Bott vanishing for Fano threefolds

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A smooth projective variety X satisfies *Bott vanishing* if

$$H^j(X, \Omega^i_X \otimes L) = 0$$

for all $j > 0$, $i \geq 0$, and all ample line bundles L. This is a strong property, useful when it holds. Combined with Riemann-Roch, Bott vanishing gives complete information about the sections of many natural vector bundles on X.

Bott proved Bott vanishing for projective space. An important generalization, by Danilov and Steenbrink, is that every smooth projective toric variety satisfies Bott vanishing; proofs can be found in [5, 12, 32, 18]. The first non-toric Fano variety found to satisfy Bott vanishing is the quintic del Pezzo surface [37]. That paper also analyzes Bott vanishing among some varieties that are not rationally connected, such as K3 surfaces. Generalizing the quintic del Pezzo surface, Torres showed that all GIT quotients of $(\mathbf{P}^1)^n$ by the action of $PGL(2)$ satisfy Bott vanishing [36]. Torres's examples include one new Fano variety (not just a product) in each even dimension.

A Fano variety X that satisfies Bott vanishing is rigid, since $H^1(X, TX) = H^1(X, \Omega^{n-1}_X \otimes K^*_X) = 0$. As a result, there are only finitely many smooth complex Fano varieties in each dimension (up to isomorphism) that satisfy Bott vanishing. We can view them as a generalization of toric Fano varieties; they should have some kind of combinatorial classification.

In this paper, we classify the smooth Fano 3-folds that satisfy Bott vanishing. There are many more than expected.

Theorem 0.1. *Bott vanishing holds for exactly 37 smooth complex Fano 3-folds, up to isomorphism. These consist of the 18 toric Fano 3-folds and 19 others. In Mori-Mukai's numbering, the toric Fano 3-folds are (1.17), (2.33)–(2.36), (3.25)–(3.31), (4.9)–(4.12), (5.2)–(5.3). The non-toric Fano 3-folds that satisfy Bott vanishing are (2.26), (2.30), (3.15)–(3.16), (3.18)–(3.24), (4.3)–(4.8), (5.1), and (6.1).*

Here (6.1) is $\mathbf{P}^1$ times the quintic del Pezzo surface, but the other 18 non-toric examples are new. Bott vanishing fails for the quadric 3-fold, but, surprisingly, it holds for the blow-up of the quadric at a point, (2.30). Likewise, Bott vanishing fails for the flag manifold $W = GL(3)/B$, but it holds for several blow-ups of W such as (3.16). In order to prove Bott vanishing in all cases of Theorem 0.1, we find that the fastest approach is to reduce systematically to calculations on toric varieties. Many of the calculations in characteristic zero were first made by Belmans, Fatighenti, and Tanturri [7, 6], as explained in section 2.

We actually prove Theorem 0.1 in any characteristic not 2, using the recent classification of smooth Fano 3-folds in every characteristic by Tanaka [35, Theorem 1.1]. (Without using Tanaka's result, we classify which of the *known* Fano 3-folds satisfy

Bott vanishing.) We also find the 27 Fano 3-folds for which Bott vanishing persists in characteristic 2. We give meaningful proofs for all the cohomology calculations, although computers are useful for checking that nothing has gone wrong.

Our arguments suggested a conjecture on vanishing. For every projective birational morphism $\pi\colon X \to Y$ of smooth varieties, and every line bundle A on X that is ample over Y, we conjectured that the higher direct image sheaf $R^j\pi_*(\Omega^i_X \otimes A)$ should be zero for all $j > 0$ and $i \geq 0$ (Conjecture 4.1). However, this was disproved by Chuanhao Wei [38, section 3].

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Contents

1 Vanishing theorems

In this section, we recall the known Bott vanishing property for toric varieties, and we prove a variant (Proposition 1.3). The basic result (attributed to Danilov and Steenbrink) is that for a smooth projective toric variety X over a field, we have $H^j(X, \Omega^i_X(L)) = 0$ for $j > 0$, $i \geq 0$, and L an ample line bundle. Fujino proved several generalizations, such as the following [19, Theorem 1.3].

Theorem 1.1. *Let X be a smooth projective toric variety over a field. Let D be a reduced toric divisor in X, and let E be a reduced divisor with $0 \leq E \leq D$. Then*

$$H^j(X, \Omega^i_X(\log D)(A - E)) = 0$$

for $j > 0$, $i \geq 0$, and A an ample line bundle.

Remark 1.2. More generally, Fujino proves Theorem 1.1 even when the toric variety X is singular, with the sheaf $\Omega^i_X(\log D)(A-E)$ replaced by its double dual. Even more generally, the result holds with A the class of an ample Weil divisor rather than an ample line bundle, by the proof of [18, Proposition 3.2].

Here is a related statement that may be new. It is helpful for some later arguments. The case $i=0$ is known [20, p. 68, p. 74].

Proposition 1.3. *Let X be a smooth proper toric variety over a field. Let L be a nef line bundle on X. Then $H^j(X,\Omega^i_X\otimes L)=0$ for $j>i$.*

Unlike Theorem 1.1, Proposition 1.3 does not extend to singular toric varieties using the sheaf of reflexive differentials, $\Omega^{[i]}_X := (\Omega^i_X)^{**}$. For example, Danilov found a complex projective toric variety X with $H^2(X,\Omega^{[1]}_X)\neq 0$ [14, Example 12.12].

Proof. Use the resolution (which applies to any divisor with simple normal crossings in a smooth variety):

$$0\to\Omega^i_X\to\Omega^i_X(\log\partial X)\to\oplus\Omega^{i-1}_{D_j}(\log\partial D_j)\to\cdots\to\oplus O_{D_{j_1\cdots j_i}}(\log\partial D_{j_1\cdots j_i})\to 0.$$

Here each term but the first involves logarithmic differentials with respect to the full toric boundary, and the sums are over all the torus-invariant subvarieties of a given dimension in X. The vector bundles of logarithmic differentials on a toric variety with respect to the full toric boundary are all trivial [20, p. 87]. Tensor this exact sequence with a nef line bundle L. For each toric subvariety D_J of X, we have $H^j(D_J,L)=0$ for all $j>0$ [20, p. 68, p. 74]. Applying that fact to this resolution of $\Omega^i_X\otimes L$, we conclude that $H^j(X,\Omega^i_X\otimes L)=0$ for all $j>i$. $\square$

Next, we state the Kodaira-Akizuki-Nakano (KAN) vanishing theorem [29, Theorem 4.2.3], [15], [1, Theorem A.2]. (Petrov strengthened Theorem 1.4(2) to say that a smooth projective globally F-split variety of any dimension satisfies KAN vanishing [9, Corollary 2.7.6].)

Theorem 1.4. *(1) Let X be a smooth projective variety over a field of characteristic zero. Then*

$$H^j(X,\Omega^i_X\otimes L)=0$$

for all ample line bundles L and all $i+j>\dim(X)$.

(2) Let X be a smooth projective variety over a perfect field k of characteristic $p>0$. If X lifts to $W_2(k)$ and X has dimension $\leq p$, then X satisfies KAN vanishing (as in (1)). Also, if X is globally F-split and X has dimension $\leq p+1$, then X satisfies KAN vanishing.

To check Bott vanishing in positive characteristic for the new cases in this paper, the following lemma is helpful.

Lemma 1.5. *The 18 smooth Fano 3-folds studied in this paper satisfy KAN vanishing in every characteristic.*

Proof. All the known smooth Fano 3-folds in characteristic $p>0$ lift to W_2, which implies KAN vanishing for $p>2$. Moreover, the 18 smooth Fano 3-folds studied in this paper are globally F-split in every characteristic $p>0$, and so they satisfy KAN vanishing even in characteristic 2. Indeed, Mehta and Ramanathan showed that a smooth projective variety X in positive characteristic is globally F-split if there is a section of $-K_X$ whose zero scheme has completion at some point isomorphic to $x_1 \cdots x_n = 0$, where $n = \dim(X)$ [30, Proposition 7].

This condition is easy to check for all 18 of the Fano 3-folds studied in this paper. Indeed, in all cases except (2.26), $-K_X$ can be written as a sum of three basepoint-free line bundles $L_1 + L_2 + L_3$. Then general sections of L_1, L_2, L_3 are smooth. (In characteristic $p > 0$, this is not automatic from Bertini's theorem, but one checks it easily in each case.) One also checks in each case that the resulting three smooth divisors can be taken to meet transversely at some point. So X is globally F-split. For the rigid Fano 3-fold of type (2.26) (section 7), $-K_X$ is the sum of three effective line bundles $(H - E) + H + H$, with each one representable by a smooth divisor in X. These three divisors can be taken to meet transversely at some point. So again X is globally F-split. □

By KAN vanishing, the 18 Fano 3-folds studied in this paper have $H^j(X, TX) = H^j(X, \Omega^2_X \otimes K^*_X) = 0$ for $j > 1$. These 18 Fano 3-folds are also known (in every characteristic not 2) to be rigid, meaning that $H^1(X, TX) = 0$. These results will be part of the proof that these 18 Fano 3-folds satisfy Bott vanishing in every characteristic not 2.

2 Cases where Bott vanishing fails

In this section, we give Talon Stark's proof that Bott vanishing fails for all smooth Fano 3-folds other than the 37 in Theorem 0.1. This will appear in Stark's Ph.D. thesis. We also give the analogous results in positive characteristic. Finally, we explain the related calculations by Belmans, Fatighenti, and Tanturri, and the negative answer by Belmans and Smirnov to a question I asked.

The smooth complex Fano 3-folds were classified into 105 deformation types by Iskovskikh and Mori-Mukai [24, 25, 31]. A standard reference is [26, Tables 12.3-12.6]. The Big Table in [4, section 6] and the web site [6] are also convenient.

If a smooth Fano 3-fold satisfies Bott vanishing, then the vector bundle $TX = \Omega^2_X \otimes K^*_X$ has zero cohomology in positive degrees. In particular, the Euler characteristic $\chi(X, TX)$ must be nonnegative. By Riemann-Roch, this is a characteristic number of X, namely

$$\chi(X, TX) = \frac{1}{2}c_1^3 - \frac{19}{24}c_1c_2 + \frac{1}{2}c_3.$$

The invariants of Fano 3-folds tabulated in [26] are the degree $(-K_X)^3$, the Picard number ρ (equal to the second Betti number b_2), and the Hodge number $h^{2,1}$. We have $c_1c_2 = 24$ (since $\chi(X, O) = c_1c_2/24$), $c_1^3 = (-K_X)^3$, and $c_3 = \chi_{\text{top}}(X) = 2 + 2\rho - 2h^{2,1}$. Therefore, $\chi(X, TX) = \frac{1}{2}(-K_X)^3 - 18 + b_2 - h^{2,1}$. This is negative for 57 of the 105 deformation classes of Fano 3-folds; so those do not satisfy Bott vanishing. The 48 deformation classes of Fano 3-folds with $\chi(X, TX) \geq 0$ are: (1.15)–(1.17), (2.26)–(2.36), (3.13)–(3.31), (4.3)–(4.12), (5.1)–(5.3), (6.1), and (7.1).

Next, some Fano 3-folds are not rigid even though $\chi(X, TX) \geq 0$. Bott vanishing implies that $H^1(X, TX) = 0$, and so it fails in such a case. Let $V_5 \subset \mathbf{P}^6$ denote the quintic del Pezzo 3-fold (which is rigid), a smooth codimension-3 linear section of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbf{P}^9$. First, (2.26) is the blow-up of $V_5 \subset \mathbf{P}^6$ along a line. The Hilbert scheme of lines on V_5 is isomorphic to $\mathbf{P}^2$, and there are two orbits of lines under $\mathrm{Aut}(V_5)$. For special lines, the normal bundle is $O(1) \oplus O(-1)$, while for general lines the normal bundle is $O \oplus O$ [28, Lemma 2.2.6]. Thus, the special blow-up is not rigid and hence does not satisfy Bott vanishing. We will see that the general blow-up, which is rigid, does satisfy Bott vanishing (section 7).

Next, (2.28) is the blow-up of $\mathbf{P}^3$ along a plane cubic curve; that is clearly not rigid, because plane cubics have a 1-dimensional moduli space. Likewise, (3.14) is the blow-up of $\mathbf{P}^3$ at a plane cubic and a disjoint point, and so it is not rigid. Next, (3.13) is the intersection of three divisors in $(\mathbf{P}^2)^3$ of degrees $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$. There is a 1-dimensional moduli space of such 3-folds, described in [4, Lemma 5.19.7], and so they are not rigid. Finally, (7.1) is $\mathbf{P}^1$ times a quartic del Pezzo surface, which is not rigid.

Thus there are 44 rigid Fano 3-folds, up to isomorphism: (1.15)–(1.17), (2.26)–(2.27), (2.29)–(2.36), (3.15)–(3.31), (4.3)–(4.12), (5.1)–(5.3), and (6.1). This information can also be found in [4].

For seven of these, Bott vanishing fails, using an ample line bundle other than $-K_X$. The failure of Bott vanishing was known for the quadric 3-fold (1.16) [12, section 4.1], the flag manifold $W = GL(3)/B$ (2.32) [12, section 4.2], and the quintic del Pezzo 3-fold V_5 (1.15) ([2, Lemma 7.10] or [36, section 7]).

The earlier arguments can be simplified a bit: a Riemann-Roch calculation suffices to disprove Bott vanishing. (The Macaulay2 package Schubert2 is convenient for these calculations [21].) Namely, for the quadric 3-fold Q, we have $-K_X \cong O(3)$ and $\chi(X, \Omega^2(1)) = -1$. For the flag manifold W, a divisor of degree $(1,1)$ in $\mathbf{P}^2 \times \mathbf{P}^2$, we have $-K_W = 2A + 2B$ (with A and B the pullbacks of $O(1)$ from the two $\mathbf{P}^2$'s), and $\chi(W, \Omega^2(A+B)) = -1$. For the quintic del Pezzo 3-fold $X = V_5 \subset \mathbf{P}^6$, we have $-K_X = O(2)$ and $\chi(X, \Omega^2_X(1)) = -3$.

Bott vanishing fails for four other rigid Fano 3-folds, as follows. (2.27) is the blow-up X of $\mathbf{P}^3$ along a twisted cubic curve. Here $-K_X = 4H - E$ (where H is the pullback of $O(1)$ from $\mathbf{P}^3$ and E is the exceptional divisor). But the "smaller" line bundle $L = 3H - E$ is also ample, and $\chi(X, \Omega^2_X \otimes L) = -2 < 0$. (2.29) is the blow-up of the quadric 3-fold Q along a conic. Here $L = 2H - E$ is ample and "smaller" than $-K_X = 3H - E$, and $\chi(X, \Omega^2_X \otimes L) = -2$. (2.31) is the blow-up of Q along a line. Here $L = 2H - E$ is ample and "smaller" than $-K_X = 3H - E$, and $\chi(X, \Omega^2_X \otimes L) = -1$. Finally, (3.17) is a divisor of degree $(1,1,1)$ in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$. The line bundle $L = A + B + C$ is ample and "smaller" than $-K_X = A + B + 2C$, and $\chi(X, \Omega^2_X \otimes L) = -1$.

We will show that the 37 other rigid Fano 3-folds satisfy Bott vanishing. We know this for the 18 toric Fanos (listed in Theorem 0.1). We also know Bott vanishing for (6.1), the product of $\mathbf{P}^1$ with the quintic del Pezzo surface [37, Theorem 2.1, Lemma 2.3]. It remains to prove Bott vanishing for the 18 other Fano 3-folds listed in Theorem 0.1.

We will prove the same results for the Fano 3-folds over an algebraically closed field of any characteristic other than 2. Indeed, by Tanaka, the classification of smooth Fano 3-folds takes essentially the same form in every characteristic [35, Theorem 1.1], [34, Theorem 1.1].

