Bott vanishing for algebraic surfaces

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For William Fulton on his eightieth birthday

A smooth projective variety X over a field is said to satisfy *Bott vanishing* if

$$H^j(X, \Omega^i_X \otimes L) = 0$$

for all ample line bundles L, all $i \geq 0$, and all $j > 0$. Bott proved this when X is projective space. Danilov and Steenbrink extended Bott vanishing to all smooth projective toric varieties; proofs can be found in [4, 7, 28, 15].

What does Bott vanishing mean? It does not have a clear geometric interpretation in terms of the classification of algebraic varieties. But it is useful when it holds, as a sort of preprocessing step, since the vanishing of higher cohomology lets us compute the spaces of sections of various important vector bundles. Bott vanishing includes Kodaira vanishing as a special case (where i equals $n := \dim X$), but it says much more.

For example, any Fano variety that satisfies Bott vanishing must be rigid, since $H^1(X, TX) = H^1(X, \Omega^{n-1}_X \otimes K^*_X) = 0$ for X Fano. So Bott vanishing holds for only finitely many smooth complex Fano varieties in each dimension. Even among rigid Fano varieties, Bott vanishing fails for quadrics of dimension at least 3 and for Grassmannians other than projective space [7, section 4]. As a result, Achinger, Witaszek, and Zdanowicz asked whether a rationally connected variety that satisfies Bott vanishing must be a toric variety [1, after Theorem 4].

In this paper, we exhibit several new classes of varieties that satisfy Bott vanishing. First, we answer Achinger-Witaszek-Zdanowicz's question: there are non-toric rationally connected varieties that satisfy Bott vanishing, since *Bott vanishing holds for the quintic del Pezzo surface* (Theorem 2.1). Over an algebraically closed field, a quintic del Pezzo surface is isomorphic to the moduli space $\overline{M_{0,5}}$ of 5-pointed stable curves of genus zero. It is the only rigid del Pezzo surface that is not toric: del Pezzo surfaces of degree at least 5 are rigid, and those of degree at least 6 are toric. (The quintic del Pezzo surface also does not have a lift of the Frobenius endomorphism from $\mathbf{Z}/p$ to $\mathbf{Z}/p^2$, a property known to imply Bott vanishing [7], [1, Proposition 7.1.4].) In view of this example, there is a good hope of finding more Fano or rationally connected varieties that satisfy Bott vanishing.

We also consider varieties that are not rationally connected, with most of the paper devoted to K3 surfaces. Bott vanishing holds for abelian varieties over any field: it reduces to Kodaira vanishing, since the tangent bundle is trivial. On the other hand, Riemann-Roch shows that Bott vanishing fails for all K3 surfaces of degree less than 20 (Theorem 3.1). But recent work of Ciliberto-Dedieu-Sernesi and Feyzbakhsh [9, 14] implies: *Bott vanishing holds for all K3 surfaces of degree 20 or at least 24 with Picard number 1* (Theorem 3.2). Version 2 of this paper on the arXiv gave a more elementary proof, not using Feyzbakhsh's work on Mukai's program (reconstructing a K3 surface from a curve), but here we give a short proof using her work. Surprisingly, Bott vanishing fails in degree 22.

More strongly, we end up with a clear geometric understanding of the meaning of Bott vanishing for a K3 surface with any Picard number; see Theorems 5.1, 6.2, and 6.1. The key question is whether $H^1(X, \Omega^1_X \otimes B)$ is zero for an ample line bundle B. This cohomology group has a direct geometric meaning, related to the map from the moduli space of curves on K3 surfaces to the moduli space of curves (section 3).

Roughly speaking, the failure of this vanishing for a K3 surface is caused *either* by elliptic curves of low degree on the surface, *or* by the existence of a (possibly singular) Fano 3-fold in which the K3 surface is a hyperplane section. The proofs build on a long development, starting with the work of Beauville, Mori, and Mukai about moduli spaces of K3 surfaces, and leading up to recent advances by Arbarello-Bruno-Sernesi and Ciliberto-Dedieu-Sernesi [5, 26, 27, 3, 9]. We give a complete description of all K3 surfaces X with an ample line bundle B of high degree such that $H^1(X, \Omega^1_X \otimes B)$ is not zero. The most novel aspect of the paper is our analysis of what happens when there is an elliptic curve of low degree (Theorem 6.1). (In other terminology, this concerns K3 surfaces that are monogonal, hyperelliptic, trigonal, or tetragonal.) It turns out that the crucial issue is whether an elliptic fibration has a certain special type of singular fiber.

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1 Notation

We take a *variety* over a field k to mean an integral separated scheme of finite type over k. A *curve* means a variety of dimension 1. So, in particular, a curve is irreducible. A property is said to hold for *general* (resp. *very general*) complex points of a variety Y if it holds outside a finite (resp. countable) union of closed subvarieties not equal to Y.

On a smooth variety, we often identify line bundles with divisors modulo linear equivalence. For example, the tensor product $A \otimes B$ of two line bundles may also be written as $A + B$. A line bundle is *primitive* if it cannot be written as a positive integer at least 2 times some line bundle.

2 Bott vanishing for the quintic del Pezzo surface

Theorem 2.1. *Let X be a del Pezzo surface of degree 5 over a field k. Then X satisfies Bott vanishing, but is not toric.*

Proof. It suffices to prove the theorem after extending k, and so we can assume that k is algebraically closed. In this case, there is a unique del Pezzo surface X (a smooth projective surface with ample anticanonical bundle K_X^*) of degree 5 over k, up to isomorphism. It can be described as the blow-up of $\mathbf{P}^2$ at any set of 4 points with no three on a line [23, Remark 24.4.1]. Here X has finite automorphism group, because any automorphism of X in the identity component of $\mathrm{Aut}(X)$ would pass to an automorphism of $\mathbf{P}^2$ (that is, an element of $PGL(3,k)$) that fixes the 4 chosen points, and such an automorphism must be the identity. In particular, X is not a toric variety. (In fact, the automorphism group of X is the symmetric group S_5, but we will not use that.)

