear to me when the number n is a multiple of the characteristic of k. The map f is separable since $f_*O_X = O_C$, but Sard's theorem still fails: the general fiber need not be smooth. I do not see how to deduce any topological relation between D_1 and D_2 in this case, although it may be possible.

I can only say something if the order of $D_1 - D_2$ in the Picard group of $X_{\overline{k}}$ is invertible in k. Then we get a map f from $X_{\overline{k}}$ onto a smooth curve $C_{\overline{k}}$ such that $f_*O_X = O_C$, $f^{-1}(p_1) = nD_1$, and $f^{-1}(p_2) = nD_2$, for some points p_1 and p_2 in C and some positive integer n dividing the order of $D_1 - D_2$ in the Picard group of $X_{\overline{k}}$, hence invertible in k. It follows that D_1 over $\overline{k}$ has an etale $\mathbf{Z}/n$-covering which is deformation equivalent to a general fiber of f and hence to an etale $\mathbf{Z}/n$-covering of D_2 over $\overline{k}$. Therefore, using the known relations between the topology of varieties in characteristic zero and their reductions to positive characteristic, the divisors D_1 and D_2 in characteristic zero have $\mathbf{Z}/n$-coverings $\widetilde{D_1}$ and $\widetilde{D_2}$ with the same pro-l homotopy type for all prime numbers l invertible in k ([1], pp. 142–144). In particular these two coverings have isomorphic $\mathbf{Z}_l$-cohomology rings for all such l.

Thus we can prove a slightly weaker form of Conjecture 6.1 if we can find a prime ideal $\mathfrak{p}$ of the number field F such that (X, D_1, D_2) reduces smoothly over $k = o_F/\mathfrak{p}$ and the order of $D_1 - D_2$ in the Picard group of $X_{\overline{k}}$ is invertible in k. It would suffice for this to know that given an abelian variety A over a number field F (the Picard variety of X over F) and a point of A over F (the class of $D_1 - D_2$, or a suitable multiple of $D_1 - D_2$ if D_1 and D_2 are only rationally homologous), there are infinitely many primes $\mathfrak{p}$ of F such that A has good reduction modulo $\mathfrak{p}$ and the reduction of x in $A(o_F/\mathfrak{p})$ has order invertible in $o_F/\mathfrak{p}$. This would follow from the following well-known conjecture on the arithmetic of abelian varieties.

Conjecture 6.2 *For any abelian variety A over a number field F, there are infinitely many primes $\mathfrak{p}$ of F such that the order of the group $A(o_F/\mathfrak{p})$ is prime to the characteristic of the field $o_F/\mathfrak{p}$.*

In fact, it is expected that the set of primes $\mathfrak{p}$ such that $A(o_F/\mathfrak{p})$ has order a multiple of the characteristic p of $o_F/\mathfrak{p}$, called anomalous primes in Mazur [10], has density zero. But even the much weaker statement of Conjecture 6.2 seems inaccessible in general.

It is known for elliptic curves. For example, it follows from a result of Serre's on the distribution of eigenvalues of Frobenius for an elliptic curve as $\mathfrak{p}$ varies ([12], exercise 1, p. IV-13). There is also a more elementary argument, as follows. First, to prove Conjecture 6.2 for a given abelian variety A over a number field F, it suffices to prove it after extending the field F. Consider the case of an elliptic curve E over F; after extending the field F, we can assume that the torsion subgroup of $E(F)$ is nonzero. Let l be a prime number such that $E(F)$ has l-torsion. By the Chebotarev density theorem, the set of primes $\mathfrak{p}$ of F such that the field $o_F/\mathfrak{p}$ has prime order has positive density. For such primes $\mathfrak{p}$, by Hasse, the group $E(o_F/\mathfrak{p}) = E(\mathbf{F}_p)$ has order $p+1-a_p$ where $|a_p| \leq 2\sqrt{p}$. (This is the famous bound generalized by Weil from elliptic curves to curves of arbitrary genus.) So if $p \geq 7$ and $E(\mathbf{F}_p)$ has order a multiple of p, then it has order equal to p. But we arranged that $E(F)$ has l-torsion, so $E(o_F/\mathfrak{p})$ has order a multiple of l for all but finitely many primes $\mathfrak{p}$ of F. So, for all but finitely many of the primes $\mathfrak{p}$ of F with $o_F/\mathfrak{p}$ of prime order p, the group $E(o_F/\mathfrak{p})$ cannot have order p and hence does not have order a multiple of p. This proves Conjecture 6.2 for elliptic curves.

The same argument proves Conjecture 6.2 for any abelian variety A which is a product of elliptic curves. It follows easily that Conjecture 6.2 holds whenever A is isogenous to a product of elliptic curves. As we have said, it suffices to prove Conjecture 6.2 after a finite extension of the number field F, so it suffices that A is isogenous to a product of elliptic curves over the algebraic closure of $\mathbf{Q}$. Thus we have proved:

Theorem 6.3 *Let D_1 and D_2 be smooth connected divisors in a smooth complex projective variety X which represent the same element of $H^2(X, \mathbf{Q})$. Suppose that the Picard variety of X is isogenous to a product of elliptic curves. Then there are etale $\mathbf{Z}/n$-coverings $\widetilde{D_1}$ and $\widetilde{D_2}$ of D_1 and D_2, for some positive integer n, which are deformation equivalent via passage to some characteristic $p > 0$. It follows that $\widetilde{D_1}$ and $\widetilde{D_2}$ have the same pro-l homotopy type in the sense of [1] for all prime numbers $l \neq p$, hence, for example, isomorphic $\mathbf{Z}_l$-cohomology rings.*

The assumption that the Picard variety of X is isogenous to a product of elliptic curves is strange. It should certainly be unnecessary; this would follow from Conjecture 6.2 on abelian varieties, which is universally believed to be true but which seems inaccessible. It would be very interesting to find some geometric approach to at least some weaker version of Conjecture 6.1, for example only showing that the universal coverings of D_1 and D_2 are homotopy equivalent, which avoids reducing to characteristic p and thereby avoids the assumption on the Picard variety of X.

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