In more detail, Tanaka classifies the Fano 3-folds in every characteristic into 105 classes, numbered as in Mori-Mukai's list, and he gives a geometric description of all varieties in each class. For the 18 classes of Fano 3-folds which are toric in characteristic zero, Tanaka shows that they have the same description in every characteristic, and in particular they are toric in every characteristic [35, section 7]. Likewise, for the 19 classes of non-toric Fanos that satisfy Bott vanishing in characteristic zero, Tanaka gives the same geometric description as in characteristic zero. We will analyze Bott vanishing for these 19 classes in the main part of this paper.

We also need to observe that the Fano 3-folds that do not satisfy Bott vanishing in characteristic zero, as shown earlier in this section, do not satisfy Bott vanishing in any characteristic $p > 0$. A priori, the cone of curves could be bigger in characteristic p; then the ample cone would be smaller, and then it might happen that Bott vanishing holds in characteristic p but not in characteristic zero. In fact, that does not happen. First, in the cases where $H^1(X, TX) \neq 0$ in characteristic zero, we also have $H^1(X, TX) \neq 0$ in characteristic p by semi-continuity, and so Bott vanishing does not hold in characteristic p.

That leaves only the four rigid Fanos for which we disproved Bott vanishing in characteristic zero: (2.27), (2.29), (2.31), and (3.17). In each of these cases, Tanaka gives the same geometric description of X as above, and we read off that the ample line bundle L we used to disprove Bott vanishing remains ample in any characteristic. Then the Euler characteristic calculations above show that these four varieties do not satisfy Bott vanishing in any characteristic.

In characteristic 2, some of the 37 Fano 3-folds that satisfy Bott vanishing in characteristic not 2 have non-reduced automorphism group scheme, essentially because of the distinctive features of conics in characteristic 2. (In particular, the subgroup scheme of $PGL(3)$ that preserves a conic and a general point in $\mathbf{P}^2$ is not reduced, in characteristic 2.) It follows that X is not rigid in such a case, meaning that $H^1(X, TX) \neq 0$. These cases are (2.26), (3.15), (3.18), (3.21), (4.3), (4.4), (4.5), and (5.1). (They are still "set-theoretically rigid" in the sense that they are isomorphic to nearby varieties over an algebraically closed field. This situation is analyzed in [16, Theorem 0.2].) Two others are rigid but do not satisfy Bott vanishing in characteristic 2, (2.30) and (3.19). Our arguments give that the remaining 27 Fano 3-folds do satisfy Bott vanishing in characteristic 2.

The examples above suggest the following question (but see the discussion below):

Question 2.1. Let X be a smooth Fano variety which is rigid, meaning that $H^1(X, TX) = 0$. Is $H^j(X, \Omega^i_X \otimes K^*_X \otimes L) = 0$ for all $j > 0$, $i \geq 0$, and nef line bundles L?

Here rigidity is a necessary assumption, since $H^1(X, TX) = H^1(X, \Omega^{n-1}_X \otimes K^*_X)$. Question 2.1 was suggested by the results in this paper. Namely, in each case (above) where Bott vanishing fails for some rigid Fano 3-fold, it always involves an ample line bundle which is "smaller" than $-K_X$, in particular not of the form $-K_X + L$ with L nef.

In view of the isomorphism $\Lambda^i TX \cong \Omega^{n-i}_X \otimes K^*_X$, Question 2.1 can be rephrased in terms of the groups $H^j(X, \Lambda^i TX \otimes L)$ for L nef. The arguments in this paper work by reducing this problem to the case where L is trivial, that is, computing the cohomology groups of "polyvector fields", $H^j(X, \Lambda^i TX)$. These groups are closely related to the Hochschild cohomology of X. The cohomology of polyvector fields was computed for all smooth Fano 3-folds in characteristic zero by Belmans, Fatighenti, and Tanturri [7, 6].

It turns out that Question 2.1 has a negative answer in general. Namely, Belmans and Smirnov showed that many projective homogeneous varieties $X = G/P$ in characteristic zero have $H^j(X, \Lambda^i TX) \neq 0$ for some $j > 0$ and $i \geq 0$ [8, Proposition D]. (All such varieties are rigid and Fano.) They say that such varieties are not "Hochschild affine". For example, the symplectic Grassmannian $X = \mathrm{SGr}(3, 2n)$ with $n \geq 4$, which has dimension $6(n-2)$, has $H^1(X, \Lambda^2 TX) \neq 0$. This is a more extreme failure of Bott vanishing than what happens for the usual Grassmannians $\mathrm{Gr}(a, b)$ (other than projective space) and for quadrics of dimension at least 3. According to Belmans, there is even a homogeneous variety X with $H^1(X, \Lambda^i TX \otimes L) \neq 0$ for a very ample line bundle L: $X = F_4/P_2$ (in Bourbaki's numbering) which has dimension 20, with $i = 5$ and $L = O_X(1)$.

3 Inductive approach to Bott vanishing

To prove Bott vanishing for a given variety X means proving the vanishing of higher cohomology for the bundles Ω^i_X tensored with all ample line bundles. Intuitively, this should be hardest for the "smallest" ample line bundles on X. In this section, we give a simple procedure for deducing Bott vanishing for "bigger" ample line bundles from smaller ones. The method works well in all our examples.

Lemma 3.1. *Let X be a smooth projective variety over a field. Let D be a smooth divisor in X that satisfies Bott vanishing. (For example, D could be a toric variety.) Let L be a line bundle on X such that $L - D$ and L are ample. If X satisfies Bott vanishing for $L - D$, then it satisfies Bott vanishing for L.*

Proof. The assumption means that $H^j(X, \Omega^i_X \otimes L(-D)) = 0$ for all $j > 0$ and $i \geq 0$. We have an exact sequence of coherent sheaves, $0 \to O_X(-D) \to O_X \to O_D \to 0$. Tensoring with $\Omega^i_X \otimes L$ and taking cohomology, we get an exact sequence

$$H^j(X, \Omega^i_X \otimes L(-D)) \to H^j(X, \Omega^i_X \otimes L) \to H^j(D, \Omega^i_X \otimes L).$$

Let $j > 0$ and $i \geq 0$; then we are given that the first group here is zero. In order to show that $H^j(X, \Omega^i_X \otimes L) = 0$ as we want, it suffices to show that $H^j(D, \Omega^i_X \otimes L) = 0$.

We have an exact sequence $0 \to O_D(-D) \to \Omega^1_X|_D \to \Omega^1_D \to 0$ of vector bundles on D. Taking exterior powers, it follows that $0 \to O_D(-D) \otimes \Omega^{i-1}_D \to \Omega^i_X|_D \to \Omega^i_D \to 0$. So the vanishing we want follows if $H^j(D, \Omega^{i-1}_D \otimes L(-D))$ and $H^j(D, \Omega^i_D \otimes L)$ are zero. Since L and $L(-D)$ are ample, both groups vanish by Bott vanishing on D. ☐

Remark 3.2. Lemma 3.1 simplifies the proof of Bott vanishing for the quintic del Pezzo surface [37, Theorem 2.1]. Namely, this induction reduces the problem to the vanishing of $H^1(X, \Omega^1_X \otimes K^*_X) = H^1(X, TX)$, which holds because X is rigid (or by a direct calculation). We will give more details of the analogous reduction for Fano 3-folds in the rest of the paper.

4 Higher direct images of differential forms

Most Fano 3-folds arise as blow-ups of a simpler variety along a point or a curve. In order to prove Bott vanishing in such a case, we need to analyze the cohomology of bundles of differential forms twisted by a line bundle on a smooth blow-up.

I made the following vanishing conjecture, recently disproved (even in characteristic zero) by Wei [38, section 3]. However, some cases are true and will be useful in what follows, as discussed below.

Conjecture 4.1. *Let $\pi\colon X \to Y$ be a projective birational morphism between smooth varieties over a field, and let A be a line bundle on X that is ample over Y. Then $R^j\pi_*(\Omega^i_X \otimes A) = 0$ for all $j > 0$ and $i \geq 0$.*

For $i + j > n = \dim(X)$ (KAN-type vanishing), the conjecture holds in characteristic zero with no assumptions on the singularities of Y [3, Corollary 2.1.2]. For $i = 1$ and $j = n - 1$, the Steenbrink-type vanishing theorem of [22, Theorem 14.1] is a similar statement.

The main evidence for Conjecture 4.1 was that it holds for toric morphisms, by Fujino (Theorem 4.2). (Toric morphisms need not be birational, but our main interest here is in the birational case.) Since Conjecture 4.1 can be checked after completing the base at a point, the case of toric morphisms implies the conjecture for the blow-up of any smooth subvariety along a smooth subvariety (Corollary 4.4). Theorem 4.2 implies the conjecture for some iterated blow-ups as well. Such iterated blow-ups occur in several examples where we check Bott vanishing, including the hardest case, the Fano 3-fold (5.1) (section 11).

Theorem 4.2. *(Fujino [18, Theorem 5.2]) Let $\pi\colon X \to Y$ be a projective toric morphism over a field k, with X smooth over k. Let A be a line bundle on X that is ample over Y. Then $R^j\pi_*(\Omega^i_X \otimes A) = 0$ for all $j > 0$ and $i \geq 0$.*

Remark 4.3. A further generalization is that Theorem 4.2 holds even when the toric variety X is singular, with Ω^i_X replaced by the sheaf of reflexive differentials. Moreover, for X singular, we can allow A to be an ample Weil divisor, replacing $\Omega^i_X \otimes A$ in the statement by the reflexive sheaf $(\Omega^i_X \otimes O(A))^{**}$. For projective toric varieties, Fujino proved Bott vanishing in this generality [18, Proposition 3.2].

Corollary 4.4. *Let Y be a smooth variety over a field k, S a smooth subvariety of Y, and $\pi\colon X \to Y$ the blow-up along S. Let E be the exceptional divisor in X, and let m be a positive integer. Then the higher direct image sheaves $R^j\pi_*(\Omega^i_X(-mE))$ are zero for all $j > 0$, $i \geq 0$, and $m > 0$.*

Proof. It suffices to prove this after passing to the algebraic closure of k and completing Y at a point. So we can assume that S is a linear subspace of an affine space Y. In that case, the blow-up $X \to Y$ is a toric morphism. Also, the line bundle $O(-E)$ on X is ample over Y. So Theorem 4.2 implies that $R^j\pi_*(\Omega^i_X(-mE)) = 0$ for all $j > 0$, $i \geq 0$, and $m > 0$. $\square$

For applications, we also want to compute $\pi_*(\Omega^i_X(-E))$. We first consider the blow-up at a point. Proposition 6.1 will extend this to the blow-up along a higher-dimensional smooth subvariety.

Proposition 4.5. *Let Y be a smooth variety of dimension n over a field k, p a k-point of Y, and $\pi\colon X \to Y$ the blow-up at p. Let E be the exceptional divisor in X. Then $\pi_*(\Omega^1_X(-E))$ is the subsheaf $\Omega^1_Y \otimes I_{p/Y}$ of Ω^1_Y. For $i \geq 2$, $\pi_*(\Omega^i_X(-E))$ is equal to Ω^i_Y.*

It would be interesting to compute explicitly the filtration of each vector bundle Ω^i_Y by the subsheaves $\pi_*(\Omega^i_X(-mE))$ for $m \geq 0$. That is, filter the differential forms on Y by their order of vanishing along $E \subset X$. We will need only a few cases, Propositions 4.5 and 6.1.

Proof. (Proposition 4.5) We first show that $\pi_*(\Omega^1_X(-E))$ is the subsheaf $\Omega^1_Y \otimes I_{p/Y} \subset \Omega^1_Y$. In other words, we want to show that a 1-form on a neighborhood of p in Y vanishes at p if and only if its pullback to X vanishes as a section of Ω^1_X on E. Clearly, if a 1-form vanishes at p, then its pullback vanishes along E. For the converse, let $x_1, \ldots, x_n$ be regular functions near p that form a basis for $\mathfrak{m}_p/\mathfrak{m}_p^2$. On one affine chart of the blow-up X (which is enough to consider), the morphism $\pi\colon X \to Y$ is given by $(x_1, u_2, \ldots, u_n) \mapsto (x_1, x_1u_2, \ldots, x_1u_n)$. So dx_1 pulls back to dx_1 and dx_i for $2 \leq i \leq n$ pulls back to $x_1du_i + u_idx_1$. Restricting to sections of Ω^1_X on the exceptional divisor $E = \{x_1 = 0\}$, these sections become $dx_1, u_2dx_1, \ldots, u_ndx_1$. Since these are linearly independent over k, the 1-forms on Y whose pullback vanishes along E are only those that vanish at p.

Next, for $i \geq 2$, let us show that the subsheaf $\pi_*(\Omega^i_X(-E))$ of Ω^i_Y is equal to Ω^i_Y. That is, we want to show that the pullback of every i-form on Y vanishes as a section of Ω^i_X on E. This follows from the previous paragraph's calculation: the 1-forms $dx_1, \ldots, dx_n$ pull back to $dx_1, u_2dx_1, \ldots, u_ndx_1$ as sections of Ω^1_X on E, and any wedge product of $i \geq 2$ such 1-forms vanishes on E (since $dx_1 \wedge dx_1 = 0$). ☐

5 The Fano 3-folds (2.30) and (3.19)

We now check Bott vanishing for the first new cases, (2.30) and (3.19). These are the blow-up of the quadric 3-fold at one point, or at two non-collinear points. Part of the argument involves reducing to properties of toric varieties, although other parts are special to the quadric. Most of the later cases will reduce completely to results on toric varieties.

The proof of Bott vanishing for (2.30) and (3.19) works in characteristic not 2. Bott vanishing actually fails for these two varieties in characteristic 2.

For each smooth complex Fano 3-fold, Coates, Corti, Galkin, and Kasprzyk determined the nef cone [13]. For the Fano 3-folds in this paper, we will compute the nef cone again. One reason is to make sure that the proofs work in any characteristic. Another reason is that it is convenient for our arguments to find explicit generators for the cone of curves.

Consider case (2.30) first. Here X is the blow-up of the smooth quadric 3-fold Q at a point p. (It is also the blow-up of $\mathbf{P}^3$ along a conic.) The Picard group of X is $\mathbf{Z}\{H, E\}$, where H denotes the pullback of $O(1)$ from Q and E is the exceptional divisor. We have $-K_X = 3H - 2E$. Let C be a line in $E \cong \mathbf{P}^2$. Let D be the strict transform of a line on Q through p. We have the intersection numbers:

	C	D
H	0	1
E	-1	1

So the dual basis to C, D is given by $H-E, H$. Here $H-E$ and H are basepoint-free, hence nef, giving contractions of X to $\mathbf{P}^3$ and Q. (Sections of the line bundle $H-E$ on X correspond to sections of H on Q that vanish at the point p. Basepoint-freeness of $H-E$ on X follows from the fact that p is cut out by sections of H, as a subscheme of Q.) It follows that the closed cone of curves $\overline{\mathrm{Curv}(X)}$ is $\mathbf{R}^{\geq 0}\{C, D\}$, and the nef cone is spanned by $H-E$ and H. More strongly, the monoid of nef classes in $\mathrm{Pic}(X)$ is generated by H and $H-E$.

Since $(2H-E)\cdot C = (2H-E)\cdot D = 1$, the line bundle $2H-E$ is ample, and every ample line bundle is $2H-E$ plus a nef divisor, hence $2H-E$ plus an $\mathbf{N}$-linear combination of H and $H-E$. A general divisor in the linear system $|H|$ on X is a quadric surface $\mathbf{P}^1\times\mathbf{P}^1$, and a general divisor in $|H-E|$ is also a quadric surface. Since these are both toric, Lemma 3.1 reduces Bott vanishing to the single ample line bundle $2H-E$.

