The cohomology ring of the space of rational functions

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In this paper we consider three spaces which can be viewed as finite-dimensional approximations to the 2-fold loop space $\Omega^2 S^2$. These are $Rat_k(\mathbf{C}P^1)$, the space of based holomorphic maps $S^2 \to S^2$ of degree k; $B\beta_q$, the classifying space of Artin's braid group on q strings; and $C_k(\mathbf{R}^2, S^1)$, which is the space of configurations of $\leq k$ distinct points in $\mathbf{R}^2$ with labels in S^1, with some identifications. The space Rat_k can be described more explicitly as the space of rational functions

$$\frac{p(z)}{q(z)} = \frac{z^k + a_{k-1}z^{k-1} + \cdots + a_0}{z^k + b_{k-1}z^{k-1} + \cdots + b_0}$$

where the polynomials $p(z)$ and $q(z)$ are relatively prime. Extending a fundamental theorem of G. Segal [8], Cohen, Cohen, Mann, and Milgram ([3], [4]) have shown that the spaces Rat_k, $B\beta_{2k}$, and C_k are all stably homotopy equivalent; in fact they all split stably as a wedge $\vee_{j \leq k} D_j(S^1)$, where $D_j = C_j / C_{j-1}$ is a well-known space related to Brown-Gitler spectra. In this paper we show that for most values of k, the space Rat_k is not homotopy equivalent to $B\beta_{2k}$ or to C_k.

More precisely, we prove:

Theorem 1 *For all $k \geq 0$ such that $k+1$ is not a power of 2, the $\mathbf{Z}/2$ cohomology ring of Rat_k is not isomorphic (as a graded ring) to that of $B\beta_{2k}$ or C_k.*

So, for $k+1$ not a power of 2, there are no maps between Rat_k and one of $B\beta_{2k}$ or C_k which induce isomorphisms on homology. What happens when $k+1$ is a power of 2? For $k = 1$ there are homotopy equivalences $Rat_1 \simeq B\beta_2 \simeq C_1 \simeq S^1$, but I don't know what happens for other such k.

For $k = 3$, all three spaces Rat_3, $B\beta_6$, C_3 have isomorphic $\mathbf{Z}/2$-cohomology rings. For all k and all $p \geq 3$, the three spaces Rat_k, $B\beta_{2k}$, and C_k all have isomorphic $\mathbf{Z}/p$-cohomology rings.

We can also ask how $B\beta_{2k}$ differs from C_k. We have $\pi_1(B\beta_{2k}) = \beta_{2k}$ and $\pi_1(C_k) = \mathbf{Z}$, so they are not homotopy equivalent for $k \geq 2$ because the braid group β_{2k} is not abelian. Less trivially, R. Cohen [6] asked whether there is a map $B\beta_{2k} \rightarrow C_k$ which gives an isomorphism on homology. As evidence for this, we prove that $B\beta_{2k}$ and C_k have isomorphic $\mathbf{Z}/p$-cohomology rings for all primes p. Nonetheless, if one considers homology with non-constant coefficients, then R. Cohen's statement can fail. This is easier to state in terms of the natural $\mathbf{Z}$-covers $\tilde{Rat_k}$, $\tilde{B\beta_{2k}}$, and $\tilde{C_k}$ of these spaces: namely, in general, all three of these covering spaces have different homology groups. The proof will appear in the final version of this paper. For example $H_*(\tilde{C_k})$ is sometimes infinite-dimensional, while $\tilde{B\beta_{2k}}$ and $\tilde{Rat_k}$ always have finite-dimensional homology.

This paper was inspired by Cohen-Shimamoto [7]. They proved that Rat_2 and C_2 are not homotopy equivalent. Then I proved Theorem 1 for $k = 2$, using results from their paper. R. Cohen suggested that one could give a better proof by comparing the action of the Araki-Kudo operations on different components of $\Omega^2 S^2$. This idea allowed me to generalize Theorem 1 to its current form.