The Picard group of X is isomorphic to $\mathbf{Z}^5$, and so Bott vanishing must be checked for a fairly large (infinite) class of ample line bundles. We argue as follows. Recall the Kodaira-Akizuki-Nakano vanishing theorem [22, Theorem 4.2.3], [13]:

Theorem 2.2. *(1) Every smooth projective variety over a field of characteristic zero satisfies Kodaira-Akizuki-Nakano vanishing:*

$$H^j(X, \Omega^i \otimes L) = 0$$

for all ample line bundles L and all $i + j > \dim(X)$.

(2) Let X be a smooth projective variety over a perfect field of characteristic $p > 0$. If X lifts to $W_2(k)$ and X has dimension $\leq p$, then X satisfies KAN vanishing (as in (1)).

It follows that the quintic del Pezzo surface X satisfies KAN vanishing: the hypotheses of (2) hold if the algebraically closed field k has characteristic p. (For example, if we view X as the blow-up of $\mathbf{P}^2$ at four k-points, then those points can be lifted to $W_2(k)$, and so X lifts to $W_2(k)$.) Thus we know that $H^j(X, \Omega^2 \otimes L) = 0$ for all ample line bundles L and all $j > 0$. Since $K_X^* = (\Omega_X^2)^*$ is ample, it follows that $H^j(X, L) = 0$ for ample L and $j > 0$. Also by KAN vanishing, we have $H^2(X, \Omega^1 \otimes L) = 0$ for ample L. To prove Bott vanishing, it remains to show that $H^1(X, \Omega^1 \otimes L) = 0$ for ample L.

For any del Pezzo surface X of degree at most 7, the cone of curves is spanned by the finitely many lines in X (or equivalently, (-1)-curves, meaning curves C in X isomorphic to $\mathbf{P}^1$ with $C^2 = -1$; then $(-K_X) \cdot C = 1$) [12, section 6.5]. Therefore, a line bundle L on X is nef if and only if it has nonnegative degree on all (-1)-curves in X, and it is ample if and only if it has positive degree on all (-1)-curves in X.

We return to the del Pezzo surface X of degree 5 (in which case there are 10 (-1)-curves, shown in Figure 1). Let L be any ample line bundle on X, and let a be the minimum degree of L on the (-1)-curves, which is a positive integer. Since $-K_X$ has degree 1 on each (-1)-curve, L can be written (using additive notation for line bundles) as

$$L = a(-K_X) + M$$

for some nef line bundle M on X which has degree zero on some (-1)-curve.

Choose a (-1)-curve E on which M has degree zero, and let Y be the smooth projective surface obtained by contracting E (by Castelnuovo's contraction theorem). Then Y is a del Pezzo surface of degree 6, and such a surface is toric. Since M has degree 0 on E, the isomorphism $\mathrm{Pic}(X) = \mathrm{Pic}(Y) \oplus \mathbf{Z}$ for a blow-up implies that M is pulled back from a line bundle on Y, which we also call M. Clearly M is nef on Y. By Bott vanishing on Y, we have

$$H^1(Y, \Omega^1 \otimes K_Y^* \otimes M) = 0,$$

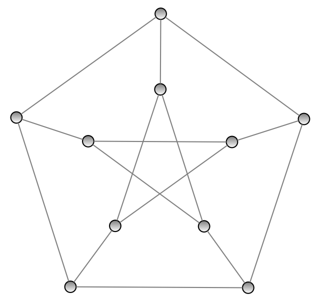

Figure 1: Dual graph of the 10 (-1)-curves on the quintic del Pezzo surface

using that K_Y^* is ample. Here $\Omega^1_Y \otimes K_Y^* \cong TY$ (as on any surface), and so

$$H^1(Y, TY \otimes M) = 0.$$

For any blow-up $\pi\colon X \to Y$ of a point y on a smooth surface Y, we have $R\pi_*(TX) \cong \pi_*(TX) \cong TY \otimes I_{y/Y}$, where $I_{y/Y}$ is the ideal sheaf of y in Y. (That is, vector fields on X are equivalent to vector fields on Y that vanish at y.) We have an exact sequence of coherent sheaves on Y,

$$0 \to I_{y/Y} \to O_Y \to O_y \to 0.$$

Tensoring with the vector bundle $TY \otimes M$ gives another exact sequence,

$$0 \to TY \otimes M \otimes I_{y/Y} \to TY \otimes M \to (TY \otimes M)|_y \to 0.$$

Combining this with the isomorphism above gives a long exact sequence of cohomology:

$$H^0(Y, TY \otimes M) \to (TY \otimes M)|_y \to H^1(X, TX \otimes \pi^*(M)) \to H^1(Y, TY \otimes M).$$

Here $H^1(Y, TY \otimes M) = 0$ by Bott vanishing as above. Therefore, to show that $H^1(X, TX \otimes \pi^*(M)) = 0$, it suffices to show that the rank-2 vector bundle $TY \otimes M$ is spanned at the point y by its global sections. This follows if we can show that TY and M are spanned at y by their global sections. For TY, this is clear by the vector fields coming from the action of the torus $T = (G_m)^2$ on Y, since y must be in the open T-orbit. (The blow-up of Y at a point not in the open T-orbit would contain a (-2)-curve and hence could not be a del Pezzo surface.) Also, every nef line bundle M on a toric variety Y is basepoint-free [16, section 3.4]. Thus we have shown that $H^1(X, TX \otimes \pi^*(M)) = 0$.

To prove Bott vanishing for X, as discussed above, we have to show that $H^1(X, \Omega^1 \otimes (K_X^*)^{\otimes a} \otimes \pi^*(M)) = 0$ for all positive integers a. Equivalently, we want $H^1(X, TX \otimes (K_X^*)^{\otimes a-1} \otimes \pi^*(M)) = 0$ for all positive integers a. We have proved this for $a = 1$. By induction, suppose we know this statement for a, and then we will show that $H^1(X, TX \otimes (K_X^*)^{\otimes a} \otimes \pi^*(M)) = 0$.

