Commutative Algebra (M24)

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Commutative algebra is the study of commutative rings, the basic examples being the ring of integers $\mathbf{Z}$ and the ring of polynomials $k[x_1, \dots, x_n]$ over a field. These examples correspond to the two main subjects where commutative algebra is used: number theory and algebraic geometry. Commutative algebra is also fundamental for algebraists. My interests lie in algebraic geometry, and one aim of the course is to prove the algebraic results used in Part III Algebraic Geometry.

We try to think of any commutative ring as the ring of functions on some space, so that there is a constant interplay between algebra and geometry. In that spirit, the course starts by defining the prime spectrum of a ring, which is a topological space. Localization of rings and modules is an algebraic notion that corresponds to working in a neighborhood of a point. We define noetherian rings, which include all the rings we are concerned with. We prove the Hilbert basis theorem, the Hilbert Nullstellensatz, and the Noether normalization lemma: these are foundational results for algebraic geometry. We study Dedekind domains and discrete valuation rings, which are the well-behaved rings of dimension 1.

We define regular local rings, which correspond in geometric terms to smooth varieties (or manifolds). We develop the rich theory of dimension for commutative rings. Integral closure is an algebraic concept which describes both the arithmetic notion of algebraic integers and the geometric process of "normalization".

For much of the course, our approach will be close to the beautiful short book by Atiyah and Macdonald. That corresponds to about the first half of Eisenbud's book. Compared to Atiyah-Macdonald, we include more homological algebra (flat modules, projective modules, Ext and Tor), and we prove some of the deeper results on regular local rings.

Pre-requisite Mathematics

Undergraduate algebra is the main background needed (especially fields, rings, and modules). Lang's *Algebra* is a good graduate-level reference that includes all the background needed, as well as some of the course material.

Literature

1. M. Atiyah and I. Macdonald. *Introduction to commutative algebra.* Westview (1994), £42.99.

2. D. Eisenbud. *Commutative algebra with a view toward algebraic geometry.* Springer (1995), £27.19.

3. S. Lang. *Algebra.* Springer (2002), £53.99.

4. M. Reid. *Undergraduate commutative algebra.* Cambridge (1993), £20.39.