I thank Ralph Cohen for his help. [1]

1 Homology of $\Omega^2 S^2$ with $\mathbf{Z}/2$ coefficients

The standard May-Milgram-Segal approximation to $\Omega^2 S^2$ is the disjoint union of braid spaces, $\coprod_{q\geq 0} B\beta_q$. See Fred Cohen [2], for example. We can describe $B\beta_q$ geometrically as follows. Define the configuration space $F(\mathbf{R}^2, q)$ of q distinct points in $\mathbf{R}^2$ to be $\{(x_1, \ldots, x_q) : x_i \in \mathbf{R}^2, x_i \neq x_j\}$. The symmetric group Σ_q acts on $F(\mathbf{R}^2, q)$ in a natural way. Then $B\beta_q = F(\mathbf{R}^2, q)/\Sigma_q$. Thus $B\beta_q$ is the space of q distinct unordered points in $\mathbf{R}^2$.

Cohen computed the $\mathbf{Z}/2$-homology of $\coprod_{q\geq 0} B\beta_q$ and $\Omega^2 S^2$ as follows. (In this section, all homology will have $\mathbf{Z}/2$ coefficients.) To begin, we note that $\pi_0 \Omega^2 S^2 = \pi_2(S^2) \cong \mathbf{Z}$, where the isomorphism is given by the degree of a map $S^2 \rightarrow S^2$. Let $\Omega^2_i S^2$ denote the ith component of $\Omega^2 S^2$, $i \in \mathbf{Z}$.

Both spaces are "C_2-spaces," so they each have an Araki-Kudo operation $Q : H_q \rightarrow H_{2q+1}$ as well as a Pontrjagin product which makes the homology

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a commutative ring. More precisely, the map $\coprod_{q\geq 0} B\beta_q \to \Omega^2 S^2$ of C_2-spaces maps $B\beta_q$ into $\Omega^2_q S^2$, and Q maps $H_q\Omega^2_i S^2$ to $H_{2q+1}\Omega^2_{2i} S^2$. Let $g \in H_0(B\beta_1)$ be the generator of this group; we also let g denote the image in $H_0(\Omega^2_1 S^2)$ of this element. (Thus $g \in H_0\Omega^2 S^2$ represents the component of $\Omega^2 S^2$ containing the identity map $S^2 \to S^2$ (viewing this as a point in $\Omega^2 S^2$).) Then we have algebra isomorphisms as follows:

$$H_*(\coprod B\beta_q) \cong \mathbf{Z}/2[g, Qg, Q^2 g, \ldots]$$

$$H_*(\Omega^2 S^2) \cong \mathbf{Z}/2[g, g^{-1}] \otimes \mathbf{Z}/2[Qg, Q^2 g, \ldots]$$

In particular, we can view $H_*(\coprod B\beta_q)$ as a subring of $H_*(\Omega^2 S^2)$, as I do from now on. We note that $Q^i g$ is a homology class of dimension $2^i - 1$ living in the 2^ith component of $\Omega^2 S^2$.

Thus $H_*(B\beta_q)$ is the span of the monomials in g, Qg, $Q^2 g, \ldots$ of weight $= q$, where $Q^i g$ is given weight 2^i and dimension $2^i - 1$.

Using the fact that the "Browder operation" on $H_*\Omega^2 S^2$ is 0, F. Cohen shows ([2], p. 372) that the Araki-Kudo operation $Q : H_q\Omega^2 S^2 \to H_{2q+1}\Omega^2 S^2$ is linear, and that the Cartan formula $Q(xy) = x^2 Qy + Qx \cdot y^2$ holds with no error terms. This indicates that $\Omega^2 S^2$ is easier to understand than a general 2-fold loop space.