On X (as on any del Pezzo surface of degree at least 3), the line bundle K_X^* is very ample, and so it has a section whose zero locus is a smooth curve C. By the adjunction formula, K_C is trivial; that is, C has genus 1. We have an exact sequence

$$0 \to O_X(-C) \to O_X \to O_C \to 0$$

of coherent sheaves on X, where $O(-C) \cong K_X$. Tensoring with the vector bundle $TX \otimes (K_X^*)^{\otimes a} \otimes \pi^*(M)$ gives another exact sequence of sheaves, and hence a long exact sequence of cohomology:

$$H^1(X, TX \otimes (K_X^*)^{\otimes a-1} \otimes \pi^*(M)) \to H^1(X, TX \otimes (K_X^*)^{\otimes a} \otimes \pi^*(M)) \\ \to H^1(C, (TX \otimes (K_X^*)^{\otimes a} \otimes \pi^*(M))|_C).$$

By induction, the first group shown is zero. Also, the restriction of TX to C is an extension

$$0 \to TC \to TX|_C \to N_{C/X} \to 0,$$

where $N_{C/X} \cong (K_X^*)|_C$ by definition of C. Since C has genus 1, this says that the restriction of TX to C is an extension of two line bundles of nonnegative degree. Since K_X^* is ample on X, $\pi^*(M)$ is nef, and a is positive, it follows that $TX \otimes (K_X^*)^{\otimes a} \otimes \pi^*(M)$ restricted to C is an extension of two line bundles of positive degree. Since C has genus 1, H^1 of every line bundle of positive degree on C is zero. We conclude that the group on the right of the exact sequence above is zero (like the group on the left). Therefore,

$$H^1(X, TX \otimes (K_X^*)^{\otimes a} \otimes \pi^*(M)) = 0,$$

which completes the induction. We have shown that X satisfies Bott vanishing. $\square$

There are also higher-dimensional Fano varieties which satisfy Bott vanishing but are not toric, in view of:

Lemma 2.3. *Let X and Y be smooth projective varieties over an algebraically closed field. Suppose that $H^1(X, O) = 0$. If X and Y satisfy Bott vanishing, then so does $X \times Y$.*

Proof. Since $H^1(X, O) = 0$, we have $\mathrm{Pic}(X \times Y) = \mathrm{Pic}(X) \oplus \mathrm{Pic}(Y)$ [18, exercise III.12.6]. That is, every line bundle on $X \times Y$ has the form $\pi_1^* L \otimes \pi_2^* M$ for some line bundles L on X and M on Y, where π_1 and π_2 are the two projections of $X \times Y$.

By the Künneth formula [33, Tag 0BEC],

$$H^0(X \times Y, \pi_1^* L \otimes \pi_2^* M) \cong H^0(X, L) \otimes_k H^0(Y, M).$$

Therefore, $\pi_1^* L \otimes \pi_2^* M$ is very ample on $X \times Y$ if and only if L and M are very ample. It follows that $\pi_1^* L \otimes \pi_2^* M$ is ample on $X \times Y$ if and only if L and M are ample.

Assume that X and Y satisfy Bott vanishing. We need to show that

$$H^j(X \times Y, \Omega^i_{X \times Y} \otimes \pi_1^* L \otimes \pi_2^* M) = 0$$

for all $j > 0$, $i \geq 0$, and L and M ample line bundles. Here $\Omega^i_{X \times Y} = \oplus_m \pi_1^* \Omega^m_X \otimes \pi_2^* \Omega^{i-m}_Y$. So the desired vanishing follows from Bott vanishing on X and Y using the Künneth formula. $\square$

Remark 2.4. The Kawamata-Viehweg vanishing theorem extends Kodaira vanishing to nef and big line bundles, but it seems unreasonable to ask when Bott vanishing holds for nef and big line bundles. Indeed, Bott vanishing fails for a nef and big line bundle on the blow-up of $\mathbf{P}^2$ at a point, which is about as simple as you can get. Even KAN vanishing fails for a nef and big line bundle on the blow-up of $\mathbf{P}^3$ at a point [22, Example 4.3.4].

3 Bott vanishing for K3 surfaces of Picard number 1

We now show that Bott vanishing fails for K3 surfaces of degree less than 20 or equal to 22, while it holds for all K3 surfaces of degree 20 or at least 24 with Picard number 1. We give a quick proof by applying recent work of Ciliberto-Dedieu-Sernesi, Arbarello-Bruno-Sernesi, and Feyzbakhsh, which we discuss in more detail in section 4. In later sections, we consider what happens for K3 surfaces with Picard number greater than 1.

The less precise statement that Bott vanishing holds for very general K3 surfaces of degree 20 or at least 24 follows from the work of Beauville, Mori, and Mukai in the 1980s [5, section 5.2], [26, Theorem 1], [27, Theorem 7].

Note that Ciliberto, Dedieu, Galati, and Knutsen recently proved the analog of Beauville-Mori-Mukai's result for Enriques surfaces, in particular computing $H^1(X, \Omega^1 \otimes B)$ for (X, B) a general member of any component of the moduli space of polarized Enriques surfaces [8]. By analogy with the results in this paper for K3 surfaces, it would be interesting to describe the precise locus where $H^1(X, \Omega^1 \otimes B)$ is not zero.

We define a *K3 surface* to be a smooth projective surface X with trivial canonical bundle and $H^1(X, O) = 0$. A *polarized K3 surface* of degree $2a$ is a K3 surface X together with a primitive ample line bundle B such that $B^2 = 2a$. The degree of a polarized K3 surface must be even, because the intersection form on $H^2(X, \mathbf{Z})$ is even. Sometimes we call (X, B) simply a K3 surface of degree $2a$.

Theorem 3.1. *Let X be a K3 surface with an ample line bundle A of degree A^2 less than 20. Then Bott vanishing fails for X.*

Proof. It suffices to show that $H^1(X, \Omega^1_X \otimes A)$ is not zero. That holds if the Euler characteristic $\chi(X, \Omega^1_X \otimes A)$ is negative. Writing z for the class of a point in $H^4(X)$, Riemann-Roch gives:

$$\begin{aligned}\chi(X, \Omega^1_X \otimes A) &= \int_X \mathrm{td}(TX)\,\mathrm{ch}(\Omega^1_X \otimes A)\\ &= \int_X (1 + 0 + 2z)(2 + 0 - 24z)(1 + c_1(A) + c_1(A)^2/2)\\ &= c_1(A)^2 - 20.\end{aligned}$$

□

We deduce the following result from the work of Ciliberto-Dedieu-Sernesi and Feyzbakhsh [9, 14].