The vanishing of $H^j(X,\Omega^i_X\otimes L)$ is clear whenever X is a Fano 3-fold, $j>0$, L is ample, and i is 0 or 3. Indeed, the case $i=3$ is Kodaira vanishing (part of KAN vanishing, Lemma 1.5). The case $i=0$ follows from Kodaira vanishing since $-K_X$ is ample. So, throughout this paper, we only need to consider Bott vanishing with i equal to 1 or 2.

For $i=2$, Proposition 4.5 gives that

$$\begin{aligned} H^j(X,\Omega^2_X\otimes O(2H-E)) &= H^j(Q,\pi_*(\Omega^2_X(-E))\otimes O(2))\\ &= H^j(Q,\Omega^2_Q(2)). \end{aligned}$$

For k of characteristic 2, Macaulay2 shows that $h^1(Q,\Omega^2_Q(2))=1$, and so $h^1(X,\Omega^2_X\otimes O(2H-E))=1$. It follows that Bott vanishing fails for (2.30) in characteristic 2.

Let us analyze the tangent bundle of the quadric $Q\subset\mathbf{P}^4$. Here Q is defined by a quadratic form q on k^5, which we can assume is $x_0x_2+x_1x_3+x_4^2$. Since we assume that the characteristic of the base field k is not 2, the associated bilinear form on k^5 is nondegenerate. Think of Q as the space of isotropic lines L in k^5. By the Euler sequence, the tangent bundle of $\mathbf{P}^4$ at a point $[L]$ has $T\mathbf{P}^4(-1)\cong k^5/L$. It follows that $TQ(-1)$ at a point $[L]$ in Q is $L^\perp/L$. As a result, there is a canonical nondegenerate symmetric bilinear form on $TQ(-1)$, $TQ(-1)\otimes TQ(-1)\to O_Q$. So $\Omega^1_Q\cong TQ(-2)$. We also have $TQ\cong\Omega^2_Q\otimes K_Q^*=\Omega^2_Q(3)$.

The vector bundle $\Omega^2_Q(2)=TQ(-1)$ has zero cohomology in all degrees. To check this by hand, first note that $L:=O_Q(-1)$ has zero cohomology in all degrees, by reducing to the known cohomology of line bundles on $\mathbf{P}^4$. Then observe that $L^\perp\subset O_Q^{\oplus 5}$ has zero cohomology in all degrees, using that $O_Q^{\oplus 5}/L^\perp$ is isomorphic to $L^*=O_Q(1)$. So $\Omega^2_Q(2)=TQ(-1)=L^\perp/L$ has zero cohomology in all degrees, as we want. We remark that $h^1(Q,\Omega^2_Q(1))=1$, which checks again that Q itself does not satisfy Bott vanishing.

For $i=1$, Proposition 4.5 gives that

$$H^j(X,\Omega^1_X\otimes O(2H-E)) = H^j(Q,\pi_*(\Omega^1_X(-E))\otimes O(2)).$$

The Proposition also gives a short exact sequence

$$0\to\pi_*(\Omega^1_X(-E))\to\Omega^1_Q\to\Omega^1_Q|_p\to 0.$$

So Bott vanishing for $i=1$ and the ample line bundle $2H-E$ on X follows if $H^j(Q,\Omega^1_Q(2))=0$ for $j>0$ and $H^0(Q,\Omega^1_Q(2))$ maps onto its fiber at p. By the isomorphism $\Omega^1_Q \cong TQ(-2)$ above, we can rephrase the problem in terms of the tangent bundle, which seems easier to visualize. Namely, we want to show that $H^j(Q,TQ) = 0$ for $j > 0$ and that TQ is spanned at p by its global sections.

The fact that $H^j(Q,TQ) = 0$ for $j > 0$ is immediate from Q being a rigid Fano variety. Namely, rigidity means that $H^1(Q,TQ) = 0$, and KAN vanishing implies that $H^j(Q,TQ) = 0$ for $j \geq 2$. (KAN vanishing holds for the quadric 3-fold Q in any characteristic, by the same argument as in Lemma 1.5.) The fact that TQ is spanned at p by global sections follows in characteristic zero from the homogeneity of the quadric Q. In any characteristic, one can check by hand that the subspace of the Lie algebra $\mathfrak{so}(5)$ that fixes the point p in Q has codimension 3, which proves that the map $\mathfrak{so}(5) \to H^0(Q,TQ)$ maps onto the fiber of Q at p. Explicitly, for the quadratic form $q = x_0x_2 + x_1x_3 + x_4^2$ over any field, $\mathfrak{so}(5)$ is the space of 5×5 matrices

$$\begin{pmatrix} X_1 & X_2 & b_1 \\ X_3 & X_4 & b_2 \\ a_1 & a_2 & 0 \end{pmatrix}$$

with each X_i a 2×2 matrix such that $X_4 = -X_1^t$, X_2 and X_3 are skew-symmetric with zeros on the diagonal, $b_1 = -2a_2^t$, and $b_2 = -2a_1^t$ [10, sections 23.4, 23.6]. We read off that the stabilizer of $p = [1,0,0,0,0]$ in Q has codimension 3, in any characteristic. That completes the proof (in characteristic not 2) of Bott vanishing for (2.30), the blow-up of Q at a point.

We can also handle (3.19), the blow-up X of the quadric 3-fold Q at two non-collinear points p_1 and p_2. Again, Bott vanishing fails for X in characteristic 2, and so we assume that the base field k has characteristic not 2. Let E_1 and E_2 be the exceptional divisors over p_1 and p_2. We have $\mathrm{Pic}(X) = \mathbf{Z}\{H, E_1, E_2\}$. Since $-K_Q = 3H$, we have $-K_W = 3H - 2E_1 - 2E_2$.

Let $C_1 \subset E_1 \cong \mathbf{P}^2$ and $C_2 \subset E_2 \cong \mathbf{P}^2$ be lines. Let G_1 be the strict transform of a line in Q through p_1, and G_2 the strict transform of a line in Q through p_2. We have the intersection numbers:

	C_1	C_2	G_1	G_2
H	0	0	1	1
E_1	-1	0	1	0
E_2	0	-1	0	1

It follows that the dual cone to $\mathbf{R}^{\geq 0}\{C_1, C_2, G_1, G_2\}$ is spanned by H, $H - E_1$, $H - E_2$, and $H - E_1 - E_2$. (This calculation is immediate for Magma [11].) These four divisors are basepoint-free, hence nef, giving contractions of X to Q, $\mathbf{P}^3$, $\mathbf{P}^3$, and $\mathbf{P}^2$. It follows that $\overline{\mathrm{Curv}(X)} = \mathbf{R}^{\geq 0}\{C_1, C_2, G_1, G_2\}$, and the nef cone is spanned by H, $H - E_1$, $H - E_2$, and $H - E_1 - E_2$. This is the first non-simplicial cone we have encountered. Using Magma, we check that the nef monoid in $\mathrm{Pic}(X)$ is also generated by those four divisors.

Let $M = 2H - E_1 - E_2$; then M has degree 1 on C_1, C_2, G_1, and G_2. So M is ample, and every ample line bundle is M plus a nef divisor, hence M plus an $\mathbf{N}$-linear combination of H, $H - E_1$, $H - E_2$, and $H - E_1 - E_2$.

A general divisor in $|H|$ is a quadric surface $\mathbf{P}^1 \times \mathbf{P}^1$, which is toric. A general divisor in $|H - E_1|$ or $|H - E_2|$ is a quadric surface blown up at a point, which is also toric. A general divisor in $|H - E_1 - E_2|$ is a quadric surface blown up at 2 non-collinear points, which is also toric. By Lemma 3.1, this reduces Bott vanishing on X to the ample line bundle M.

That is, we want to show that $H^j(X,\Omega^1_X\otimes M)$ and $H^j(X,\Omega^2_X\otimes M)$ are zero for $j>0$. By Proposition 4.5, for Ω^2_X, it is equivalent to show that $H^j(Q,\Omega^2_Q(2))=0$ for $j>0$. We already showed this in characteristic not 2, in analyzing (2.30) above. As mentioned there, $h^1(Q,\Omega^2_Q(2))=1$ in characteristic 2, and so Bott vanishing actually fails for (3.19) in characteristic 2.

For Ω^1_X, Proposition 4.5 gives that $R\pi_*(\Omega^1_X\otimes M)\cong\Omega^1_Q(2)\otimes I_{p_1\cup p_2/Q}$. So we have an exact sequence

$$H^0(Q,\Omega^1_Q(2))\to\Omega^1_Q|_{p_1}\oplus\Omega^1_Q|_{p_2}\to H^1(X,\Omega^1_X\otimes L)\to H^1(Q,\Omega^1_Q(2)).$$

So it suffices to show that $\Omega^1_Q(2)$ has zero cohomology in positive degrees, and that its global sections map onto the sum of its fibers at p_1 and p_2. As shown above, $\Omega^1_Q(2)$ is isomorphic to the tangent bundle of Q. Here $H^j(Q,TQ)=0$ for $j>0$ because Q is a rigid Fano, as discussed above. It remains to show that the global sections of the tangent bundle of Q map onto the direct sum of its fibers at p_1 and p_2.

In characteristic zero, this holds because the group $O(5)$ over $\mathbf{C}$, acting on the quadric $Q\subset\mathbf{P}^4$, acts transitively on pairs of non-collinear points. Indeed, in terms of the given quadratic form on $\mathbf{C}^5$, such a pair corresponds to a hyperbolic plane in $\mathbf{C}^5$, and the Witt Extension Theorem gives that $O(5)$ acts transitively on the set of hyperbolic planes in $\mathbf{C}^5$ [17, Theorem 8.3]. In any characteristic, we can assume that $q=x_0x_2+x_1x_3+x_4^2$, $p_1=[1,0,0,0,0]$, and $p_2=[0,0,1,0,0]$. By the description of the Lie algebra $\mathfrak{so}(5)$ above, the subspace that fixes p_1 and p_2 in Q has codimension $3+3=6$, in any characteristic. That is, the map $\mathfrak{so}(5)\to H^0(Q,TQ)$ maps onto the direct sum of its fibers at p_1 and p_2. That completes the proof of Bott vanishing (in characteristic not 2) for (3.19).

6 Higher direct images of differential forms, continued

We now compute the higher direct images of differential forms twisted by a line bundle for a blow-up along a smooth subvariety (not just a point). We will use this to check Bott vanishing for the remaining Fano 3-folds, since each of them is the blow-up of a simpler variety along a curve.

Proposition 6.1. *Let Y be a smooth variety over a field k, F a smooth subvariety of Y, and $\pi\colon X\to Y$ the blow-up along F. Let E be the exceptional divisor in X, and let m be a positive integer. Then the higher direct image sheaves $R^j\pi_*(\Omega^i_X(-mE))$ are zero for all $j>0$ and $i\geq 0$. For $j=0$ and $m=1$, $\pi_*(\Omega^i_X(-E))$ is the subsheaf of Ω^i_Y whose sections restricted to $\Omega^i_Y|_F$ lie in the image of the product map $\Omega^{i-2}_Y|_F\otimes_{O_F}\Lambda^2N^*_{F/Y}\to\Omega^i_Y|_F$.*

To describe the subsheaf $\pi_*(\Omega^i_X(-E))$ in more detail: the vector bundle $\Omega^i_Y|_F$ is filtered with quotients Ω^i_F (on top), then $\Omega^{i-1}_F\otimes N^*_{F/Y}$, then $\Omega^{i-2}_F\otimes\Lambda^2N^*_{F/Y}$, and so on. The quotient sheaf $\Omega^i_Y/\pi_*(\Omega^i_X(-E))$ is a vector bundle on F, consisting of the top two steps of the filtration. That is, it is an extension of Ω^i_F by $\Omega^{i-1}_F\otimes N^*_{F/Y}$.

Proof. The line bundle $O(-E)$ on X is ample over Y. So Corollary 4.4 gives that $R^j\pi_*(\Omega^i_X(-mE))$ is zero for all $j>0$, $i\geq 0$, and $m>0$.

Next, we want to describe the subsheaf $\pi_*(\Omega^i_X(-E))$ of Ω^i_Y. That is, which i-forms on Y pull back to i-forms on X that vanish (as sections of Ω^i_X) on E? We want to show that the answer is the subsheaf of Ω^i_Y of i-forms whose restriction to $\Omega^i_Y|_F$ lies in the image of $\Omega^{i-2}_Y|_F \otimes \Lambda^2 N^*_{F/Y} \to \Omega^i_Y|_F$. By replacing k by its algebraic closure and completing in the neighborhood of a point, it suffices to check this statement when $F = A^b \times \{0\}$ and $X = A^b \times A^{n-b}$, as above. Let $x_1, \ldots, x_{n-b}$ be coordinates on A^{n-b} and $y_1, \ldots, y_b$ coordinates on F.

Clearly i-forms on Y that vanish in $\Omega^i_Y|_F$ pull back to i-forms that vanish in $\Omega^i_X|_E$. So it suffices to consider i-forms on Y that are O_F-linear combinations of wedge products of dx_i's and dy_j's. To see whether such a form pulls back to one that vanishes on E, we can work in a single affine chart of the blow-up of Y along F, as in the proof of Proposition 4.5. The blow-up map $\pi\colon X \to Y$ is given in this chart by $(x_1, u_2, \ldots, u_{n-b}, y_1, \ldots, y_b) \to (x_1, x_1u_2, \ldots, x_1u_{n-b}, y_1, \ldots, y_b)$. So, pulling back and restricting to $\Omega^i_X|_E$, dx_1 maps to dx_1, dx_i maps to $u_i dx_1$ for $1 \leq i \leq n-b$, and dy_i pulls back to dy_i. It follows that the i-forms that pull back to zero in $\Omega^i_X|_E$ are spanned by those with at least two dx_i factors. That is the statement we want. $\square$

Let us spell out what Proposition 6.1 says in the main case used in this paper, the blow-up of a 3-fold along a curve.

Corollary 6.2. *Let $\pi\colon X \to Y$ be the blow-up of a smooth 3-fold along a smooth curve F over a field. Then $R^j\pi_*(\Omega^i_X(-mE)) = 0$ for $j > 0$, $i \geq 0$, and $m > 0$. Also, $\pi_*(\Omega^1_X(-E))$ is the kernel of the surjection $\Omega^1_Y \to \Omega^1_Y|_F$, and $\pi_*(\Omega^2_X(-E))$ is the kernel of the surjection $\Omega^2_Y \to \Omega^1_F \otimes N^*_{F/Y}$.*

To explain the last statement of Corollary 6.2: Proposition 6.1 says that $\pi_*(\Omega^2_X(-E))$ is the subsheaf of Ω^2_Y whose sections restricted to $\Omega^2_Y|_F$ lie in the image of $\Lambda^2 N^*_{F/Y} \to \Omega^2_Y|_F$. Equivalently, the quotient sheaf $\Omega^2_Y/\pi_*(\Omega^2_X(-E))$ is an extension of Ω^2_F by $\Omega^1_F \otimes N^*_{F/Y}$. Since F is a curve, Ω^2_F is zero, which gives the statement in Corollary 6.2.

The following lemma works very efficiently to prove the base case of Bott vanishing in most of our examples.

Lemma 6.3. *Let X be the blow-up of a smooth projective toric variety Y along a smooth codimension-2 subvariety F that is a complete intersection $S_1 \cap S_2$ in Y. Let E be the exceptional divisor on X.*

1. *Suppose that $-K_Y$, $-K_Y - S_1$, and $-K_Y - S_2$ are ample, and $-K_Y - S_1 - S_2$ is nef. Then $H^j(X, \Omega^1_X(-K_X)) = 0$ for $j > 0$.*
2. *Let L be a line bundle on Y such that L, $L - S_1$, and $L - S_2$ are ample, and $L - S_1 - S_2$ is nef. Then $H^j(X, \Omega^1_X(\pi^*(L) - E)) = 0$ for $j > 0$.*
3. *Let L be a line bundle on Y such that L and $L - S_1$ are ample. Suppose that S_1 is a toric variety (not necessarily torically embedded in Y), $(L - S_2)|_{S_1}$ is ample, and $(L - S_1 - S_2)|_{S_1}$ is nef. Then $H^j(X, \Omega^1_X(\pi^*(L) - E)) = 0$ for $j > 0$.*

Proof. We have $-K_X = \pi^*(-K_Y) - E$, and so part 1 follows from part 2. Let us prove part 2. By Corollary 6.2, it suffices to show that (1) $H^j(Y, \Omega^1_Y(L)) = 0$ for

$j > 0$; (2) $H^0(Y, \Omega^1_Y(L)) \to H^0(F, \Omega^1_Y(L))$ is surjective; and (3) $H^j(F, \Omega^1_Y(L)) = 0$ for $j > 0$.