Cohen also computes the coproduct on the homology of these spaces (or, equivalently, he computes the cup product on cohomology). The result is that $H_*(\coprod B\beta_q)$ is a primitively generated Hopf algebra. More precisely, if we denote the coproduct by $\psi : H_* \to H_* \otimes H_*$, we have $\psi(g) = g \otimes g$, and $Q^i g$ for $i \geq 1$ is primitive in its component, that is, $\psi(Q^i g) = g^{2^i} \otimes Q^i g + Q^i g \otimes g^{2^i}$. To verify this formula, use the diagonal Cartan formula for Q, [2], p. 217. In particular, it helps to notice that, by the Cartan formula for Q, we have $Q(x^2) = x^2 Qx + Qx \cdot x^2 = 0$ for all x in $H_*\Omega^2 S^2$, since we are using $\mathbf{Z}/2$ coefficients. (This is a sort of Adem relation, since $x \mapsto x^2$ is an Araki-Kudo operation just as Q is.)

2 $B\beta_{2k}$ and C_k have isomorphic $\mathbf{Z}/2$-cohomology rings

We first prove that $B\beta_{2k}$ and C_k have isomorphic $\mathbf{Z}/2$-cohomology rings. The next three paragraphs quote [7]'s description of C_k verbatim.

Let $C(\mathbf{R}^2, Y)$ denote the space of all configurations of distinct points in $\mathbf{R}^2$ with labels in Y. That is,

$$C(\mathbf{R}^2, Y) = \cup_{q=1}^{\infty} F(\mathbf{R}^2, q) \times_{\Sigma_q} Y^q / \sim$$

where $F(\mathbf{R}^2, q) = \{(x_1, \ldots, x_q) : x_i \in \mathbf{R}^2, x_i \neq x_j\}$ and Σ_j is the symmetric group on q letters. The relation is generated by setting

$$(x_1, \ldots, x_q) \times_{\Sigma_q} (t_1, \ldots, t_{q-1}, *) \sim (x_1, \ldots, x_{q-1}) \times_{\Sigma_{q-1}} (t_1, \ldots, t_{q-1})$$

where $* \in Y$ is a fixed basepoint.

A well-known result of May, Milgram, and Segal states that, when Y is a connected CW-complex, $C(\mathbf{R}^2, Y)$ is homotopy equivalent to the based loop space

$$\Omega^2 \Sigma^2 Y = \{f : S^2 \to \Sigma^2 Y : f(\infty) = * \in Y\}.$$

Now let $C_k(\mathbf{R}^2, Y) \subset C(\mathbf{R}^2, Y)$ denote the subspace of configurations of length $\leq k$. That is,

$$C_k(\mathbf{R}^2, Y) = \cup_{q=1}^{k} F(\mathbf{R}^2, q) \times_{\Sigma_q} Y^q / \sim .$$

The space we want to consider is $C_k := C_k(\mathbf{R}^2, S^1)$, a finite-dimensional approximation to $\Omega^2 S^3$. F. Cohen's calculations show that

$$H_*(\cup_{k \geq 0} C_k) = H_*(\Omega^2 S^3) = \mathbf{Z}/2[h, Qh, Q^2 h, \ldots]$$

where $h \in H_1(\Omega^2 S^3)$ is the image of the generator of $H_1 S^1$ via the standard map $S^1 \to \Omega^2 \Sigma^2 (S^1) = \Omega^2 S^3$. This is also a primitively generated Hopf algebra; namely, $Q^i h$ is primitive for all $i \geq 0$.

Define a weight on the monomials in $H_*(\Omega^2 S^3)$ by

$$\mathrm{wt}(Q^i h) = 2^i, \ \ \mathrm{wt}(ab) = \mathrm{wt}(a) + \mathrm{wt}(b).$$

By SLN 533, p. 239, $H_* C_k \hookrightarrow H_* \Omega^2 S^3$ is the span of the monomials of weight $\leq k$.

Now the Hopf map $S^3 \to S^2$ induces a map of 2-fold loop spaces $\Omega^2 S^3 \to \Omega^2 S^2$; in fact this map is a homotopy equivalence from $\Omega^2 S^3$ to the component $\Omega_0^2 S^2$, as follows from the long exact sequence of homotopy groups of the Hopf fibration $S^1 \to S^3 \to S^2$. From now on we identify $\Omega^2 S^3$ with the component $\Omega_0^2 S^2$.