Theorem 3.2. *Let (X, B) be a polarized complex K3 surface with Picard number 1 and of degree 20 or at least 24. Then $H^1(X, \Omega^1_X \otimes B) = 0$. On the other hand, for every polarized K3 surface (X, B) of degree 22, $H^1(X, \Omega^1_X \otimes B) \neq 0$.*

Note that B is a *primitive* ample line bundle in Theorem 3.2. There is an irreducible (19-dimensional) moduli space of polarized complex K3 surfaces of degree $2a$, for each positive integer a [19, Corollary 6.4.4]. Moreover, a very general K3 surface X in this moduli space has Picard number 1 [19, proof of Corollary 14.3.1].

Proof. Let $\mathcal{P}_g$ be the moduli stack of pairs (X, C) with X a K3 surface and C a smooth curve of genus g in X such that $O(C)$ is a primitive ample line bundle on X. (Then $O(C)$ has degree $2g - 2$ on X.) Let $\mathcal{M}_g$ be the moduli stack of curves of genus g. There is a morphism of stacks

$$f_g \colon \mathcal{P}_g \to \mathcal{M}_g,$$

taking (X, C) to the curve C. Beauville observed that $H^1(X, \Omega^1_X \otimes O(C))^* \cong H^1(X, TX \otimes O(-C))$ can be identified with the kernel of the derivative of f_g at (X, C) [5, section 5.2]. Therefore, Theorem 3.2 for *general* polarized K3 surfaces reduces to Mukai's theorem (completing his work with Mori) that f_g is generically finite if and only if $g = 11$ or $g \geq 13$ (corresponding to polarized K3 surfaces of degree 20 or at least 24) [26, Theorem 1], [27, Theorem 7]. From this point of view, describing the locus where $H^1(X, \Omega^1_X \otimes B)$ is not zero amounts to determining the ramification locus of the morphism f_g.

Arbarello-Bruno-Sernesi and Feyzbakhsh recently strengthened Mukai's result by showing that when $g = 11$ or $g \geq 13$, the morphism f_g is injective at all pairs (X, C) with X of Picard number 1 [2, 14]. We want to show that when $g = 11$ or $g \geq 13$, the *derivative* of f_g is also injective at all pairs (X, C) with X of Picard number 1.

The failure of Bott vanishing for K3 surfaces (X, B) of degree 22 follows from the existence of a smooth Fano 3-fold W with Picard group generated by $-K_W$ and genus 12 [27, Proposition 6]. (The genus g is defined by $(-K_W)^3 = 2g - 2$. The possible genera of smooth Fano 3-folds with Picard group generated by $-K_W$ are $2 \leq g \leq 10$ and $g = 12$.) Indeed, Beauville showed by a short deformation-theory argument that a general hyperplane section of a general deformation W' of W gives a general K3 surface X of degree 22. But then a hyperplane section $C \subset X$ is the intersection of W' with a codimension-2 linear space. So there is a whole $\mathbf{P}^1$ of K3 surfaces (generically not isomorphic) which all have the same curve C as a hyperplane section. That is, $f_{12} \colon \mathcal{P}_{12} \to \mathcal{M}_{12}$ is not generically finite, and hence Bott vanishing fails for all K3 surfaces (X, B) of degree 22.

Now let (X, B) be any polarized K3 surface of degree 20 or at least 24 with Picard number 1. The assumptions imply that any smooth curve C in the linear system of B has Clifford index at least 3, as discussed in section 4. Using this together with $B^2 \geq 20$, Ciliberto-Dedieu-Sernesi showed that $h^1(X, \Omega^1 \otimes B) = \dim(\ker(df_g|_{(X,C)}))$ is equal to the fiber dimension $\dim(f_g^{-1}(C))$ near (X, C) [9, Theorem 2.6]. Using that X has Picard number 1 and B^2 is 20 or at least 24, Arbarello-Bruno-Sernesi and Feyzbakhsh showed that C lies on a unique K3 surface of Picard number 1 [2], [14, Theorem 1.1]. Therefore, $f_g^{-1}(C)$ is a single point in a neighborhood of (X, C). Combining these two results shows that $H^1(X, \Omega^1 \otimes B) = 0$. $\square$

We now deduce the full statement of Bott vanishing for K3 surfaces with Picard number 1:

Theorem 3.3. *Let X be a complex polarized K3 surface with Picard number 1 and of degree 20 or at least 24. Then X satisfies Bott vanishing.*

Note that, without the assumption of Picard number 1, Bott vanishing does not hold for any nonempty Zariski open subset of the moduli space of K3 surfaces of given degree $2a \geq 20$. Indeed, there is a countably infinite set of divisors in that moduli space corresponding to K3 surfaces that also have an ample line bundle of degree < 20, and Bott vanishing fails for those K3 surfaces by Theorem 3.1.

On the other hand, it is arguably more natural to ask when Bott vanishing holds for positive multiples of the given line bundle B, rather than for all ample line bundles. By Lemma 3.5, it is equivalent to determine the locus of polarized K3 surfaces (X, B) such that $H^1(X, \Omega^1_X \otimes B)$ is not zero. The rest of the paper will focus on that problem.

Proof. (Theorem 3.3) Write $\text{Pic}(X) = \mathbf{Z} \cdot B$ with B ample. Then every ample line bundle on X is a positive multiple of B.

Kodaira vanishing (Theorem 2.2) gives that $H^i(X, \Omega^2_X \otimes L) = 0$ for L ample and $i > 0$. Since $K_X = \Omega^2_X$ is trivial, it follows that $H^i(X, L) = 0$ for L ample and $i > 0$. Next, KAN vanishing (Theorem 2.2) gives that $H^2(X, \Omega^1_X \otimes L) = 0$ for L ample.

It remains to show that $H^1(X, \Omega^1_X \otimes L) = 0$ for every ample line bundle L on X. By Theorem 3.2, since B^2 is 20 or at least 24, we know that $H^1(X, \Omega^1_X \otimes B) = 0$, where B is the ample generator. We will go from there to the result for all positive multiples of B (thus for all ample line bundles on X).

We recall Saint-Donat's sharp results about linear systems on K3 surfaces [32], [25, Theorem 5]:

Theorem 3.4. *Let X be a K3 surface over an algebraically closed field of characteristic not 2. Let B be a nef line bundle on X. Then:*

(1) B is not basepoint-free if and only if there is a curve E in X such that $E^2 = 0$ and $B \cdot E = 1$.