First, we know (1) by Bott vanishing on the toric variety Y. Next, consider the exact sequence $H^j(Y, \Omega^1_Y(L)) \to H^j(S_1, \Omega^1_Y(L)) \to H^{j+1}(Y, \Omega^1_Y(L - S_1))$. By Bott vanishing on Y, we know that $H^j(Y, \Omega^1_Y(L)) = 0$ and $H^j(Y, \Omega^1_Y(L - S_1)) = 0$ for $j > 0$. So $H^j(S_1, \Omega^1_Y(L)) = 0$ for $j > 0$ and $H^0(Y, \Omega^1_Y(L)) \to H^0(S_1, \Omega^1_Y(L))$ is surjective. Next, consider the exact sequence $H^j(S_1, \Omega^1_Y(L)) \to H^j(F, \Omega^1_Y(L)) \to H^{j+1}(S_1, \Omega^1_Y(L - S_2))$. Statements (2) and (3) follow if we can show that $H^j(S_1, \Omega^1_Y(L - S_2)) = 0$ for $j > 0$.

To do that, consider the exact sequence $H^j(Y, \Omega^1_Y(L - S_2)) \to H^j(S_1, \Omega^1_Y(L - S_2)) \to H^{j+1}(Y, \Omega^1_Y(L - S_1 - S_2))$. By Bott vanishing on Y, we know that $H^j(Y, \Omega^1_Y(L - S_2)) = 0$ for $j > 0$. Since $L - S_1 - S_2$ is nef on the toric variety Y, Proposition 1.3 gives that $H^j(Y, \Omega^1_Y(L - S_1 - S_2)) = 0$ for $j > 1$. It follows that $H^j(S_1, \Omega^1_Y(L - S_2)) = 0$ for $j > 0$.

To prove part 3, we reduce as before to showing that $H^j(S_1, \Omega^1_Y(L - S_2)) = 0$ for $j > 0$. Consider the exact sequence $0 \to O_{S_1}(-S_1) \to \Omega^1_Y|_{S_1} \to \Omega^1_{S_1} \to 0$. Tensoring with $L - S_2$ and taking cohomology, we have an exact sequence $H^j(S_1, L - S_1 - S_2) \to H^j(S_1, \Omega^1_Y(L - S_2)) \to H^j(S_1, \Omega^1_{S_1}(L - S_2))$. Since S_1 is a toric variety, the first group is zero for $j > 0$ since $(L - S_1 - S_2)|_{S_1}$ is nef (Proposition 1.3). The last group is zero for $j > 0$ since $(L - S_2)|_{S_1}$ is ample. It follows that $H^j(S_1, \Omega^1_Y(L - S_2)) = 0$ for $j > 0$, which completes the proof. $\square$

7 First blow-up along a curve: (2.26)

Most Fano 3-folds are blow-ups of simpler varieties along a smooth curve. We now prove Bott vanishing in one such case. Although part of the method will apply to later examples, this case is relatively far from toric varieties and hence requires individual handling.

In case (2.26), X is the blow-up of the quintic del Pezzo threefold $Y := V_5 \subset \mathbf{P}^6$ along a general line. We will instead use another description, mentioned by Mori-Mukai [31, Table 3]: X is the blow-up of the quadric 3-fold Q along a twisted cubic curve F. (It is easier to work with Q than with V_5.) We assume that F is general, in the sense that the $\mathbf{P}^3$ it spans is transverse to $Q \subset \mathbf{P}^4$; otherwise, Bott vanishing fails, as discussed in section 2. We work in characteristic not 2, since X is not rigid (hence does not satisfy Bott vanishing) in characteristic 2.

We have $\text{Pic}(X) = \{H, E\}$, and $-K_X = 3H - E$. Let $C \cong \mathbf{P}^1$ be a fiber of the exceptional divisor $E \to F$. Let D be the strict transform of a line in Q that meets F in two points. We have the intersection numbers:

	C	D
H	0	1
E	-1	2

So the dual basis to C, D is given by $2H - E, H$. The line bundles $2H - E$ and H are basepoint-free, hence nef, corresponding to contractions of X to $V_5 \subset \mathbf{P}^6$ and to Q. It follows that $\overline{\text{Curv}(X)} = \mathbf{R}^{\geq 0}\{C, D\}$, and the nef cone is spanned by H and $2H - E$. More strongly, the nef monoid in $\text{Pic}(X)$ is generated by H and $2H - E$. Since $-K_X \cdot C = 1$ and $-K_X \cdot D = 1$, every ample line bundle on X is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of H and $2H - E$.

A general divisor in $|H|$ on X is the blow-up of a quadric surface $\mathbf{P}^1\times\mathbf{P}^1$ at three points with no three collinear; so it is isomorphic to the quintic del Pezzo surface. That is not toric, but it satisfies Bott vanishing. A general divisor in $|2H-E|$ is a quartic del Pezzo surface, which does not satisfy Bott vanishing. However, $|H-E|$ consists of one smooth quadric surface, which is toric and hence satisfies Bott vanishing. Since $2H-E=H+(H-E)$, Lemma 3.1 along with the previous paragraph reduces Bott vanishing to the single line bundle $-K_X$.

It is easy to show that $H^j(X,\Omega^2_X\otimes K_X^*)=H^j(X,TX)$ vanishes for $j>0$. First, this is zero for $j\geq 2$ by Kodaira-Akizuki-Nakano (KAN) vanishing (Lemma 1.5). That shows in general that deformations of smooth Fano varieties are unobstructed. The fact that $H^1(X,TX)=0$ in this case amounts to the known rigidity of this Fano 3-fold.

It remains to show that $H^j(X,\Omega^1_X\otimes K_X^*)=0$ for $j>0$. Corollary 6.2 gives an exact sequence

$$H^{j-1}(F,\Omega^1_Q(-K_Q))\to H^j(X,\Omega^1_X(-K_X))\to H^j(Q,\Omega^1_Q(-K_Q))\to H^j(F,\Omega^1_Q(-K_Q)).$$

So the desired vanishing would follow if (1) $H^j(Q,\Omega^1_Q(-K_Q))=0$ for $j>0$, (2) $H^0(Q,\Omega^1_Q(-K_Q))\to H^0(F,\Omega^1_Q(-K_Q))$ is surjective, and (3) $H^1(F,\Omega^1_Q(-K_Q))=0$. By section 5, Ω^1_Q is isomorphic to $TQ(-2)$ and $-K_Q$ is $O(3)$; so we can restate (1)–(3) in terms of the vector bundle $TQ(1)$.

Let S be the intersection of $Q\subset\mathbf{P}^4$ with the $\mathbf{P}^3$ spanned by F; so S is a smooth quadric surface, isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$. Write A and B for the two pullbacks of $O(1)$ to S. Then F is linearly equivalent to $A+2B$ or $2A+B$ on S; without loss of generality, we can assume that $F\sim 2A+B$ on S. Although F is not a complete intersection in Q, we can work with the chain $F\subset S\subset Q$, as follows.

First, we have an exact sequence $0\to TS\to TQ|_S\to O_S(A+B)\to 0$ on S, and $TS\cong O(2A)\oplus O(2B)$ since $S=\mathbf{P}^1\times\mathbf{P}^1$. Since $O(1)$ restricted to S is $O(A+B)$, we read off that $H^j(S,TQ(1))=0$ for $j>0$. Since Q is a rigid Fano variety, we have $H^j(Q,TQ)=0$ for $j>0$. Then the exact sequence

$$H^j(Q,TQ)\to H^j(Q,TQ(1))\to H^j(S,TQ(1))\to H^{j+1}(Q,TQ)$$

implies that $H^j(Q,TQ(1))=0$ for $j>0$, which is statement (1). Since $H^1(Q,TQ)=0$, the sequence also gives that $H^0(Q,TQ(1))\to H^0(S,TQ(1))$ is surjective.

Next, since $O_S(1)\sim A+B$ and $F\sim 2A+B$ on S, we have an exact sequence:

$$H^j(S,TQ(1))\to H^j(F,TQ(1))\to H^{j+1}(S,TQ|_S(-A)).\qquad (*)$$

By the previous paragraph's description of $TQ|_S$, we have an exact sequence $0\to O(A)\oplus O(2B-A)\to TQ|_S(-A)\to O(B)\to 0$ on S. By Kodaira vanishing (or just the Künneth formula) on $S=\mathbf{P}^1\times\mathbf{P}^1$, we read off that $H^j(S,TQ|_S(-A))=0$ for $j>0$. So the exact sequence (*) gives that $H^1(F,TQ(1))=0$ (statement (3)) and also that $H^0(S,TQ(1))\to H^0(F,TQ(1))$ is surjective. Together with the previous paragraph, this proves (2). Thus we have shown that $H^j(X,\Omega^1_X\otimes K_X^*)=0$ for $j>0$. This completes the proof of Bott vanishing for (2.26).

8 The Fano 3-fold (3.24)

We now prove Bott vanishing for the Fano 3-fold (3.24). Here X is the blow-up of the flag manifold W, a smooth divisor of degree $(1,1)$ in $\mathbf{P}^2\times\mathbf{P}^2$, along a fiber of the first projection. We will instead use a different description, mentioned by Mori-Mukai [31, Table 3]. Namely, X is the blow-up of $Y = \mathbf{P}^1 \times \mathbf{P}^2$ along a curve F of degree $(1,1)$. The advantage of this description is that it expresses X as the blow-up of a toric variety. The proof works in any characteristic.

Writing f_1 and f_2 for the projections to $\mathbf{P}^1$ and $\mathbf{P}^2$, let $A = f_1^*O(1)$ and $B = f_2^*O(1)$. Let E be the exceptional divisor in X. Then $\text{Pic}(X) = \mathbf{Z}\{A, B, E\}$ and $-K_X = 2A+3B-E$. Let C be a fiber of $E \to F$. Let D_1 be the strict transform of a curve $\mathbf{P}^1 \times \text{pt.}$ that meets F. Let D_2 be the strict transform of a curve pt. $\times$line that meets F. We have the intersection numbers:

	C	D_1	D_2
A	0	1	0
B	0	0	1
E	-1	1	1

It follows that the dual basis to C, D_1, D_2 is given by $A+B-E, A, B$. These three line bundles are basepoint-free, hence nef, giving contractions of X to $\mathbf{P}^2$, $\mathbf{P}^1$, and $\mathbf{P}^2$. It follows that $\overline{\text{Curv}(X)} = \mathbf{R}^{\geq 0}\{C, D_1, D_2\}$, and the nef cone is spanned by $A+B-E, A, B$. More strongly, the nef monoid in $\text{Pic}(X)$ is generated by $A+B-E$, A, and B.

We have $-K_X = 2A+3B-E$. Let $M = 2A+2B-E$; then $M \cdot C = 1$, $M \cdot D_1 = 1$, and $M \cdot D_2 = 1$. So M is ample, and every ample line bundle on X is M plus a nef divisor, hence M plus an $\mathbf{N}$-linear combination of $A+B-E$, A, and B.

A general divisor in $|A|$ or $|A+B-E|$ is the blow-up of $\mathbf{P}^2$ at one point, which is toric. A general divisor in $|B|$ is the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1$ at one point, which is also toric. By Lemma 3.1 plus the previous paragraph, this reduces Bott vanishing on X to the single line bundle M.

For M and Ω^1_X, we need to show that $H^j(X, \Omega^1_X \otimes M) = 0$ for $j > 0$, where $M = 2A+2B-E$. Let L be the line bundle $2A+2B$ on $Y = \mathbf{P}^1 \times \mathbf{P}^2$. The nef cone of Y is spanned by A and B. The curve F is a complete intersection $S_1 \cap S_2$ in Y, with $S_1 \sim B$ and $S_2 \sim A+B$. By Lemma 6.3, it suffices to show that L, $L - S_1$, and $L - S_2$ are ample, and that $L - S_1 - S_2$ is nef. By the nef cone of Y, these things are true, with $L - S_1 - S_2 \sim A$.

It remains to prove Bott vanishing for the ample line bundle M and Ω^2_X. Here $M = \pi^*(L) - E$, and $L = -K_Y - B$. By Corollary 6.2, it suffices to show: (1) $H^j(Y, \Omega^2_Y \otimes L) = 0$ for $j > 0$, (2) $H^0(Y, \Omega^2_Y \otimes K_Y^*(-B)) \to H^0(F, \Omega^1_F \otimes N^*_{F/Y} \otimes K_Y^*(-B))$ is surjective, and (3) $H^1(F, \Omega^1_F \otimes N^*_{F/Y} \otimes K_Y^*(-B)) = 0$. Use the canonical isomorphism $V^* \cong \Lambda^{r-1}V \otimes \det(V)^*$ for a vector bundle V of rank r, and the isomorphism $K_Y^*|_F \cong K_F^* \otimes \det(N_{F/Y})$. So we can replace (2) and (3) by the equivalent statements: (2) $H^0(Y, TY(-B)) \to H^0(F, N_{F/Y}(-B))$ is surjective, and (3) $H^1(F, N_{F/Y}(-B)) = 0$.

Here (1) is immediate from Bott vanishing on the toric variety $Y = \mathbf{P}^1 \times \mathbf{P}^2$. Next, the tangent bundle of Y is $f_1^*(T\mathbf{P}^1) \oplus f_2^*(T\mathbf{P}^2)$, and so $TY(-B) = f_1^*(T\mathbf{P}^1) \otimes O(-B) \oplus f_2^*(T\mathbf{P}^2(-1))$.

By the chain of inclusions $F \subset \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^1 \times \mathbf{P}^2$, the normal bundle $N_{F/Y}$ is an extension $0 \to O(2) \to N_{F/Y} \to O(1) \to 0$, and so $N_{F/Y} \cong O(2) \oplus O(1)$. It follows that $N_{F/Y}(-B) \cong O(1) \oplus O$. This proves (3). For (2), we will just use the second summand of TY; that is, we will prove that $H^0(\mathbf{P}^2, T\mathbf{P}^2(-1)) =$

$H^0(Y, f_2^*(T\mathbf{P}^2(-1)))$ maps onto $H^0(F, N_{F/Y}(-B))$.

Here F embeds as a line in $\mathbf{P}^2$, so we can first use that $H^0(\mathbf{P}^2, T\mathbf{P}^2(-1))$ maps onto $H^0(F, T\mathbf{P}^2(-1))$, since $H^1(\mathbf{P}^2, T\mathbf{P}^2(-2)) = 0$. Next, the composition $T\mathbf{P}^2|_F \subset TY|_F \twoheadrightarrow N_{F/Y}$ is an isomorphism, because the derivative of $f_1 \colon F \to \mathbf{P}^1$ is nonzero at every point. So $H^0(F, T\mathbf{P}^2(-1))$ maps isomorphically to $H^0(F, N_{F/Y}(-B))$. Thus (2) is proved. This completes the proof of Bott vanishing for the Fano 3-fold (3.24).

9 The Fano 3-folds (3.15), (3.16), (3.18), (3.20), (3.21), (3.22), (3.23)

In these seven cases with Picard number 3, we prove Bott vanishing efficiently by relating each Fano variety to a toric variety. In each case except (3.20), the Fano variety is the blow-up of a smooth toric variety along a smooth curve.