In these terms, the generator $h \in H_1(\Omega^2 S^3)$ is equal to $g^{-2} Q g$, the generator of $H_1(\Omega_0^2 S^2) \cong \mathbf{Z}/2$. So we can say that:

$$H_*(\cup_{k \geq 0} C_k) = H_*(\Omega_0^2 S^2) = \mathbf{Z}/2[g^{-2} Qg, Q(g^{-2} Qg), Q^2(g^{-2} Qg), \ldots]$$

And H_*C_k is the span of the monomials of weight $\leq k$, where $Q^i(g^{-2}Qg)$ is given weight 2^i and dimension $2^{i+1}-1$.

Proposition 1 *The homotopy equivalence* $\cdot g^{-2k} : \Omega^2_{2k}S^2 \to \Omega^2_0 S^2$ *sends* $H_*B\beta_{2k} \subset H_*\Omega^2_{2k}S^2$ *onto* $H_*C_k \subset H_*\Omega^2_0 S^2$. *Therefore* $H_*(B\beta_{2k}, \mathbf{Z}/2) \cong H_*(C_k, \mathbf{Z}/2)$ *as coalgebras.*

This proposition says in particular that $B\beta_{2k}$ and C_k have isomorphic $\mathbf{Z}/2$-cohomology rings, as I claimed at the beginning of this section.

The key computation for the proof of this proposition is that $Q^i(g^{-2}Qg) = g^{-2^{i+1}}Q^{i+1}g$ for all $i \geq 0$. This is clear for $i = 0$, and the induction uses the Cartan formula for Q and the fact that $Q(x^2) = 0$ for all $x \in H_*\Omega^2S^2$.

By Section 1, the homology $H_*(B\beta_q)$ is the span of the monomials in g, Qg, Q^2g, ... of weight $= q$, where Q^ig is given weight 2^i and dimension $2^i - 1$. Therefore the image of $H_*B\beta_{2k}$ in $H_*\Omega^2_0S^2$ under the map $\cdot g^{-2k}$: $H_*\Omega^2_{2k}S^2 \xrightarrow{\sim} H_*\Omega^2_0S^2$ is the span of the monomials in 1, $g^{-2}Qg$, $g^{-4}Qg$, ... of weight $= 2k$, where $g^{-2^i}Q^ig$ is given weight 2^i and dimension $2^i - 1$. Equivalently, by the previous paragraph, this is the span of the monomials in $g^{-2}Qg$, $Q(g^{-2}Qg)$, $Q^2(g^{-2}Qg)$, ... of weight $\leq k$, where (after we multiply the previous weights by 1/2) $Q^i(g^{-2}Qg)$ is given weight 2^i and dimension $2^{i+1} - 1$. But this is exactly the space $H_*C_k \subset H_*\Omega^2_0S^2$.

This proves the proposition.

3 Rat_1

It is easy to see that the space Rat_1 of based holomorphic maps $S^2 \to S^2$ of degree 1 is homotopy equivalent to the circle S^1. For example, take the basing condition to be $\infty \mapsto \infty$; then Rat_1 is the group of affine tranformations $z \mapsto az + b$, a, $b \in \mathbf{C}$, $a \neq 0$, which is homotopy equivalent to $\mathbf{C} - 0$ and hence to S^1. In particular $H_1(Rat_1, \mathbf{Z}) \cong \mathbf{Z}$.

By Segal's stability theorem [8], the inclusion $Rat_1 \to \Omega^2_1S^2$ induces a surjection $H_1(Rat_1, \mathbf{Z}) \to H_1(\Omega^2_1S^2, \mathbf{Z})$. Since both groups are isomorphic to $\mathbf{Z}$, this map is actually an isomorphism. We deduce that $H_1(Rat_1, \mathbf{Z}/p) \to H_1(\Omega^2_1S^2, \mathbf{Z}/p) \cong \mathbf{Z}/p$ is an isomorphism for all primes p, a fact which we will need later.