(2) Assume that $B^2 \geq 4$. Then B is not very ample if and only if there is (a) a curve E with $E^2 = 0$ such that $B \cdot E$ is 1 or 2, (b) a curve E such that $E^2 = 2$ and $B \sim 2E$, or (c) a curve E such that $E^2 = -2$ and $B \cdot E = 0$. (So, if B is ample and $B^2 \geq 10$, B fails to be very ample if and only if there is a curve E in X such that $E^2 = 0$ and $B \cdot E$ is 1 or 2.)

The following lemma completes the proof of Theorem 3.3. □

Lemma 3.5. *Let X be a complex K3 surface with a basepoint-free ample line bundle B. (In particular, this holds if $\text{Pic}(X) = \mathbf{Z} \cdot B$ and B is ample.) If $H^1(X, \Omega^1_X \otimes B) = 0$, then $H^1(X, \Omega^1_X \otimes B^{\otimes j}) = 0$ for all $j \geq 1$.*

Proof. First, if $\text{Pic}(X) = \mathbf{Z} \cdot B$ and B is ample, then B is basepoint-free by Theorem 3.4.

Let X be a complex K3 surface with a basepoint-free ample line bundle B. By Bertini's theorem, there is a smooth curve D in the linear system $|B|$. This gives a short exact sequence of sheaves $0 \to O_X \to B \to B|_D \to 0$. Tensoring with Ω^1_X and taking cohomology gives an exact sequence

$$H^1(X, \Omega^1_X \otimes B) \to H^1(D, \Omega^1_X \otimes B) \to H^2(X, \Omega^1_X).$$

We are given that $H^1(X, \Omega^1 \otimes B)$ is zero, and $H^2(X, \Omega^1_X)$ is zero since X is a K3 surface; so $H^1(D, \Omega^1_X \otimes B) = 0$. Next, since B restricted to the curve D is basepoint-free, it is represented by an effective divisor S on D. This gives a short exact sequence of sheaves $0 \to O_D \to B|_D \to B|_S \to 0$, and hence a surjection

$$H^1(D, \Omega^1_X \otimes B^{\otimes j-1}) \to H^1(D, \Omega^1_X \otimes B^{\otimes j})$$

for any $j \in \mathbf{Z}$ (using that S has dimension 0). By induction on j, it follows that $H^1(D, \Omega^1_X \otimes B^{\otimes j}) = 0$ for all $j \geq 1$. We now make another induction on j using the analogous exact sequence on X:

$$H^1(X, \Omega^1_X \otimes B^{\otimes j-1}) \to H^1(X, \Omega^1_X \otimes B^{\otimes j}) \to H^1(D, \Omega^1_X \otimes B^{\otimes j}).$$

Since $H^1(X, \Omega^1_X \otimes B) = 0$, it follows that $H^1(X, \Omega^1_X \otimes B^{\otimes j}) = 0$ for all $j \geq 1$. $\square$

4 Failure of Bott vanishing on a K3 surface in terms of elliptic curves of low degree

Theorem 4.1 clarifies the meaning of Bott vanishing for a K3 surface X. Namely, if $H^1(X, \Omega^1_X \otimes B) \neq 0$ for an ample line bundle B, then one of three conditions must hold: B^2 is less than 20, there is an elliptic curve of low degree with respect to B, or X is an anticanonical divisor in a singular Fano 3-fold Y with $B = -K_Y|_X$. The proof is based on recent work of Ciliberto, Dedieu, and Sernesi, which in turn buids on the work of Arbarello, Bruno, and Sernesi [3, 9].

The main result of Arbarello-Bruno-Sernesi was that a Brill-Noether general curve C of genus at least 12 is the hyperplane section of a (possibly singular) K3 surface, or of a limit of K3 surfaces (a "fake K3"), if and only if the Wahl map of C is not surjective. The proof was based on a new vanishing theorem for the square of the ideal sheaf of a projective curve. Ciliberto-Dedieu-Sernesi applied that work on curves to give criteria for a projective K3 surface to be a hyperplane section of a (possibly singular) Fano 3-fold.

Theorem 4.1 characterizes exactly when $H^1(X, \Omega^1 \otimes B)$ is zero except when X contains an elliptic curve of low degree. That case is studied in section 6, which includes a complete answer for B of high degree.

The classification of Fano 3-folds with canonical Gorenstein singularities remains open. As a result, Theorem 4.1 is not as explicit an answer as one might like. Nonetheless, it is a strong statement, from which we draw more specific consequences in the rest of the paper. For our purpose, we only want the classification of Fano 3-folds with *isolated* canonical Gorenstein singularities, which may be within reach.

In particular, Theorem 4.1 implies that for K3 surfaces (X, B) with no elliptic curve of low degree, the nonvanishing of $H^1(X, \Omega^1 \otimes B)$ is a Noether-Lefschetz condition. More precisely, this group is nonzero if and only if (X, B) belongs to certain irreducible components of the space of K3 surfaces with Picard group containing one of a finite list of lattices. (This follows from Theorem 4.1 by Beauville's deformation-theory argument, which works with no change for Fano 3-folds with isolated singularities [5, Theorem].) The lattices that occur are exactly the Picard lattices of the Fano 3-folds with isolated canonical Gorenstein singularities, these being not yet known.

Theorem 4.1. *Let X be a complex K3 surface with an ample line bundle B such that there is no curve E in X with $E^2 = 0$ and $1 \leq B \cdot E \leq 4$. Then $H^1(X, \Omega^1 \otimes B) \neq$*

0 *if and only if* $B^2 < 20$ *or* X *is a smooth anticanonical divisor in some Fano 3-fold* Y *with at most isolated canonical Gorenstein singularities such that* $B = -K_Y|_X$.

Note that any curve E in a K3 surface with $E^2 = 0$ is a fiber of an elliptic fibration, for example by Theorem 3.4. Using work of Prokhorov, the final case of Theorem 4.1 implies that $B^2 \leq 72$ (Theorem 5.1).