We first prove Bott vanishing for (3.15), the blow-up of the quadric 3-fold Q along a disjoint line and conic. We will instead use a different description, mentioned by Mori-Mukai [31, Table 3]. Namely, X is the blow-up of $Y = \mathbf{P}^1 \times \mathbf{P}^2$ along a smooth curve F of degree $(2, 2)$. (The advantage of this description is that it expresses X as the blow-up of a toric variety along a curve.) Here X is not rigid in characteristic 2, and so we work over a field of characteristic not 2. We can take F to be given by the equations $y_0y_1 = y_2^2, x_0y_1 = x_1y_0 = 0$ in $\mathbf{P}^1 \times \mathbf{P}^2 = \{([x_0, x_1], [y_0, y_1, y_2])\}$.

Here $\text{Pic}(X) = \mathbf{Z}\{A, B, E\}$ and $-K_X = 2A + 3B - E$. Let C be a fiber of $E \to F$. Let D_1 be the strict transform of a curve $\mathbf{P}^1 \times \text{pt.}$ that meets F in one point. Let D_2 be the strict transform of a curve pt. $\times$line that meets F in two points. We have the intersection numbers:

	C	D_1	D_2
A	0	1	0
B	0	0	1
E	-1	1	2

It follows that the dual basis to C, D_1, D_2 is $A + 2B - E, A, B$. The line bundles $A + 2B - E$, A, and B are basepoint-free, hence nef. Namely, they contract X to the quadric 3-fold Q, $\mathbf{P}^1$, and $\mathbf{P}^2$. It follows that $\overline{\text{Curv}(X)} = \mathbf{R}^{\geq 0}\{C, D_1, D_2\}$, and the nef cone is spanned by $A + 2B - E$, A, and B. More strongly, the nef monoid in $\text{Pic}(X)$ is generated by these three divisors. Since $-K_X \cdot C = -K_X \cdot D_1 = -K_X \cdot D_2 = 1$, every ample line bundle is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of $A + 2B - E$, A, and B.

A general divisor in $|A|$ is $\mathbf{P}^2$ blown up at two points, which is toric. A general divisor in $|B|$ is a quadric surface $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at two non-collinear points, which is toric. Finally, a general divisor in $|A + 2B - E|$ is $\mathbf{P}^2$ blown up at 4 points with no two collinear. So it is the quintic del Pezzo surface (which is not toric). All these surfaces satisfy Bott vanishing. By Lemma 3.1, this reduces Bott vanishing on X to the ample line bundle $-K_X$.

Since X is rigid (in characteristic not 2), we have $H^j(X, \Omega_X^2 \otimes K_X^*) = H^j(X, TX) = 0$ for $j > 0$. It remains to show that $H^j(X, \Omega_X^1 \otimes K_X^*) = 0$ for $j > 0$. Lemma 6.3 does exactly what we need. Namely, the curve F in $Y = \mathbf{P}^1 \times \mathbf{P}^2$ was defined (above) as a complete intersection $S_1 \cap S_2$ with $S_1 \sim A + B$ and $S_2 \sim 2B$ on

Y. Because $-K_Y = 2A + 3B$, the line bundles $-K_Y$, $-K_Y - S_1 = A + 2B$, and $-K_Y - S_2 = 2A + B$ are ample, and $-K_Y - S_1 - S_2 = A$ is nef. By Lemma 6.3, it follows that $H^j(X, \Omega^1_X \otimes K_X^*) = 0$ for $j > 0$. That proves Bott vanishing for (3.15).

We now prove Bott vanishing for the Fano 3-fold (3.16). The proof works in any characteristic. Namely, X is the blow-up of the toric variety $Y = \mathrm{Bl}_{\mathrm{pt.}} \mathbf{P}^3$ along the strict transform F of a twisted cubic through the point p in $\mathbf{P}^3$. We have $\mathrm{Pic}(Y) = \mathbf{Z}\{H, E_1\}$, where E_1 is the exceptional divisor over p, and $\mathrm{Pic}(X) = \mathbf{Z}\{H, E_1, E_2\}$, where E_2 is the exceptional divisor over F. (We write E_1 in X for the strict transform of E_1 in Y.) We have $-K_Y = 4H - 2E_1$ and $-K_X = 4H - 2E_1 - E_2$. Let C_1 be the strict transform of a line in $E_1 \cong \mathbf{P}^2 \subset Y$ through the point $E_1 \cap F$. Let C_2 be a fiber of $E_2 \to F$. Let D be the strict transform of a line in $\mathbf{P}^3$ through p that meets the twisted cubic in another point. We have the intersection numbers:

	C_1	C_2	D
H	0	0	1
E_1	-1	0	1
E_2	1	-1	1

The dual basis to C_1, C_2, D is given by $H - E_1, 2H - E_1 - E_2, H$. These three divisors are basepoint-free, giving contractions of X to $\mathbf{P}^2$, $\mathbf{P}^2$, and $\mathbf{P}^3$. It follows that $\overline{\mathrm{Curv}(X)} = \mathbf{R}^{\geq 0}\{C_1, C_2, D\}$, and the nef cone is spanned by $H - E_1, 2H - E_1 - E_2, H$. More strongly, the nef monoid in $\mathrm{Pic}(X)$ is generated by these three divisors. Since $-K_X \cdot C_1 = 1$, $-K_X \cdot C_2 = 1$, and $-K_X \cdot D = 1$, every ample line bundle on X is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of $H - E_1, 2H - E_1 - E_2, H$.

A general divisor in $|H|$ or $|H - E_1|$ is isomorphic to $\mathbf{P}^2$ blown up at 3 non-collinear points, which is toric. A general divisor in $|2H - E_1 - E_2|$ is a quadric surface $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at one point, which is toric. By Lemma 3.1, this reduces Bott vanishing on X to the ample line bundle $-K_X$.

For $-K_X$ and Ω^2_X, this is easy: since X is rigid, we have $H^1(X, \Omega^2_X \otimes K_X^*) = H^1(X, TX) = 0$, and KAN vanishing gives that $H^j = 0$ for $j \geq 2$ (Lemma 1.5). It remains to prove Bott vanishing for $-K_X$ and Ω^1_X, meaning that $H^j(X, \Omega^1_X \otimes K_X^*) = 0$ for $j > 0$. By Corollary 6.2, this would follow if (1) $H^j(Y, \Omega^1_Y(-K_Y)) = 0$ for $j > 0$, (2) $H^0(Y, \Omega^1_Y(-K_Y)) \to H^0(F, \Omega^1_Y(-K_Y))$ is surjective, and (1) $H^1(F, \Omega^1_Y(-K_Y)) = 0$.

We cannot apply Lemma 6.3, because the curve F is not a complete intersection in Y (because the twisted cubic is not a complete intersection in $\mathbf{P}^3$). Instead, note that the twisted cubic is a curve of bidegree $(1, 2)$ on a smooth quadric surface $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^3$. Let S be the strict transform of that surface in $Y = \mathrm{Bl}_p \mathbf{P}^3$; then we can work with the chain $F \subset S \subset Y$. The surface S is isomorphic to $\mathrm{Bl}_p(\mathbf{P}^1 \times \mathbf{P}^1)$.

Statement (1) is immediate from Bott vanishing on the toric variety Y. Next, consider the exact sequence

$$H^j(Y, \Omega^1_Y(-K_Y)) \to H^0(S, \Omega^1_Y(-K_Y)) \to H^{j+1}(Y, \Omega^1_Y(-K_Y - S)).$$

Here $\mathrm{Pic}(Y) = \mathbf{Z}\{H, E\}$, $-K_Y = 4H - 2E$, and $S \sim 2H - E$ on Y. The nef cone of Y is spanned by H and $H - E$. So $-K_Y - S = 2H - E$ is ample, and hence $H^j(Y, \Omega^1_Y(-K_Y - S)) = 0$ for $j > 0$ by Bott vanishing on Y again. It follows that $H^j(S, \Omega^1_Y(-K_Y)) = 0$ for $j > 0$, and that $H^0(Y, \Omega^1_Y(-K_Y)) \to H^0(S, \Omega^1_Y(-K_Y))$ is surjective.

Next, consider the exact sequence:

$$H^j(S,\Omega^1_Y(-K_Y)) \to H^j(F,\Omega^1_Y(-K_Y)) \to H^{j+1}(S,\Omega^1_Y|_S(-K_Y-F)). \qquad (*)$$

Here $\mathrm{Pic}(S) \cong \mathbf{Z}\{A,B,E\}$, $F \sim 2A+B-E$ on S, and $-K_Y|_S = 4A+4B-2E$. Also, $\mathrm{Nef}(S) = \mathbf{R}^{\geq 0}\{A,B,A+B-E\}$. So $-K_Y - F = 2A+3B-E$ is ample on S.

To analyze Ω^1_Y restricted to S, use the exact sequence

$$0 \to O_S(-S) \to \Omega^1_Y|_S \to \Omega^1_S \to 0, \qquad (**)$$

where $O_S(-S) = -2A-2B+E$. The surface $S = \mathrm{Bl}_p(\mathbf{P}^1 \times \mathbf{P}^1)$ is a toric variety, although not torically embedded in Y. So Bott vanishing on S gives that $H^j(S,\Omega^1_S(-K_Y-F)) = 0$. Also, $-S|_S - K_Y - F = B$, which is nef. So $H^j(S,O(-S-K_Y-F)) = 0$ for $j>0$ by Proposition 1.3. By (**), we conclude that $H^j(S,\Omega^1_Y|_S(-K_Y-F)) = 0$ for $j>0$. By (*), we deduce that $H^1(F,\Omega^1_Y(-K_Y)) = 0$ (which is statement (3)) and that $H^0(S,\Omega^1_Y(-K_Y)) \to H^0(F,\Omega^1_Y(-K_Y))$ is surjective. That completes the proof of statement (2). Thus we have shown that $H^j(X,\Omega^1_X(-K_X)) = 0$ for $j>0$. This completes the proof of Bott vanishing for the Fano 3-fold (3.16).

We now prove Bott vanishing for (3.18). We work in characteristic not 2, since X is not rigid in characteristic 2. Here X is the blow-up of $\mathbf{P}^3$ along a disjoint line F_1 and conic F_2. We have $\mathrm{Pic}(X) = \mathbf{Z}\{H,E_1,E_2\}$ and $-K_X = 4H-E_1-E_2$. Let C_1 be a fiber of $E_1 \to F_1$ and C_2 a fiber of $E_2 \to F_2$. Let D be the strict transform of a line meeting the line in one point and the conic in two points (which exists). We have the following intersection numbers.

	C_1	C_2	D
H	0	0	1
E_1	-1	0	1
E_2	0	-1	2

It follows that the dual basis to C_1,C_2,D is $H-E_1$, $2H-E_2$, H. These three divisors are basepoint-free, hence nef, giving contractions of X to $\mathbf{P}^1$, the quadric 3-fold Q, and $\mathbf{P}^3$. So $\overline{\mathrm{Curv}(X)} = \mathbf{R}^{\geq 0}\{C_1,C_2,D\}$, and the nef cone is spanned by $H-E_1, 2H-E_2, H$. More strongly, the nef monoid in $\mathrm{Pic}(X)$ is generated by these three divisors. Also, $-K_X$ has degree 1 on C_1, C_2, and D. So every ample line bundle is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of $H-E_1$, $2H-E_2$, H.

A general divisor in $|H-E|$ is the blow-up of $\mathbf{P}^2$ at 2 points, which is toric. A general divisor in $|2H-E_2|$ is the blow-up of a quadric surface $\mathbf{P}^1 \times \mathbf{P}^1$ at 2 points, which is toric. A general divisor in $|H|$ is the blow-up of $\mathbf{P}^2$ at 3 non-collinear points, which is toric. By Lemma 3.1, Bott vanishing for X reduces to the single line bundle $-K_X$.

Since X is rigid, we know that $H^j(X,\Omega^2_X \otimes K^*_X) = H^j(X,TX) = 0$ for $j>0$. It remains to show that $H^j(X,\Omega^1_X \otimes K^*_X) = 0$ for $j>0$. Let Y be the blow-up of $\mathbf{P}^3$ along a line, so that X is the blow-up of Y along the disjoint conic F_2. This description has the advantage that Y is toric. Here $-K_Y = 4H-E_1$, and F_2 is a complete intersection $S_1 \cap S_2$ in Y with $S_1 \sim H$ and $S_2 \sim 2H$. So $-K_Y$, $-K_Y - S_1 = 3H-E_1$, and $-K_Y - S_2 = 2H-E_1$ are ample, and $-K_Y - S_1 - S_2 = H-E_1$ is nef. By Lemmas 6.3 and 1.3, it follows that $H^j(X,\Omega^1_X \otimes K^*_X) = 0$ for $j>0$. That completes the proof of Bott vanishing for (3.18).

We now prove Bott vanishing for (3.20). The proof works in any characteristic. Here X is the blow-up of the quadric 3-fold Q along two disjoint lines, F_1 and F_2. We have $\mathrm{Pic}(X) = \mathbf{Z}\{H, E_1, E_2\}$ and $-K_X = 3H - E_1 - E_2$. Let C_i be a fiber of $E_i \to F_i$, for $i = 1, 2$. Let D be the strict transform of a line in Q meeting F_1 and F_2, which exists. We have the following intersection numbers.

	C_1	C_2	D
H	0	0	1
E_1	-1	0	1
E_2	0	-1	1

It follows that the dual basis to C_1, C_2, D is $H - E_1$, $H - E_2$, H. These three divisors are basepoint-free, hence nef, giving contractions of X to $\mathbf{P}^2$, $\mathbf{P}^2$, and Q. So $\overline{\mathrm{Curv}(X)} = \mathbf{R}^{\geq 0}\{C_1, C_2, D\}$, and the nef cone is spanned by $H - E_1, H - E_2, H$. More strongly, the nef monoid in $\mathrm{Pic}(X)$ is generated by these three divisors. Also, $-K_X$ has degree 1 on C_1, C_2, and D. So every ample line bundle is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of $H - E_1$, $H - E_2$, H.

A general divisor in $|H - E_1|$ or $|H - E_2|$ is the blow-up of a quadric surface $\mathbf{P}^1 \times \mathbf{P}^1$ at one point, which is toric. A general divisor in $|H|$ is the blow-up of a quadric surface at two points, which is also toric. By Lemma 3.1, Bott vanishing for X reduces to the single line bundle $-K_X$.

Since X is rigid, we know that $H^j(X, \Omega^2_X \otimes K^*_X) = H^j(X, TX) = 0$ for $j > 0$. It remains to show that $H^j(X, \Omega^1_X \otimes K^*_X) = 0$ for $j > 0$. We cannot apply our usual method, since X is not the blow-up of a toric variety along a curve. Instead, we will view X as a hypersurface in a toric variety Z: the blow-up of $\mathbf{P}^4$ along two disjoint lines. We have $\mathrm{Pic}(Z) = \mathbf{Z}\{H, E_1, E_2\}$, with $-K_Z = 5H - 2E_1 - 2E_2$, and X is linearly equivalent to $2H - E_1 - E_1$ on Z.

Let $L = 3H - E_1 - E_2$ on Z (so that L restricted to X is $-K_X$). Consider the exact sequence $0 \to O(L - X)|_X \to \Omega^1_Z(L)|_X \to \Omega^1_X(L) \to 0$. The line bundle $O(L-X)|_X = O(H)|_X = O(K_X + (4H - E_1 - E_2))$ has zero cohomology in degrees >0 on X, by Kodaira vanishing. (In fact, this holds in any characteristic, by reducing to Kodaira vanishing on the toric variety Z.) So the vanishing of cohomology of degree > 0 for $\Omega^1_X(L)$ would follow from the same statement for $\Omega^1_Z(L)|_X$. Now consider the exact sequence $0 \to \Omega^1_Z(L - X) \to \Omega^1_Z(L) \to \Omega^1_Z(L)|_X \to 0$. Here $\Omega^1_Z(L)$ has zero cohomology in degrees > 0 by Bott vanishing on the toric variety Z (since L is ample), and $\Omega^1_Z(L - X)$ has zero cohomology in degrees > 1 by Proposition 1.3 (since $L - X = H$ is nef). It follows that $\Omega^1_Z(L)|_X$ has zero cohomology in degrees > 0. That completes the proof of Bott vanishing for (3.20).