4 Proof of Theorem 1

We have proved that H_*C_k and $H_*B\beta_{2k}$ are isomorphic as coalgebras. To complete the proof of Theorem 1, we have to show that H_*Rat_k is not isomorphic to $H_*B\beta_{2k}$ as a coalgebra, assuming that $k+1$ is not a power of 2.

By F. Cohen and Boyer-Mann [1], $\coprod_{k\geq 0} Rat_k$ is a C_2-space, and the natural map $\coprod_{k\geq 0} Rat_k \to \Omega^2 S^2$ is a map of C_2-spaces. The image of $H_1 Rat_1 \to H_1\Omega_1^2 S^2 \cong \mathbf{Z}/2$ is not zero, so it contains the generator of the latter group, $g^{-1}Qg$. So the image of $H_*(\coprod_k Rat_k) \to H_*\Omega^2 S^2$ contains the subring A of $H_*\Omega^2 S^2$ generated by g and $Q^{i-1}(g^{-1}Qg)$, $i \geq 1$. Here g has weight 1, dimension 0, and $Q^{i-1}(g^{-1}Qg)$, $i \geq 1$, has weight 2^{i-1} and dimension $2^i - 1$. We need to check that A is actually a polynomial ring on the generators g and $Q^{i-1}(g^{-1}Qg)$, $i \geq 1$.

To see this, define subrings B_i of the ring $H_*\Omega^2 S^2 = \mathbf{Z}/2[g, g^{-1}, Qg, Q^2 g, \ldots]$ to be $B_i = \mathbf{Z}/2[g, g^{-1}, Qg, \ldots, Q^i g]$. Then the Cartan formula for Q shows that $Q^{i-1}(g^{-1}Qg) \equiv g^{-2^{i-1}} Q^i g (\text{mod } B_{i-1})$, for $i \geq 1$. Although this is not an equality in $H_*\Omega^2 S^2$, the equality modulo B_{i-1} suffices to imply that A is a polynomial ring with generators g and $Q^{i-1}(g^{-1}Qg)$, $i \geq 1$. (To see this, filter the ring B_i by the subspaces $B_{ik} = \sum_{j\leq k} B_{i-1} \cdot (Q^i g)^j$.)

In particular, since A is a polynomial ring, we can compute dim A_k^l, where $A_k^l = \text{dim } A \cap H_l \Omega_k^2 S^2$. The weights of the generators of A are exactly 1/2 times the corresponding generators of $H_*(\coprod_q B\beta_q)$, with $Q^{i-1}(g^{-1}Qg) \in A$ corresponding to $Q^i g \in H_*(\coprod_q B\beta_q)$ for $i \geq 1$; so dim $A_k^* = \text{dim } H_*B\beta_{2k}$. By [3], dim $H_*B\beta_{2k} = \text{dim } H_*Rat_k$. Thus

$$\text{dim } H_*B\beta_{2k} = \text{dim } H_*Rat_k \geq \text{dim } A_k^* = \text{dim } H_*B\beta_{2k}.$$

So dim $H_*Rat_k = \text{dim } A_k^*$. Thus $H_*(\coprod_k Rat_k) \to H_*\Omega^2 S^2$ is injective and the image is exactly the polynomial ring $A = \mathbf{Z}/2[g, Q^{i-1}(g^{-1}Qg) : i \geq 1] \subset H_*\Omega^2 S^2$.

Thus the homology of Rat_k is "built up" using the Araki-Kudo operation Q just as the homology of C_k is; the difference between them, as coalgebras, arises because the basic homology class for Rat_1 lies in $H_1\Omega_1^2 S^2$, while the basic homology class for C_1 lies in $H_1\Omega_0^2 S^2$. This makes a difference because of the Cartan formula for Q and the fact that Q is nonzero on $H_0\Omega^2 S^2$.