Proof. If $B^2 < 20$, then $H^1(X, \Omega^1_X \otimes B) \neq 0$ by Riemann-Roch (Theorem 3.1). If X is a smooth anticanonical divisor in some Fano 3-fold Y with at most isolated canonical Gorenstein singularities such that $B = -K_Y|_X$, then $H^1(X, \Omega^1_X \otimes B) \neq 0$. This follows from Lvovski's theorem on extensions of projective varieties [9, Theorem 0.1, Lemma 3.5]. (One could also prove this by extending Mukai's argument from section 3.) Conversely, assume that $B^2 \geq 20$. We want to show that if $H^1(X, \Omega^1_X \otimes B) \neq 0$, then X is an anticanonical divisor.

By Theorem 3.4, B is very ample, giving an embedding $X \subset \mathbf{P}^g$ where $B^2 = 2g - 2$. Choose a smooth hyperplane section C in X (so C has genus g, and the embedding $C \to \mathbf{P}^{g-1}$ is the canonical embedding).

For a line bundle L on a smooth projective curve C, the *Clifford index* $\text{Cliff}(C, L)$ is $\deg(L) - 2h^0(C, L) + 2$. For C of genus at least 4, the *Clifford index* of C is

$$\text{Cliff}(C) := \min\{\text{Cliff}(C, L) : h^0(C, L) \geq 2 \text{ and } h^1(C, L) \geq 2\}.$$

I claim that the curve $C \subset X$ above has Clifford index at least 3. Several approaches are possible, but we use the following result of Knutsen, inspired by earlier work of Green-Lazarsfeld and Martens [20, Lemma 8.3], [17, 24].

Lemma 4.2. *Let* B *be a basepoint-free line bundle on a K3 surface* X *with* $B^2 = 2g - 2 \geq 2$. *Let* c *be the Clifford index of a smooth curve* C *in* $|B|$.

If $c < \lfloor (g-1)/2 \rfloor$, *then there is a smooth curve* E *on* X *such that* $0 \leq E^2 \leq c+2$ *and* $B \cdot E = E^2 + c + 2$.

In our case, we have $B^2 \geq 20$. Also, the Hodge index theorem gives that $(B^2)(E^2) - (B \cdot E)^2 \leq 0$ [18, Remark V.1.9.1]. Combining these results with Lemma 4.2 shows that if C has Clifford index c at most 2, then the curve E given by the lemma has $E^2 = 0$. (Otherwise, $(c, E^2, B \cdot E)$ is either $(0, 2, 4)$, $(1, 2, 5)$, $(2, 2, 6)$, or $(2, 4, 8)$, all of which are ruled out by the Hodge index theorem since $B^2 \geq 20$.) But then $1 \leq B \cdot E \leq 4$ by Lemma 4.2, contradicting our assumptions. So C has Clifford index at least 3.

For a smooth projective curve C, the *Wahl map*

$$\Phi_C \colon \Lambda^2 H^0(C, K_C) \to H^0(C, K_C^{\otimes 3})$$

is defined by $s \wedge t \mapsto s\, dt - t\, ds$. Wahl showed that the Wahl map of a curve of genus at least 2 contained in some K3 surface is not surjective; that is, $\text{corank}(\Phi_C) \geq 1$ [35]. When $g \geq 11$ and $\text{Cliff}(C) \geq 3$ (as here), Ciliberto-Dedieu-Sernesi proved the more precise statement:

$$\text{corank}(\Phi_C) = h^1(X, \Omega^1_X \otimes B) + 1$$

[9, Corollary 2.8].

Let $r = h^1(X, \Omega^1 \otimes B)$. By Ciliberto-Dedieu-Sernesi, again using that $g \geq 11$ and $\text{Cliff}(C) \geq 3$, there is an arithmetically Gorenstein normal variety Z of dimension $r+2$ in $\mathbf{P}^{g+r}$, not a cone, containing the curve $C \subset \mathbf{P}^{g-1}$ as the section by a linear space of dimension $g-1$. Moreover, Z contains $X \subset \mathbf{P}^g$ as the section by a linear space of dimension g [9, section 2.2].

Thus, if $H^1(X, \Omega^1 \otimes B) \neq 0$, then Z has dimension at least 3. Let Y be the intersection of Z with a general linear space of dimension $g+1$ that contains X; then $Y \subset \mathbf{P}^{g+1}$ is an arithmetically Gorenstein 3-fold with $-K_Y = O(1)$. Because Y has a smooth hyperplane section X and Z is not a cone, Y has at most isolated canonical singularities [9, Corollary 5.6]. Theorem 4.1 is proved. $\square$

5 K3 surfaces of high degree

In the rest of the paper, we analyze which ample line bundles B on a K3 surface X have $H^1(X, \Omega^1 \otimes B) = 0$, without assuming that X has Picard number 1. We give complete answers when B has high enough degree. Ciliberto-Dedieu-Sernesi proved a first step, using Prokhorov's work on Fano 3-folds: in high degrees, the locus where $H^1(X, \Omega^1 \otimes B) \neq 0$ is contained in the locus of K3 surfaces that contain "low-degree" elliptic curves [9, Corollaries 2.8 and 2.10]. They used slightly different language, and so we formulate the statement as Theorem 5.1. The result is analogous to Saint-Donat's theorem on very ampleness, Theorem 3.4. Theorem 6.1 will analyze the case when there is a low-degree elliptic curve.

Theorem 5.1. *Let B be an ample line bundle on a complex K3 surface X with $B^2 \geq 74$. If $H^1(X, \Omega^1_X \otimes B) \neq 0$, then there is a curve E in X with $E^2 = 0$ and $1 \leq B \cdot E \leq 4$.*

As mentioned earlier, any such curve E is a fiber of an elliptic fibration of X.

Proof. Suppose that there is no curve E in X with $E^2 = 0$ and $1 \leq B \cdot E \leq 4$, and that $H^1(X, \Omega^1_X \otimes B)$ is not zero. By Theorem 4.1, X is a smooth anticanonical divisor in some Fano 3-fold Y with at most isolated canonical Gorenstein singularities such that $B = -K_Y|_X$.