We now prove Bott vanishing for (3.21). We work in characteristic not 2, since X is not rigid in characteristic 2. Here X is the blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ along a curve F of degree $(2, 1)$. We can take $F = \{y_2 = 0, x_1 y_0^2 = x_0 y_1^2\}$ in $\mathbf{P}^1 \times \mathbf{P}^2 = \{([x_0, x_1], [y_0, y_1, y_2])\}$. Thus F is contained in $\mathbf{P}^1 \times l$, for a line l in $\mathbf{P}^2$, We have $\mathrm{Pic}(X) = \mathbf{Z}\{A, B, E\}$ and $-K_X = 2A + 3B - E$. Let C be a fiber of $E \to F$. Let q be a point on the line l, and let D_1 be the strict transform of $\mathbf{P}^1 \times q$. Let D_2 be the strict transform of $p \times l$, for some point p in $\mathbf{P}^1$. We have the intersection numbers:

	C	D_1	D_2
A	0	1	0
B	0	0	1
E	-1	1	2

We compute that the dual basis to C, D_1, D_2 is $A + 2B - E, A, B$. These three divisors are basepoint-free, hence nef, using the equation for F above. They give contractions of X to the quintic del Pezzo 3-fold with one node in $\mathbf{P}^6$ [33, section 5.4.2], $\mathbf{P}^1$, and $\mathbf{P}^2$. It follows that $\overline{\mathrm{Curv}(X)} = \mathbf{R}^{\geq 0}\{C, D_1, D_2\}$, and the nef cone is spanned by the three divisors mentioned. By the intersection numbers, the nef monoid is also generated by those three divisors. The line bundle $-K_X = 2A + 3B - E$ has degree 1 on all three curves C, D_1, D_2. So every ample line bundle is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of the three divisors mentioned.

A general divisor in $|A|$ is $\mathbf{P}^2$ blown up at 2 points, which is toric. A general divisor in $|B|$ is the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1$ at one point, which is toric. A general divisor in $|A + 2B - E|$ is $\mathbf{P}^2$ blown up at 4 points with no 3 collinear, hence a quintic del Pezzo surface. That is not toric, but it satisfies Bott vanishing. By Lemma 3.1, that reduces Bott vanishing for (3.21) to the single line bundle $-K_X$.

Since X is rigid, we know that $H^j(X, \Omega^2_X \otimes K_X^*) = H^j(X, TX)$ is zero for $j > 0$. It remains to show that $H^j(X, \Omega^1_X \otimes K_X^*)$ is zero for $j > 0$. The curve F is a complete intersection $S_1 \cap S_2$ in $Y = \mathbf{P}^1 \times \mathbf{P}^2$, with $S_1 \sim B$ and $S_2 \sim A + 2B$. Since $-K_Y = 2A+3B$, we see that $-K_Y$, $-K_Y - S_1$, and $-K_Y - S_2$ are ample. By Lemma 6.3, the desired vanishing would follow if $H^j(Y, \Omega^1_Y(-K_Y - S_1 - S_2)) = 0$ for $j > 1$. Here $-K_Y - S_1 - S_2 = A$ is nef, and so this follows from Proposition 1.3. That completes the proof of Bott vanishing for (3.21).

We now prove Bott vanishing for (3.22). The proof works in any characteristic. Here X is the blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ along a conic F in $p \times \mathbf{P}^2$, for a point p in $\mathbf{P}^1$. We can take $F = \{x_1 = 0, y_0 y_1 = y_2^2\}$ in $\mathbf{P}^1 \times \mathbf{P}^2 = \{([x_0, x_1], [y_0, y_1, y_2])\}$. We have $\mathrm{Pic}(X) = \mathbf{Z}\{A, B, E\}$ and $-K_X = 2A + 3B - E$. Let C be a fiber of $E \to F$. Let D_1 be the strict transform of a line in $p \times \mathbf{P}^2$. Let D_2 be the strict transform of $\mathbf{P}^1 \times q$, for a point q in the conic. We have the intersection numbers:

	C	D_1	D_2
A	0	0	1
B	0	1	0
E	-1	2	1

We compute that the dual basis to C, D_1, D_2 is $A + 2B - E, B, A$. These three divisors are basepoint-free, hence nef, using the equation for F above. They give contractions of X to the cone in $\mathbf{P}^6$ over the Veronese surface in $\mathbf{P}^5$, to $\mathbf{P}^2$, and to $\mathbf{P}^1$. It follows that $\overline{\mathrm{Curv}(X)} = \mathbf{R}^{\geq 0}\{C, D_1, D_2\}$, and the nef cone is spanned by the three divisors mentioned. By the intersection numbers, the nef monoid is also generated by those three divisors. The line bundle $-K_X = 2A+3B-E$ has degree 1 on all three curves C, D_1, D_2. So every ample line bundle is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of the three divisors mentioned.

A general divisor in $|A|$ is $\mathbf{P}^2$, which is toric. A general divisor in $|B|$ is the blow-up of the quadric surface $\mathbf{P}^1 \times \mathbf{P}^1$ at two collinear points, which is toric. A general divisor in $|A+2B-E|$ is $\mathbf{P}^2$ blown up at 4 points with no 3 collinear, hence a quintic del Pezzo surface. That is not toric, but it satisfies Bott vanishing. By Lemma 3.1, this reduces Bott vanishing for (3.22) to the single line bundle $-K_X$.

Since X is rigid, we know that $H^j(X, \Omega^2_X \otimes K_X^*) = H^j(X, TX)$ is zero for $j > 0$. It remains to show that $H^j(X, \Omega^1_X \otimes K_X^*)$ is zero for $j > 0$. Let $Y = \mathbf{P}^1 \times \mathbf{P}^2$; then $-K_Y = 2A+3B$. The curve F is a complete intersection $S_1 \cap S_2$ in Y with $S_1 \sim A$ and $S_2 = 2B$. So $-K_Y$, $-K_Y - S_1$, and $-K_Y - S_2$ are ample. By Lemma 6.3, the desired vanishing would follow if we have $H^j(Y, \Omega^1_Y(-K_Y - S_1 - S_2)) = 0$ for $j > 1$.

Here $-K_Y - S_1 - S_2 = A + 2B$ is ample, and so this cohomology is actually zero for $j > 0$ (Theorem 4.2). That completes the proof of Bott vanishing for (3.22).

We now prove Bott vanishing for (3.23). The proof works in any characteristic. Here X is the blow-up of $Y := V_7 = \mathrm{Bl}_p\,\mathbf{P}^3$ along the strict transform F of a conic through the point $p \in \mathbf{P}^3$. We can take p to be $[1,0,0,0]$ and the conic to be $\{x_3 = 0, x_1^2 - x_0x_2 = 0\}$. We have $\mathrm{Pic}(Y) = \mathbf{Z}\{H, E_1\}$ with $-K_Y = 4H - 2E_1$, and so $\mathrm{Pic}(X) = \mathbf{Z}\{H, E_1, E_2\}$ and $-K_X = 4H - 2E_1 - E_2$. Let C_1 be the strict transform in $E_1 \subset X$ of a line through the point $E_1 \cap F$ in Y. (I am using the same name for the surface $E_1 \cong \mathbf{P}^2$ in Y and its strict transform in X.) Let C_2 be a fiber of $E_2 \to F$. Let D be the strict transform of a line through p in $\mathbf{P}^3$ that meets the conic at another point. We have the intersection numbers:

	C_1	C_2	D
H	0	0	1
E_1	-1	0	1
E_2	1	-1	1

We compute that the dual basis to C_1, C_2, D is $H - E_1, 2H - E_1 - E_2, H$. These three divisors are basepoint-free, hence nef, giving contractions of X to $\mathbf{P}^2$, the quadric 3-fold Q, and $\mathbf{P}^3$. It follows that $\overline{\mathrm{Curv}(X)} = \mathbf{R}^{\geq 0}\{C_1, C_2, D\}$, and the nef cone is spanned by the three divisors mentioned. By the intersection numbers, the nef monoid is also generated by those three divisors. The line bundle $-K_X = 4H - 2E_1 - E_2$ has degree 1 on all three curves C_1, C_2, D. So every ample line bundle is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of the three divisors mentioned.

A general divisor in $|H|$ or $|H - E|$ is $\mathbf{P}^2$ blown up at two points, which is toric. A general divisor in $|2H - E_1 - E_2|$ is a quadric surface $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at one point, which is also toric. By Lemma 3.1, this reduces Bott vanishing for (3.23) to the single line bundle $-K_X$.

Since X is rigid, we know that $H^j(X, \Omega_X^2 \otimes K_X^*) = H^j(X, TX)$ is zero for $j > 0$. It remains to show that $H^j(X, \Omega_X^1 \otimes K_X^*)$ is zero for $j > 0$. We have $Y \cong \mathrm{Bl}_p\,\mathbf{P}^3$ and $-K_Y = 4H - 2E_1$. The curve F is a complete intersection $S_1 \cap S_2$ in Y with $S_1 \sim H - E_1$ and $S_2 \sim 2H - E_1$. So $-K_Y$, $-K_Y - S_1 = 3H - E_1$, and $-K_Y - S_2 = 2H - E_1$ are ample. By Lemma 6.3, the desired vanishing would follow if we have $H^j(Y, \Omega_Y^1(-K_Y - S_1 - S_2)) = 0$ for $j > 1$. Here $-K_Y - S_1 - S_1 = H$ is nef, and so that follows from Proposition 1.3. This completes the proof of Bott vanishing for (3.23).

10 The Fano 3-folds (4.3), (4.4), (4.5), (4.6), (4.7), (4.8)

For these Fano 3-folds with Picard number 4, the proof of Bott vanishing is again efficient by our methods. Each variety is the blow-up of a smooth toric variety along a smooth curve.

Let us prove Bott vanishing for the Fano 3-fold (4.3). We work in characteristic not 2, since X is not rigid in characteristic 2. Here X is the blow-up of $(\mathbf{P}^1)^3$ along a curve F of degree $(1,1,2)$. We can take $F = \{x_0y_1 = x_1y_0, x_0^2z_1 = x_1^2z_0\}$ in $(\mathbf{P}^1)^3 = \{([x_0,x_1],[y_0,y_1],[z_0,z_1])\}$. We have $\mathrm{Pic}(X) = \mathbf{Z}\{A, B, C, E\}$ and $-K_X = 2A + 2B + 2C - E$. Let D_4 be a fiber of $E \to F$. Let D_1, D_2, D_3 be strict transforms of curves $\mathbf{P}^1 \times \text{pt.} \times \text{pt.}$, $\text{pt.} \times \mathbf{P}^1 \times \text{pt.}$, $\text{pt.} \times \text{pt.} \times \mathbf{P}^1$ that meet the curve F in one point each. We have the intersection numbers:

	D_1	D_2	D_3	D_4
A	1	0	0	0
B	0	1	0	0
C	0	0	1	0
E	1	1	1	-1

We compute that the dual basis to D_1, D_2, D_3, D_4 is $A, B, C, A+B+C-E$. These four divisors are basepoint-free, hence nef, using the equation for F above. They give contractions of X to $\mathbf{P}^1$, $\mathbf{P}^1$, $\mathbf{P}^1$, and $\mathbf{P}^2$. It follows that $\overline{\mathrm{Curv}(X)} = \mathbf{R}^{\geq 0}\{D_1, D_2, D_3, D_4\}$, and the nef cone is spanned by the four divisors mentioned. By the intersection numbers, the nef monoid is also generated by those four divisors. The line bundle $-K_X = 2A+2B+2C-E$ has degree 1 on all four curves D_1, D_2, D_3, D_4. So every ample line bundle is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of the four divisors mentioned.

A general divisor in $|A|$ or $|B|$ is $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at one point, which is toric. A general divisor in $|C|$ if $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at 2 points, which is toric. A general divisor in $|A+B+C-E|$ is $\mathbf{P}^2$ blown up at 3 non-collinear points, which is toric. By Lemma 3.1, this reduces Bott vanishing for (4.3) to the single line bundle $-K_X$.

Since X is rigid, we know that $H^j(X, \Omega^2_X \otimes K^*_X) = H^j(X, TX)$ is zero for $j > 0$. It remains to show that $H^j(X, \Omega^1_X \otimes K^*_X)$ is zero for $j > 0$. Let $Y = (\mathbf{P}^1)^3$, which has $-K_Y = 2A+2B+2C$. The curve F is a complete intersection $S_1 \cap S_2$ in Y, with $S_1 \sim A+B$ and $S_2 \sim 2A+C$. By Lemma 6.3, the desired vanishing would follow if $-K_Y$ and $-K_Y - S_1$ are ample, the surface S_1 is toric, $(-K_Y - S_2)|_{S_1}$ is ample, and $(-K_Y - S_1 - S_2)|_{S_1}$ is nef. Indeed, $-K_Y$ and $-K_Y - S_1 = A+B+2C$ are ample, and $S_1 = \{x_0y_1 = x_1y_0\} \subset (\mathbf{P}^1)^3$ is isomorphic to $(\mathbf{P}^1)^2$, which is toric. The line bundles A and B become isomorphic on S_1, and the nef cone of S_1 is spanned by A and C. So $(-K_Y - S_1)|_{S_1} = 2A + C$ is ample, and $(-K_Y - S_1 - S_2)|_{S_1} = C$ is nef. That completes the proof of Bott vanishing for (4.3).

We first prove Bott vanishing for (4.4). We work in characteristic not 2, since X is not rigid in characteristic 2. Here X is the blow-up of $\mathrm{Bl}_{p_2,p_3} Q$ (with p_2 and p_3 non-collinear points on the quadric Q) along the strict transform of a conic through p_2 and p_3. We will use a different description of this variety, mentioned by Mori-Mukai [31, Table 3]. Namely, let Y_1 be the blow-up of $\mathbf{P}^3$ along a line F_1, and let Y be the blow-up of Y_1 along the inverse image F_2 of a point $p \in F_1$. Let F_3 be the inverse image in Y of a conic in $\mathbf{P}^3$ disjoint from the line F_1. Then X is the blow-up of Y along F_3. This description has the advantage that Y is a toric variety. We can take the point p in $\mathbf{P}^3$ to be $[0,0,1,0]$, the line F_1 to be $\{x_0 = x_1 = 0\}$, and the conic F_3 to be $\{x_2 = x_3, x_0x_1 - x_2^2 = 0\}$.

Let $E_1 \to F_1$ be the exceptional divisor in Y_1; we also write E_1 for its strict transform in Y or X. Let $E_2 \to F_2$ be the exceptional divisor in Y, or its strict transform in X. Let $E_3 \to F_3$ be the exceptional divisor in X. Then $\mathrm{Pic}(X) = \mathbf{Z}\{H, E_1, E_2, E_3\}$ and $-K_X = 4H - E_1 - 2E_2 - E_3$. (To check the formula for $-K_X$, note that the pullback of $E_1 \subset Y_1$ is $E_1 + E_2$ in X.) For $i = 1, 2, 3$, let C_i be a general fiber of $E_i \to F_i$. Let D_1 be the strict transform of a line in $\mathbf{P}^3$ through p and a point of the conic F_3. Let q in $\mathbf{P}^3$ be the intersection of the line F_1 with the plane containing the conic F_3. Let D_2 be the strict transform of a line in $\mathbf{P}^3$ through q that meets the conic F_3 in 2 points. We have the intersection numbers:

	C_1	C_2	C_3	D_1	D_2
H	0	0	0	1	1
E_1	-1	1	0	0	1
E_2	0	-1	0	1	0
E_3	0	0	-1	1	2

Using Magma, we compute that the dual cone to $\mathbf{R}^{\geq 0}\{C_1, C_2, C_3, D_1, D_2\}$ is spanned by $H, H - E_2, H - E_1 - E_2, 2H - E_3, 2H - E_2 - E_3$. These five divisors are basepoint-free, hence nef, giving contractions of X to $\mathbf{P}^3$, $\mathbf{P}^2$, $\mathbf{P}^1$, the quadric 3-fold Q, and $\mathbf{P}^3$. It follows that $\overline{\text{Curv}(X)} = \mathbf{R}^{\geq 0}\{C_1, C_2, C_3, D_2, D_3\}$, and the nef cone is spanned by the five divisors mentioned. More strongly, Magma checks that the nef monoid is generated by these five divisors. The line bundle $-K_X = 4H - E_1 - 2E_2 - E_3$ has degree 1 on all five curves C_1, C_2, C_3, D_1, D_2. So every ample line bundle is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of the five divisors mentioned.