We now compute the coproduct on H_*Rat_k. By the diagonal Cartan formula for Q,

$$\begin{aligned} \psi(g^{-1}Qg) &= \psi(g^{-1})\psi(Qg) \\ &= (g^{-1} \otimes g^{-1})(g^2 \otimes Qg + Qg \otimes G^2) \\ &= g \otimes g^{-1}Qg + g^{-1}Qg \otimes g. \end{aligned}$$

So $g^{-1}Qg$ is primitive in its component.

But $Q^i(g^{-1}Qg)$ is not primitive in its component for $i \geq 1$. In fact, we have:

$$\begin{array}{lllll} \psi Q^i(g^{-1}Qg) = & g^{2^i} \otimes Q^i(g^{-1}Qg) & +Q^i g \otimes (g^{-1}Qg)^{2^i} & +(g^{-1}Qg)^{2^i} \otimes Q^i g & +Q^i(g^{-1}Qg) \otimes g^{2^i}. \\ \text{Dimensions:} & 0, 2^{i+1}-1 & 2^i-1, 2^i & 2^i, 2^i-1 & 2^{i+1}-1, 0. \end{array}$$

For $i \geq 1$, these four terms are all in different dimensions, and each is nonzero, so we see that $Q^i(g^{-1}Qg)$ is not primitive in its component. This formula is verified by induction, using the Cartan formula for Q and the fact that $Q(x^2) = 0$, $x \in H_*(\Omega^2 S^2)$.

Although I won't need it for the proof of Theorem 1, I should explain how to compute the $\mathbf{Z}/2$ cohomology ring of Rat_k from the above formulas for $\psi Q^i(g^{-1}Qg)$. Namely, we need to expand $Q^i g$ (which appears in the formula for $\psi Q^i(g^{-1}Qg)$) as a polynomial in g and the $Q^i(g^{-1}Qg)$, since $H_*(\coprod Rat_k)$ is a polynomial ring in the latter generators. The result is, for $i \geq 1$:

$$Q^i g = \sum_{j=0}^{i-1} g^{2^j} (g^{-1}Qg)^{2^i - 2^{j+1}} Q^j (g^{-1}Qg).$$

As usual, this is easy to check by induction. Plugging this into the formula for $\psi Q^i(g^{-1}Qg)$ gives a complete, although complicated, description of $H_*(\coprod Rat_k) = \mathbf{Z}/2[g, g^{-1}Qg, Q(g^{-1}Qg), \ldots]$ as a Hopf algebra.

We now come to the proof of Theorem 1. We observe that the top-dimensional homology group of Rat_k and $B\beta_{2k}$ is 1-dimensional. Namely, let k have binary expansion $k = \sum_{j \in J} 2^j$; then the top dimension is $H_{2k-|J|}$, where $|J|$ is the order of the set J. For Rat_k, this homology group is spanned by $x = \prod_{j \in J} Q^j(g^{-1}Qg)$; for $B\beta_{2k}$ it is spanned by $y = \prod_{j \in J} Q^{j+1} g$. The idea is to consider the sets $S(x) = \{r \geq 0 : \psi(x)|_{H_r \otimes H_{2k-|J|-r}} \neq 0\}$, and the set $S(y)$ defined the same way for y. If $S(x) \neq S(y)$, then the cohomology rings $H^* Rat_k$ and $H^* B\beta_{2k}$ are not isomorphic.

If $k+1$ is not a power of 2, then there exists an integer $j \geq 1$ such that $j \in J$ but $j - 1 \notin J$; let j be the smallest such integer. (Recall that J is the set of nonzero digits in the binary expansion of k.)

Then one can check that $\psi(x)$ is nonzero in $H_{2^j-1} \otimes H_{\dim(x)-(2^j-1)}$, as follows. One has $\psi(x) = \prod_{l \in J} \psi(Q^l(g^{-1}Qg))$, and we can use the formulas above for $\psi Q^l(g^{-1}Qg)$ to express $\psi(x)$ as a sum of various products of monomials. In this sum there is only one term in $H_{2^j-1} \otimes H_{\dim(x)-(2^j-1)}$, namely

$$(Q^j g) \prod_{\substack{l \in J \\ l \neq j}} g^{2^l} \otimes (g^{-1}Qg)^{2^j} \prod_{\substack{l \in J \\ l \neq j}} Q^l(g^{-1}Qg)$$

And this is not zero, given the description of $H_*\Omega^2S^2$ as a polynomial ring. Since $j-1 \notin J$, there is no other term in this dimension when $\psi(x)$ is expanded. So $\psi(x)$ is not zero in $H_{2^j-1} \otimes H_{\dim(x)-(2^j-1)}$.