Prokhorov showed that a Fano 3-fold Y with canonical Gorenstein singularities has $(-K_Y)^3 \leq 72$ [29, Theorem 1.5], [30, Lemma 5.9]. (For comparison, a smooth Fano 3-fold Y has $(-K_Y)^3 \leq 64$.) So we reach a contradiction if $B^2 \geq 74$. $\square$

The degree bound 74 in Theorem 5.1 is sharp, by the following example.

Example 5.2. Let X be the double cover of $\mathbf{P}^2$ ramified over a very general sextic curve. Let A be the pullback of the line bundle $O_{\mathbf{P}^2}(1)$. Here (X, A) is a polarized K3 surface of degree 2 and Picard number 1. I claim that the line bundle $B = 6A$ has $B^2 = 72$ and $H^1(X, \Omega^1_X \otimes B) \neq 0$, while there is no curve E in X with $E^2 = 0$ and $1 \leq B \cdot E \leq 4$. (Thus Theorem 5.1 fails in degree 72.)

Proof: By the assumption of generality, $\text{Pic}(X) = \mathbf{Z} \cdot A$. So there is no curve E in X with $E^2 = 0$.

By considering the graded ring associated to A, X embeds as a hypersurface of degree 6 in the weighted projective space $Y = P(3, 1, 1, 1)$. Here Y is a Fano 3-fold with canonical Gorenstein singularities, $-K_Y = O(6)$, and $(-K_Y)^3 = 72$ [29, Theorem 1.5]. In particular, $B = -K_Y|_X$.

By Lvovski's theorem as in the proof of Theorem 4.1, because (X, B) is an anticanonical section of a Fano 3-fold Y, we have $H^1(X, \Omega^1_X \otimes B) \neq 0$. (Ciliberto-Lopez-Miranda claimed that $H^1(X, \Omega^1_X \otimes B) = 0$ in this case, because of an error in the proof of [10, Lemma 2.3(e)]: in the description of the tangent bundle of a ramified cover, N_π should be $\pi^* O_B(6)$, not $\pi^* O_B(3)$.)

Thus we have examples of K3 surfaces X with Picard number 1 and an ample line bundle B of degree 72 such that $H^1(X, \Omega^1_X \otimes B) \neq 0$, showing the optimality of Theorem 5.1. This does not contradict Theorem 3.2, because $B = 6A$ is not primitive.

Example 5.3. There is a K3 surface X with a *primitive* ample line bundle B of degree 62 such that $H^1(X, \Omega^1_X \otimes B) \neq 0$ and there is no curve E in X with $E^2 = 0$ and $1 \leq B \cdot E \leq 4$. (Thus Theorem 5.1 fails for primitive ample line bundles of degree 62.)

Proof: Let Y be the Fano 3-fold $P(O \oplus O(2)) \to \mathbf{P}^2$, which has $(-K_Y)^3 = 62$ and $-K_Y$ primitive. Let X be a smooth divisor in the linear system of $-K_Y$. Then X is a K3 surface with a primitive ample line bundle $B = -K_Y|_X$ of degree 62, and $H^1(X, \Omega^1 \otimes B) \neq 0$ by Lvovski's theorem again. For X very general, the restriction homomorphism $\text{Pic}(Y) = \mathbf{Z}\{R, S\} \to \text{Pic}(X)$ is an isomorphism. Given that, it is straightforward to compute the intersection form on X (it has $R^2 = 2$, $RS = 5$, and $S^2 = 10$). This quadratic form does not represent zero nontrivially, and so X contains no curve E with $E^2 = 0$. Thus Theorem 5.1 fails for (X, B), as promised.

6 Elliptic K3 surfaces

We now analyze which K3 surfaces (X, B) have $H^1(X, \Omega^1_X \otimes B) \neq 0$ when there is a curve E in X with $E^2 = 0$ and $1 \leq B \cdot E \leq 4$; these are the cases left out of Theorem 5.1. The answer is complete if $B \cdot E = 1$ or also if B^2 is large enough (with explicit bounds). Surprisingly, the answer depends on whether an elliptic fibration of X has a certain special type of singular fiber.

In particular, when $1 \leq B \cdot E \leq 3$, we give examples with B^2 arbitrarily large such that $H^1(X, \Omega^1_X \otimes B) \neq 0$, showing that these cases are genuine exceptions to Theorem 5.1. By contrast, when $B \cdot E = 4$, this cohomology group is in fact zero for $B^2 \geq 194$. (This bound is probably not optimal.)

Theorem 6.1. *Let B be an ample line bundle on a complex K3 surface X. Suppose that there is a curve E in X with $E^2 = 0$ and $r := B \cdot E$ between 1 and 4. Let $\pi \colon X \to \mathbf{P}^1$ be the elliptic fibration associated to E. If $r = 1$ and π has a fiber of type II, or $r = 2$ and π has a fiber of type III, or $r = 3$ and π has a fiber of type IV, then $H^1(X, \Omega^1_X \otimes B) \neq 0$. The converse holds if in addition $r = 1$ and $B^2 \geq 40$, or $r = 2$ and $B^2 \geq 92$, or $r = 3$ and $B^2 \geq 140$, or $r = 4$ and $B^2 \geq 194$.*

In Kodaira's classification of the singular fibers of an elliptic surface [11, Corollary 5.2.3], type II is a cuspidal cubic curve, type III is two copies of $\mathbf{P}^1$ tangent at a point, and type IV is three copies of $\mathbf{P}^1$ through a point (Figure 2).

We first consider the case where $B \cdot E = 1$, in which case (X, B) is said to be *monogonal*. In this case, we have an even stronger statement than Theorem 6.1: we can describe exactly when $H^1(X, \Omega^1_X \otimes B)$ is not zero, without having to assume

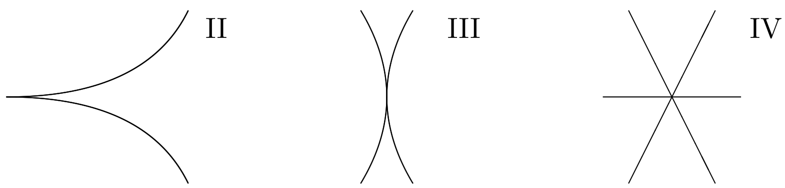

Figure 2: Singular fibers of types II, III, IV

that $B^2 \geq 40$. Most of the proof of the following theorem was suggested by Ben Bakker.