For all five divisors, a general divisor in the linear system is $\mathbf{P}^2$ blown up at 3 non-collinear points, which is toric. By Lemma 3.1, this reduces Bott vanishing for (4.4) to the single line bundle $-K_X$.

Since X is rigid, we know that $H^j(X, \Omega^2_X \otimes K^*_X) = H^j(X, TX)$ is zero for $j > 0$. It remains to show that $H^j(X, \Omega^1_X \otimes K^*_X)$ is zero for $j > 0$. The curve F_3 in Y is a complete intersection $F_1 = S_1 \cap S_2$, where $S_1 \sim H$ and $S_2 \sim 2H$. (Here, in $\mathbf{P}^3$, S_1 is the plane $\{x_2 = x_3\}$ and S_1 is the quadric cone $\{x_0x_1 - x_3^2 = 0\}$, singular at p. Since the point p is disjoint from the plane S_1 in $\mathbf{P}^3$, we could also describe F_3 in Y as a complete intersection of H and $2H - E_1$, or of H and $2H - 2E_2$. But we choose the description mentioned.)

By Lemma 6.3, the desired vanishing would follow if $-K_Y$, $-K_Y - S_1$ are ample, S_1 is a toric variety, $(-K_Y - S_2)|_{S_1}$ is ample, and $(-K_Y - S_1 - S_2)|_{S_1}$ is nef. Here $-K_Y = 4H - E_1 - 2E_2$, and the nef cone of Y is spanned by $H, H - E_2, H - E_1 - E_2$. So $-K_Y$ and $-K_Y - S_1$ are ample. The surface S_1 is $\mathbf{P}^2$ blown up at one point, which is toric. Since E_2 is disjoint from S_1, $-K_Y - S_2 = 2H - E_1 - 2E_2$ restricted to S_1 is numerically equivalent to $(1/2)H + (1/2)(H - E_2) + (H - E_1 - E_2)$, which is ample; and $-K_Y - S_1 - S_2 = H - E_1 - 2E_2$ restricted to S_1 is numerically equivalent to $H - E_1 - E_2$, which is nef. That completes the proof of Bott vanishing for (4.4).

Next, we prove Bott vanishing for (4.5). We work in characteristic not 2, since X is not rigid in characteristic 2. Here X is the blow-up of $Y := \mathbf{P}^1 \times \mathbf{P}^2$ along two disjoint curves, F_1 of degree $(2, 1)$ and F_2 of degree $(1, 0)$. Thus F_1 is contained in $\mathbf{P}^1 \times l$, for a line l in $\mathbf{P}^2$, and F_2 is equal to $\mathbf{P}^1 \times p$, for a point $p \notin l$. We can take $F_1 = \{y_0 = 0, x_1y_1^2 = x_0y_2^2\}$ and $F_2 = \{y_1 = 0, y_2 = 0\}$ in $\mathbf{P}^1 \times \mathbf{P}^2 = \{([x_0, x_1], [y_0, y_1, y_2])\}$. We have $\text{Pic}(X) = \mathbf{Z}\{A, B, E_1, E_2\}$ and $-K_X = 2A + 3B - E_1 - E_2$. Let $C_i \subset E_i$ be a fiber of $E_i \to F_i$, for $i = 1, 2$. Let D_1 be the strict transform of the curve $\mathbf{P}^1 \times q$ in Y, for a point $q \in l$. Let (s, t) be a point in F_2, and let l_2 be the line through p and t in $\mathbf{P}^2$. Then let D_2 be the strict transform of the curve $s \times l_2 \subset \mathbf{P}^1 \times \mathbf{P}^2$. Finally, let D_3 be the strict transform of the curve $s \times l$. We have the intersection numbers:

	C_1	C_2	D_1	D_2	D_3
A	0	0	1	0	0
B	0	0	0	1	1
E_1	-1	0	1	1	2
E_2	0	-1	0	1	0

We compute that the dual cone to $\mathbf{R}^{\geq 0}\{C_1, C_2, D_1, D_2, D_3\}$ is spanned by the five divisors A, B, $B - E_2$, $A + 2B - E_1$, and $A + 2B - E_1 - E_2$. These are all basepoint-free, hence nef, using the equations for F_1 and F_2 above. They give contractions of X to $\mathbf{P}^1$, $\mathbf{P}^2$, $\mathbf{P}^1$, the quintic del Pezzo 3-fold with one node in $\mathbf{P}^6$ [33, section 5.4.2], and the nodal quadric 3-fold in $\mathbf{P}^4$. It follows that $\overline{\mathrm{Curv}(X)} = \mathbf{R}^{\geq 0}\{C_1, C_2, D_1, D_2, D_3\}$, and the nef cone is spanned by the five divisors mentioned. Using Magma, we also compute that the nef monoid is generated by those five divisors. The line bundle $-K_X = 2A + 3B - E_1 - E_2$ has degree 1 on all five curves C_1, C_2, D_1, D_2, D_3. So every ample line bundle is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of the five divisors mentioned.

A general divisor in $|A|$ is $\mathbf{P}^2$ blown up at 3 non-collinear points, which is toric. A general divisor in $|B|$ or in $|B - E_2|$ is $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at one point, which is toric. A general divisor in $|A + 2B - E_1 - E_2|$ is $\mathbf{P}^2$ blown up at 4 points with no 3 collinear, hence a quintic del Pezzo surface. That is not toric, but it satisfies Bott vanishing. A general divisor in $|A + 2B - E_1|$ is the previous surface blown up at one more point; that does not satisfy Bott vanishing. Instead, we can observe that $A + 2B - E_1 = (A + 2B - E_1 - E_2) + E_2$, where E_2 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$, which is toric. By Lemma 3.1, that reduces Bott vanishing for (4.5) to the single line bundle $-K_X$.

Since X is rigid, we know that $H^j(X, \Omega^2_X \otimes K_X^*) = H^j(X, TX)$ is zero for $j > 0$. It remains to show that $H^j(X, \Omega^1_X \otimes K_X^*)$ is zero for $j > 0$. Let Z be the blow-up of $Y = \mathbf{P}^1 \times \mathbf{P}^2$ along the curve F_2 of degree $(0, 1)$; then Z is a toric variety, with $\mathrm{Pic}(Z) = \mathbf{Z}\{A, B, E_2\}$ and $\mathrm{Nef}(X) = \mathbf{R}^{\geq 0}\{A, B, B - E_2\}$. The curve F_1 is a complete intersection $S_1 \cap S_2$ in Z, with $S_1 \sim B$ and $S_2 \sim A + 2B$. By Lemma 6.3, the desired vanishing holds if $-K_Z$ and $-K_Z - S_1$ are ample, S_1 is a toric variety, $(-K_Y - S_2)_{S_1}$ is ample, and $(-K_Y - S_1 - S_2)_{S_1}$ is nef. Indeed, $-K_Z = 2A + 3B - E_2$ and $-K_Z - S_1 = 2A + 2B - E_2$ are ample. The surface $S_1 = \{y_0 = 0\}$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^2$, and the exceptional divisor E_2 in Z is disjoint from S_1. So $(-K_Y - S_2)|_{S_1} = A + B$ is ample on S_1, and $(K_Y - S_1 - S_2)|_{S_1} = A$ is nef on S_1. That completes the proof of Bott vanishing for (4.5).

Next, we prove Bott vanishing for (4.6). The proof works in any characteristic. Here X is the blow-up of $\mathbf{P}^3$ along three disjoint lines, F_1, F_2, F_3. We have $\mathrm{Pic}(X) = \mathbf{Z}\{H, E_1, E_2, E_3\}$ and $-K_X = 4H - E_1 - E_2 - E_3$. Let C_i be a fiber of $E_i \to F_i$ for $i = 1, 2, 3$, and let D be the strict transform of a line in $\mathbf{P}^3$ meeting F_1, F_2, and F_3 (which exists). Then $H \cdot C_i = 0$, $H \cdot D = 1$, $E_i \cdot C_j = -\delta_{ij}$, and $E_i \cdot D = 1$, for $1 \leq i, j \leq 3$. We have the intersection numbers:

	C_1	C_2	C_3	D
H	0	0	0	1
E_1	-1	0	0	1
E_2	0	-1	0	1
E_3	0	0	-1	1

It follows that the dual basis to $C_1, C_2, C_3, D \in N_1(X)$ is $H - E_1, H - E_2, H - E_3, H$. These four divisors are basepoint-free, hence nef, giving contractions of X to $\mathbf{P}^1$, $\mathbf{P}^1$, $\mathbf{P}^1$, and $\mathbf{P}^3$. Therefore, $\overline{\mathrm{Curv}(X)} = \mathbf{R}^{\geq 0}\{C_1, C_2, C_3, D\}$ and the nef cone is spanned by $H - E_1, H - E_2, H - E_3, H$. By the intersection numbers, the nef monoid in $\mathrm{Pic}(X)$ is spanned by those four divisors. Also, the line bundle $-K_X = 4H - E_1 - E_2 - E_3$ has degree 1 on the four curves C_1, C_2, C_3, D. It follows that every ample line bundle on X is $-K_X$ plus a nef divisor, hence $-K_X$ plus an

$\mathbf{N}$-linear combination of $H-E_1, H-E_2, H-E_3, H$.

For $1 \leq i \leq 3$, a general divisor in $|H-E_i|$ is the blow-up of $\mathbf{P}^2$ at 2 points, which is toric. A general divisor in $|H|$ is the blow-up of $\mathbf{P}^2$ at 3 non-collinear points, which is also toric. Therefore, Lemma 3.1 reduces Bott vanishing for X to the single line bundle $-K_X$. Since X is rigid, we know that $H^j(X, \Omega^2_X \otimes K^*_X) = H^j(X, TX)$ is zero for $j > 0$. It remains to show that $H^j(X, \Omega^1_X \otimes K^*_X) = 0$ for $j > 0$. Let Y be the blow-up of $\mathbf{P}^3$ along the lines F_1 and F_2; then Y is a toric variety. We have $\text{Pic}(Y) = \mathbf{Z}\{H, E_1, E_2\}$ and $\text{Nef}(Y) = \mathbf{R}^{\geq 0}\{H-E_1, H-E_2, H\}$. The curve F_3 in Y is a complete intersection $S_1 \cap S_2$ with $S_1 \sim H$ and $S_2 \sim H$. So $-K_Y = 4H - E_1 - E_2$, $-K_Y - S_1 = 3H - E_1 - E_2$, and $-K_Y - S_2 = 3H - E_1 - E_2$ are ample, and $-K_Y - S_1 - S_2 = 2H - E_1 - E_2$ is nef. By Lemma 6.3, it follows that $H^j(X, \Omega^1_X \otimes K^*_X) = 0$ for $j > 0$. That completes the proof of Bott vanishing for (4.6).

We now prove Bott vanishing for (4.7). The proof works in any characteristic. Here X is the blow-up of the flag manifold $W \subset \mathbf{P}^2 \times \mathbf{P}^2$ along disjoint curves F_1 of degree $(0,1)$ and F_2 of degree $(1,0)$. We will instead use a different description, mentioned by Mori-Mukai [31, Table 3]. Let S be the blow-up of $\mathbf{P}^2$ at a point p. Embed $F = \mathbf{P}^1$ into $Y := \mathbf{P}^1 \times S$ by the identity map on $\mathbf{P}^1$ and the inclusion into S as the strict transform of a line in $\mathbf{P}^2$ not containing p. Then X is the blow-up of Y along F. (The advantage of this description, for hand calculation, is that it expresses X as the blow-up of the toric variety Y along a single curve.) We can take S to be $\mathbf{P}^2$ blown up at the point $[0,0,1]$, with F defined by $y_2 = 0, x_0y_1 = x_1y_0$ in $\mathbf{P}^1 \times \mathbf{P}^2 = \{[x_0, x_1], [y_0, y_1, y_2]\}$.

Let A be the pullback to X of $O(1)$ on $\mathbf{P}^1$. Let B and H be the pullbacks of $O(1)$ by the contractions of S to $\mathbf{P}^1$ and $\mathbf{P}^2$. Then $\text{Pic}(X) = \mathbf{Z}\{A, B, H, E\}$ and $-K_X = 2A + B + 2H - E$. Let C be the strict transform in X of a curve $\mathbf{P}^1 \times \text{pt.}$ that meets the curve F in one point. Let D be the strict transform of a point in $\mathbf{P}^1$ times the (-1)-curve in S. (The curve D is disjoint from G.) Let G be the strict transform of a point in $\mathbf{P}^1$ times the strict transform in S of a line in $\mathbf{P}^2$ through p such that G meets F in one point. Let K be a fiber of the exceptional divisor $E \to F$. We have the intersection numbers:

	C	D	G	K
A	1	0	0	0
B	0	1	0	0
H	0	0	1	0
E	1	0	1	-1

The dual basis to C, D, G, K is given by $A, B, H, A+H-E$. These line bundles are basepoint-free, hence nef, giving contractions to $\mathbf{P}^1$ (twice) and $\mathbf{P}^2$ (twice). (Using the equation for G above, a basis for the sections of $A+H-E$ is given by $x_0y_1 - x_1y_0, x_0y_2, x_1y_2$; these equations define the curve G in Y as a scheme, which proves the basepoint-freeness of $A+H-E$ on X.)

It follows that $\overline{\text{Curv}(X)} = \mathbf{R}^{\geq 0}\{C, D, G, K\}$, and the nef cone is spanned by $A, B, H, A+H-E$. More strongly, the nef monoid in $\text{Pic}(X)$ is generated by these four divisors. Since $-K_X$ has degree 1 on C, D, G, and K, every ample line bundle on X is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of the $A, B, H, A+H-E$.

A general divisor in $|A|$, $|B|$, $|H|$, or $|A+H-E|$ is isomorphic to $\mathbf{P}^2$ blown up at 2 points, which is toric. By Lemma 3.1, this reduces Bott vanishing on X to the ample line bundle $-K_X$.

For $-K_X$ and Ω^2_X, this is easy: since X is rigid, we have $H^j(X, \Omega^2_X \otimes K_X^*) = H^j(X, TX) = 0$ for $j > 0$. It remains to prove Bott vanishing for $-K_X$ and Ω^1_X, meaning that $H^j(X, \Omega^1_X \otimes K_X^*) = 0$ for $j > 0$. Since $Y = \mathbf{P}^1 \times S$, we have $\text{Nef}(Y) = \mathbf{R}^{\geq 0}\{A, B, H\}$. We can view the curve F is a complete intersection $S_1 \cap S_2$ with $S_1 \sim H$ and $S_2 \sim A + B$. By Lemma 6.3, the desired vanishing holds if $-K_Y$, $-K_Y - S_1$, and $-K_Y - S_2$ are ample and $-K_Y - S_1 - S_2$ are nef. Indeed, $-K_Y = 2A + B + 2H$ is ample, $-K_Y - S_1 = 2A + B + H$ is ample, $-K_Y - S_2 = A + B + H$ is ample, and $-K_Y - S_1 - S_2 = A + B$ is nef. That completes the proof of Bott vanishing for (4.7).