Now $\psi(y)$ is easier to write out, and one sees that since $j \geq 1$ and $j-1 \notin J$, $\psi(y)$ is 0 in $H_{2^j-1} \otimes H_{\dim(x)-(2^j-1)}$. Therefore the cohomology rings H^*Rat_k and $H^*B\beta_{2k} \cong H^*C_k$ are not isomorphic, provided that $k+1$ is not a power of 2.

5 Homology of $\Omega^2 S^2$ with $\mathbf{Z}/p$ coefficients, $p \geq 3$

The reference for this section is [2]. Throughout this section we consider homology with $\mathbf{Z}/p$ coefficients, $p \geq 3$.

For C_2-spaces X, in particular 2-fold loop spaces, there is a function $Q : H_qX \rightarrow H_{pq+p-1}X$ defined for q odd, as well as a Pontrjagin product which makes H_*X a graded-commutative ring. In addition, Browder defined an operation

$$[\ ,\ ] : H_iX \otimes H_jX \rightarrow H_{i+j+1}X$$

which makes H^{*-1} a graded Lie algebra. The signs are such that $[x, x] = 0$ for x odd-dimensional, but not necessarily for x even-dimensional. For $X = \Omega^2S^2$, Q and the Browder operation map as follows:

$$Q : H_q\Omega^2_kS^2 \rightarrow H_{pq+p-1}\Omega^2_{pk}S^2$$

$$[\ ,\ ] : H_i\Omega^2_kS^2 \otimes H_j\Omega^2_lS^2 \rightarrow H_{i+j+1}\Omega^2_{k+l}S^2$$

F. Cohen computed $H_*(\coprod B\beta_q)$ and $H_*\Omega^2S^2$ as algebras. The result is as follows. Let g be the generator of $H_0\Omega^2_1S^2$. Let $\lambda = [g, g] \in H_1\Omega^2_2S^2$. The Lie algebra generated by g is spanned by g and λ, since $[\lambda, \lambda] = 0$ by anticommutativity and $[g, \lambda] = 0$ by the Jacobi identity. (For $p = 3$, the Jacobi identity doesn't prove this, but Cohen states explicitly, p. 216, that $[x, [x, x]] = 0$ even when $p = 3$.) Then we have:

$$\begin{aligned} H_*(\coprod B\beta_q) &= \mathbf{Z}/p[g] \quad \otimes E[\lambda, Q\lambda, Q^2\lambda, \ldots] \otimes \mathbf{Z}/p[\beta Q\lambda, \beta Q^2\lambda, \ldots] \\ H_*\Omega^2S^2 &= \mathbf{Z}/p[g, g^{-1}] \quad \otimes E[\lambda, Q\lambda, Q^2\lambda, \ldots] \otimes \mathbf{Z}/p[\beta Q\lambda, \beta Q^2\lambda, \ldots] \end{aligned}$$

Here E denotes the exterior algebra on the given generators and $\mathbf{Z}/p[\ldots]$ represents a polynomial ring. Here g has dimension 0 and weight 1, $Q^i\lambda$, $i \geq 0$ has dimension $2p^i - 1$ and weight $2p^i$, and $\beta Q^i\lambda$, $i \geq 1$ has dimension $2p^i - 2$ and weight $2p^i$. Finally, β denotes the Bockstein operation $\beta : H_iX \rightarrow H_{i-1}X$.

Using the fact that the Browder operation on $H_*\Omega^2S^2$ is 2-step nilpotent, F. Cohen shows ([2], p. 372) that the Dyer-Lashof operation $Q: H_q\Omega^2S^2 \to H_{2q+1}\Omega^2S^2$ is linear, and that the Cartan formula $Q(xy) = x^pQy$ holds with no error terms. (Here x is even-dimensional and y is odd-dimensional.) This indicates that Ω^2S^2 is easier to understand than a general 2-fold loop space.

6 Rat_k, $B\beta_{2k}$, and C_k have isomorphic $\mathbf{Z}/p$-cohomology rings, $p \geq 3$

Let $p \geq 3$. In this section we use the Cartan formula in $H_*\Omega^2S^2$ to compute the precise image of H_*Rat_k in $H_*\Omega^2_kS^2$, and to describe the coalgebra structure on H_*Rat_k. In particular, we will see that Rat_k, $B\beta_{2k}$, and C_k have isomorphic $\mathbf{Z}/p$-cohomology rings, unlike what happens for $p = 2$.

The Cartan formula for Q in $H_*\Omega^2S^2$ says that $Q(xy) = x^pQy$ for x even-dimensional, y odd-dimensional. In particular (and this is the only case we need),

$$\begin{aligned} Q^j(g^{-1}\lambda) &= g^{-p^j}Q^j\lambda \\ \beta Q^j(g^{-1}\lambda) &= g^{-p^j}\beta Q^j\lambda \end{aligned}$$

Now the image of the map $H_0Rat_1 \to H_0\Omega^2_1S^2$ obviously contains g. Also, the map $H_1Rat_1 \to H_1\Omega^2_1S^2$ is not 0, so the image contains the generator of $H_1\Omega^2_1S^2 \cong \mathbf{Z}/p$, namely $g^{-1}\lambda$. Since $\coprod_k Rat_k \to \Omega^2S^2$ is a map of C_2-spaces [1], the image of $H_*(\coprod_k Rat_k) \to H_*\Omega^2S^2$ contains the subring A of $H_*\Omega^2S^2$ generated by g, $Q^i(g^{-1}\lambda)$ for $i \geq 0$, and $\beta Q^i(g^{-1}\lambda)$ for $i \geq 1$. By the previous paragraph, A can also be described as the ring generated by g, $g^{-p^i}Q^i\lambda$, and $g^{-p^i}\beta Q^i\lambda$ for $i \geq 1$.

It is now easy to check that the isomorphism $\cdot g^k : H_*\Omega^2_kS^2 \xrightarrow{\cong} H_*\Omega^2_{2k}S^2$ maps $A^*_k \subset H_*\Omega^2_kS^2$ onto $H_*B\beta_{2k} \subset H_*\Omega^2_{2k}S^2$. We have dim $H_*B\beta_{2k} =$ dim H_*Rat_k by [3]. So

$$\dim H_*B\beta_{2k} = \dim H_*Rat_k \geq \dim A^*_k = \dim H_*B\beta_{2k},$$

so dim $H_*Rat_k =$ dim A^*_k. This implies that the map $H_*(\coprod_k Rat_k) \to H_*\Omega^2S^2$ is injective and that the image is exactly the ring

$$A = \mathbf{Z}/p[g] \otimes E[g^{-p^j}\lambda : j \geq 0] \otimes \mathbf{Z}/p[g^{-p^j}\beta Q^j\lambda : j \geq 1].$$

The isomorphism $\cdot g^k : H_*\Omega^2_k S^2 \xrightarrow{\cong} H_*\Omega^2_{2k}S^2$ maps $H_*Rat_k \subset H_*\Omega^2_k S^2$ onto $H_*B\beta_{2k} \subset H_*\Omega^2_{2k}S^2$. Since $\cdot g^k$ is induced by a map of spaces $\cdot g^k : \Omega^2_k S^2 \to \Omega^2_{2k}S^2$, it preserves the coalgebra structure on homology. So the coalgebra $H_*(Rat_k, \mathbf{Z}/p)$ is isomorphic to the coalgebra $H_*(B\beta_{2k}, \mathbf{Z}/p)$ for $p \geq 3$, as desired.

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