We now prove Bott vanishing for the Fano 3-fold (4.8). The proof works in any characteristic. Here X is the blow-up of $(\mathbf{P}^1)^3$ along a curve F of degree $(0, 1, 1)$. We can take $F = \{x_1 = 0, y_0 z_1 = y_1 z_0\}$ in $(\mathbf{P}^1)^3 = \{([x_0, x_1], [y_0, y_1], [z_0, z_1])\}$. We have $\text{Pic}(X) = \mathbf{Z}\{A, B, C, E\}$ and $-K_X = 2A + 2B + 2C - E$. Let D_4 be a fiber of $E \to F$. Let D_1, D_2, D_3 be strict transforms of curves $\mathbf{P}^1 \times \text{pt.} \times \text{pt.}$, $\text{pt.} \times \mathbf{P}^1 \times \text{pt.}$, $\text{pt.} \times \text{pt.} \times \mathbf{P}^1$ that meet the curve F in one point each. We have the intersection numbers:

	D_1	D_2	D_3	D_4
A	1	0	0	0
B	0	1	0	0
C	0	0	1	0
E	1	1	1	-1

We compute that the dual basis to D_1, D_2, D_3, D_4 is $A, B, C, A + B + C - E$. These four divisors are basepoint-free, hence nef, using the equation for F above. They give contractions of X to $\mathbf{P}^1$, $\mathbf{P}^1$, $\mathbf{P}^1$, and a nodal quadric 3-fold. It follows that $\overline{\text{Curv}(X)} = \mathbf{R}^{\geq 0}\{D_1, D_2, D_3, D_4\}$, and the nef cone is spanned by the four divisors mentioned. More strongly, the nef monoid in $\text{Pic}(X)$ is generated by those four divisors. The line bundle $-K_X = 2A + 2B + 2C - E$ has degree 1 on all four curves D_1, D_2, D_3, D_4. So every ample line bundle is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of the four divisors mentioned.

A general divisor in $|A|$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, which is toric. A general divisor in $|B|$ or $|C|$ is $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at one point, which is toric. A general divisor in $|A + B + C - E|$ is $\mathbf{P}^2$ blown up at 3 non-collinear points, which is toric. By Lemma 3.1, this reduces Bott vanishing for (4.8) to the single line bundle $-K_X$.

Since X is rigid, we know that $H^j(X, \Omega^2_X \otimes K_X^*) = H^j(X, TX)$ is zero for $j > 0$. It remains to show that $H^j(X, \Omega^1_X \otimes K_X^*)$ is zero for $j > 0$. Let $Y = (\mathbf{P}^1)^3$; then $-K_Y = 2A + 2B + 2C$ and $\text{Nef}(Y) = \mathbf{R}^{\geq 0}\{A, B, C\}$. The curve F is a complete intersection $S_1 \cap S_2$ in Y with $S_1 \sim A$ and $S_2 \sim B + C$. By Lemma 6.3, the desired vanishing holds if $-K_Y$, $-K_Y - S_1$, and $-K_Y - S_2$ are ample, and $-K_Y - S_1 - S_2$ is nef. In this case, all four line bundles are ample. That completes the proof of Bott vanishing for (4.8).

11 The Fano 3-fold (5.1)

The Fano 3-fold (5.1) (with Picard number 5) turns out to be the hardest, for proving Bott vanishing. Our technique of reducing to general properties of toric varieties seems not to be strong enough in this case. Still, we give a meaningful proof using Hodge cohomology. We will prove Bott vanishing in characteristic not 2, since X is not rigid in characteristic 2.

One construction of (5.1) is similar to (4.4). Let J be a conic in Q. Then X is the blow-up of $\mathrm{Bl}_J Q$ along three fibers of the exceptional divisor. As suggested by Coates-Corti-Galkin-Kasprzyk, we instead view X as a hypersurface in a smooth toric 4-fold G [13, section 98]. Namely, G is obtained from $\mathbf{P}^4$ by blowing up along a plane Π and then along the fibers over three non-collinear points p_2, p_3, p_4 in Π. Let H be the pullback of $O(1)$ on $\mathbf{P}^4$, E_1 the (irreducible) exceptional divisor over Π, and E_2, E_3, E_4 the exceptional divisors over the three points in Π. Then $\mathrm{Pic}(G) = \mathbf{Z}\{H, E_1, E_2, E_3, E_4\}$, and the nef cone of G is spanned by H, $H - E_2$, $H - E_3$, $H - E_4$, $H - E_1 - E_2 - E_3 - E_4$, and $2H - E_2 - E_3 - E_4$, by [13]. (In their notation, $A = H - E_1 - E_2 - E_3 - E_4$, $B = E_1$, $C = E_2$, $D = E_3$, and $E = E_4$.) These divisors are basepoint-free, giving contractions of G to $\mathbf{P}^4$, $\mathbf{P}^3$ (three times), $\mathbf{P}^1$, and another toric 4-fold.

For completeness, let us list the intersection numbers between divisors and some curves on G (which span the cone of curves). This could be used to compute the nef cone of G, if we did not already know it. Namely, let C_1, C_2, C_3, C_4 be general fibers of the exceptional divisors E_1, E_2, E_3, E_4 on G. Also, for $2 \leq i \leq 4$, D_i will be a curve in X mapping to the line through p_j and p_k, where $\{i, j, k\} = \{2, 3, 4\}$; more precisely, let D_i be the section of $E_1 \to \Pi$ associated to a general plane in $\mathbf{P}^2$ containing that line. Then we have the intersection numbers:

	C_1	C_2	C_3	C_4	D_2	D_3	D_4
H	0	0	0	0	1	1	1
E_1	-1	1	1	1	-1	-1	-1
E_2	0	-1	0	0	0	1	1
E_3	0	0	-1	0	1	0	1
E_4	0	0	0	-1	1	1	0

The Fano 3-fold X is a general divisor in the linear system $|2H - E_2 - E_3 - E_4|$; that is, X is the strict transform in G of a quadric 3-fold Q containing p_2, p_3, p_4. So X is obtained from Q by blowing up along a conic F_1 containing those 3 points, and then along the fibers over those 3 points. Here $\mathrm{Pic}(X) = \mathbf{Z}\{H, E_1, E_2, E_3, E_4\}$, but (unfortunately) the nef cone of X turns out to be bigger than $\mathrm{Nef}(G)$. To list some curves on X: we can view C_1, C_2, C_3, C_4 above as curves on X, namely fibers in the four exceptional divisors. For $i = 2, 3, 4$, let K_i be the strict transform in X of a line in Q through p_i. Finally, let V be the section of $E_1 \to F_1$ associated to a hyperplane section of Q containing the conic F_1. Then we have the intersection numbers:

	C_1	C_2	C_3	C_4	K_2	K_3	K_4	V
H	0	0	0	0	1	1	1	2
E_1	-1	1	1	1	0	0	0	-1
E_2	0	-1	0	0	1	0	0	1
E_3	0	0	-1	0	0	1	0	1
E_4	0	0	0	-1	0	0	1	1

Using Magma, we compute that the dual cone to the cone spanned by these eight curves is spanned by H, $H - E_2$, $H - E_3$, $H - E_4$, $H - E_2 - E_3$, $H - E_2 - E_4$, $H - E_3 - E_4$, and $H - E_1 - E_2 - E_3 - E_4$. These eight divisors are basepoint-free, hence nef, giving contractions of X to the quadric 3-fold Q, $\mathbf{P}^3$ (three times), $\mathbf{P}^2$ (three times), and $\mathbf{P}^1$. It follows that the cone of curves of X is spanned by the eight curves above, and the nef cone is spanned by these eight divisors. More strongly, Magma checks that the nef monoid in $\mathrm{Pic}(X)$ is generated by these eight divisors. The line bundle $-K_X = 3H - E_1 - 2E_2 - 2E_3 - 2E_4$ has degree 1 on all eight curves. So every ample line bundle is $-K_X$ plus a nef divisor, hence $-K_X$ plus an $\mathbf{N}$-linear combination of the eight divisors mentioned.

For each of those divisors except $H - E_1 - E_2 - E_3 - E_4$, a general divisor in the linear system is isomorphic to $\mathbf{P}^2$ blown up at three non-collinear points, which is toric. A general divisor in $|H - E_1 - E_2 - E_3 - E_4|$ is $\mathbf{P}^2$ blown up at four points with no three collinear; this is the quintic del Pezzo surface, which is not toric but satisfies Bott vanishing. By Lemma 3.1, this reduces Bott vanishing for (5.1) to the single line bundle $-K_X$.

Since X is rigid in characteristic not 2, we know that $H^j(X, \Omega^2_X(-K_X)) = H^j(X, TX)$ is zero for $j > 0$. It remains to show that $H^j(X, \Omega^1_X(-K_X)) = 0$ for $j > 0$, which we will prove in any characteristic. This was shown by Belmans-Fatighenti-Tanturri, in terms of the isomorphic vector bundle $\Lambda^2 TX$, when the base field has characteristic zero [7, Appendix A].

To do this, recall that X is a hypersurface in the smooth toric 4-fold G. We have $-K_G = 2(2H - E_2 - E_3 - E_4) + (H - E_1 - E_2 - E_3 - E_4) = 5H - E_1 - 3E_2 - 3E_3 - 3E_4$, which is nef and big but not ample. So $-K_X$ is the restriction of $-K_G - X = 3H - E_1 - 2E_2 - 2E_3 - 2E_4$, which is also nef and big but not ample. Since G is a toric variety, it follows that $-K_G - X$ is basepoint-free [20, p. 68]. Consider the contraction $\pi\colon G \to F$ associated to the line bundle $-K_G - X$; this is a small contraction, and all contracted curves are disjoint from X. The singular set S of F consists of three disjoint $\mathbf{P}^1$'s. (The curves D_2, D_3, D_4 in G are contracted by π to points in these three components of S.) Here X is still a smooth hypersurface in F, but now $-K_F$ and $X = 2H - E_2 - E_3 - E_4$ (which pull back to $-K_G$ and X on G) are ample line bundles on F. The description of the toric Fano 4-fold F by a fan is given in Belmans-Fatighenti-Tanturri's file about the Fano 3-fold (5.1) [7]. The singularities of F are locally isomorphic to a smooth curve times the 3-fold node. We will prove the desired cohomology vanishing by relating X to the singular toric Fano 4-fold F, although the smooth toric 4-fold G also comes up in the argument.

We have the exact sequences of coherent sheaves $0 \to O_X(-X) \to \Omega^{[1]}_F|_X \to \Omega^1_X \to 0$ on X and $0 \to O(-X) \to O_F \to O_X \to 0$ on F. Tensoring the first sequence with $-K_F - X$, we have $0 \to O_X(-K_F - 2X) \to \Omega^{[1]}_F(-K_F - X)|_X \to \Omega^1_X(-K_X) \to 0$. So the desired vanishing would follow if $\Omega^{[1]}_F(-K_F - X)|_X$ has zero cohomology in positive degrees and $O_X(-K_F - 2X)$ has zero cohomology in degrees > 1. From the second sequence above, it suffices to show that (1) $\Omega^{[1]}_F(-K_F - X)$ has zero cohomology (on F) in positive degrees; (2) $\Omega^{[1]}_F(-K_F - 2X)$ has zero cohomology in degrees > 1; (3) $-K_F - 2X$ has zero cohomology (on F) in degrees > 1; and (4) $-K_F - 3X$ has zero cohomology in degrees > 2. Since $-2K_F - 2X$ and $-2K_F - 3X$ is ample, (3) and (4) are immediate from Kodaira vanishing on the toric variety F (part of Theorem 1.1). Also, since $-K_F - X$ is ample, (1) follows from Bott vanishing on F (Theorem 1.1). (This is the advantage of working with F rather than G.)

Here $A := -K_F - 2X = H - E_1 - E_2 - E_3 - E_4$ is nef, but F is singular, and so (2) is not immediate from Proposition 1.3. On the other hand, we know that $H^j(G, \Omega^1_G(A)) = 0$ for $j > 0$ by Proposition 1.3, and so it seems natural to compare the singular variety F with its resolution G. Let S be the singular locus of F, which is the disjoint union of three $\mathbf{P}^1$'s. Near S, the morphism $\pi\colon G \to F$ is locally a smooth curve times one of the two small resolutions of the 3-fold node $xy = zw$ in A^4. It follows, for example using the theorem on formal functions [23, Theorem III.11.1], that the sheaf $R^j\pi_*\Omega^1_G$ is isomorphic to $\Omega^{[1]}_F$ for $j = 0$, O_S for $j = 1$, and zero otherwise. Equivalently, we have an exact triangle $\Omega^{[1]}_F \to R\pi_*\Omega^1_G \to O_S[-1]$ in the derived category of F. So we have a long exact sequence

$$H^1(F, \Omega^{[1]}_F) \to H^1(G, \Omega^1_G) \to H^0(S, O_S) \to H^2(F, \Omega^{[1]}_F) \to \cdots .$$

Here $H^0(S, O_S) \cong k^3$. I claim that the map from $H^1(G, \Omega^1_G)$ to $H^0(S, O_S)$ is surjective. It is equivalent to show that the image of $H^1(F, \Omega^{[1]}_F) \to H^1(G, \Omega^1_G)$ has codimension at least 3. So it suffices to find a surjection $H^1(G, \Omega^1_G) \to k^3$ that is zero on the image of $H^1(F, \Omega^{[1]}_F)$. In the notation above, the curves D_2, D_3, D_4 in G map to k-points in the three components of S. Then the restriction map from $H^1(G, \Omega^1_G)$ to $\oplus_{i=2}^4 H^1(D_i, \Omega^1_{D_i}) = k^3$ vanishes on $H^1(F, \Omega^{[1]}_F)$ (because the curves D_i map to points in F). So it suffices to show that the composition $\mathrm{Pic}(G) \to H^1(G, \Omega^1_G) \to k^3$ is surjective. This map gives the degrees of line bundles on G on the three curves D_i. The intersection numbers of H, E_3, E_4 with these three curves are $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, which has determinant 1. So, for k of any characteristic, we have shown that $H^1(G, \Omega^1_G) \to H^0(S, O_S)$ is surjective. (By the exact sequence above, it follows that $H^2(F, \Omega^{[1]}_F) = 0$.)

We want to compute the related groups $H^j(F, \Omega^{[1]}_F(A))$. By the exact triangle $\Omega^{[1]}_F(A) \to R\pi_*\Omega^1_G(A) \to A|_S[-1]$ in the derived category of F, we have a long exact sequence

$$H^1(F, \Omega^{[1]}_F(A)) \to H^1(G, \Omega^1_G(A)) \to H^0(S, A) \to H^2(F, \Omega^{[1]}_F(A)) \to \cdots .$$

The line bundle A has degree 1 on each $\mathbf{P}^1$ component of $S = S_2 \coprod S_3 \coprod S_4$, using that each of these curves on F is the image of a curve numerically equivalent to C_1 on G. Since A is nef on the toric variety G, it is basepoint-free. It follows that the restriction $H^0(G, A) \to H^0(S_i, A) \cong k^2$ is surjective for $i = 2, 3, 4$. Combining this with the previous paragraph, the composition $H^1(G, \Omega^1_G) \otimes_k H^0(G, A) \to H^1(G, \Omega^1_G(A)) \to H^0(S, A)$ is surjective. Also, $H^j(G, A) = 0$ for $j > 0$ and (by Proposition 1.3) $H^j(G, \Omega^1_G(A)) = 0$ for $j > 1$. Therefore, the exact sequence above shows that $H^j(F, \Omega^{[1]}_F(A)) = 0$ for all $j > 1$. This is statement (2), above. That completes the proof that $H^j(X, \Omega^1_X(-K_X)) = 0$ for $j > 0$. Thus Bott vanishing holds for the Fano 3-fold (5.1).

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