Theorem 6.2. *Let B be an ample line bundle on a complex K3 surface X. Suppose that there is a curve E in X with $E^2 = 0$ and $B \cdot E = 1$. Let $\pi\colon X \to \mathbf{P}^1$ be the elliptic fibration associated to E. Then $H^1(X, \Omega^1_X \otimes B) \neq 0$ if and only if $B^2 \leq 38$ or some fiber of π is of type II (a cuspidal cubic).*

In particular, there are polarized K3 surfaces (X, B) with B^2 arbitrarily large such that there is a curve E in X with $E^2 = 0$ and $B \cdot E = 1$ while $H^1(X, \Omega^1_X \otimes B) \neq 0$. To construct such examples, let $\pi\colon X \to \mathbf{P}^1$ be an elliptic K3 surface with a section B_0 such that there are 22 fibers of type I_1 (a nodal cubic) and one fiber of type II (a cuspidal cubic). Such a surface is easy to construct, using a Weierstrass equation. Let E be a fiber of π; then $B_0^2 = -2$, $E^2 = 0$, and $B_0 \cdot E = 1$. For any integer $m \geq 2$, it is straightforward to check that $B := B_0 + mE$ is ample, and we have $B^2 = 2m - 2$ and $B \cdot E = 1$. Since $B \cdot E = 1$, B is primitive. By Theorem 6.2, $H^1(X, \Omega^1_X \otimes B)$ is not zero, no matter how big B^2 is.

Theorem 6.2 shows that the locus of polarized K3 surfaces (X, B) with $H^1(X, \Omega^1_X \otimes B) \neq 0$ is not a Noether-Lefschetz locus when there is an elliptic curve of low degree. (That is, this property cannot always be read from the Picard lattice of X.) Indeed, the condition that an elliptic fibration $\pi\colon X \to \mathbf{P}^1$ has a cuspidal fiber is not determined by the Picard lattice of X. A general elliptic K3 surface as in the previous paragraph has Picard lattice $\mathbf{Z} \cdot \{B_0, E\}$ with $B_0^2 = -2$, $E^2 = 0$, and $B_0 \cdot E = 1$, whether there is a cuspidal fiber or not.

Proof. (Theorem 6.2) By the Riemann-Roch calculation in Theorem 3.1, we know that $H^1(X, \Omega^1_X \otimes B) \neq 0$ if $B^2 < 20$. So we can assume from now on that $B^2 \geq 20$.

Every fiber of $\pi\colon X \to \mathbf{P}^1$ is an effective divisor linearly equivalent to E. Since B is ample and $B \cdot E = 1$, every fiber of π is irreducible and has multiplicity 1. By Kodaira's classification, every singular fiber of π is of type I_1 (a nodal cubic) or II (a cuspidal cubic).

By Riemann-Roch, B can be represented by an effective divisor. Since $B \cdot E = 1$, this divisor must be the sum of a section B_0 of π with some curves supported in fibers. Then $B_0 \cong \mathbf{P}^1$, and so $B_0^2 = -2$. Because all fibers of π are irreducible, it follows that B is linearly equivalent to $B_0 + mE$, where $B^2 = 2m - 2$.

We have the following exact sequence of coherent sheaves on X:

$$0 \to \pi^* \Omega^1_{\mathbf{P}^1} \to \Omega^1_X \to \Omega^1_{X/\mathbf{P}^1} \to 0.$$

The sheaf $\Omega^1_{X/\mathbf{P}^1}$ of relative Kähler differentials is torsion-free but not reflexive, by a direct computation at singular fibers of π. It is related to the relative dualizing sheaf $\omega_{X/\mathbf{P}^1}$ (a line bundle) by another exact sequence:

$$0 \to \Omega^1_{X/\mathbf{P}^1} \to \omega_{X/\mathbf{P}^1} \to \omega_{X/\mathbf{P}^1}|_S \to 0,$$

where S is the non-smooth locus of π. Here S is a closed subscheme of degree 24 in X, supported at the singular points of fibers of π.

Let us compute the degree of the 0-dimensional scheme S at each singular point of a fiber of π. In local analytic coordinates, π is given by $\pi(x, y) = x^2 - y^2$ (at a node), $\pi(x, y) = x^2 - y^3$ (at a fiber of type II), $\pi(x, y) = x(x - y^2)$ (at type III), or $\pi(x, y) = x(x^2 - y^2)$ (at type IV). The scheme S is defined by $\partial\pi/\partial x = \partial\pi/\partial y = 0$. Because π is quasi-homogeneous in these coordinates, S is contained (as a scheme) in the fiber, $\pi^{-1}(0)$. The degree of S in these cases is: (I) $\dim_{\mathbf{C}} \mathbf{C}[x, y]/(2x, -2y) = 1$, (II) $\dim_{\mathbf{C}} \mathbf{C}[x, y]/(2x, -3y^2) = 2$, (III) $\dim_{\mathbf{C}} \mathbf{C}[x, y]/(2x - y^2, -2xy) = 3$, and (IV) $\dim_{\mathbf{C}} \mathbf{C}[x, y]/(3x^2 - y^2, -2xy) = 4$.

Since the line bundle $\Omega^1_{\mathbf{P}^1}$ is isomorphic to $O(-2)$, $\pi^*\Omega^1_{\mathbf{P}^1}$ is isomorphic to $O(-2E)$. Tensoring the first exact sequence with B and taking cohomology gives an exact sequence of complex vector spaces:

$$H^1(X, B - 2E) \to H^1(X, \Omega^1_X \otimes B) \to H^1(X, \Omega^1_{X/\mathbf{P}^1} \otimes B) \to H^2(X, B - 2E).$$

We arranged that $B^2 \geq 20$, and so $m \geq 11$. (For what follows, $m \geq 4$ would be enough.) Therefore, $B - 2E = B_0 + (m - 2)E$ is nef and big. So $H^1(X, B - 2E) = H^2(X, B - 2E) = 0$ by Kawamata-Viehweg vanishing. We deduce that $H^1(X, \Omega^1_X \otimes B)$ maps isomorphically to $H^1(X, \Omega^1_{X/\mathbf{P}^1} \otimes B)$.

Outside the 0-dimensional subscheme S of X, the first exact sequence above is an exact sequence of vector bundles. Taking determinants shows that $\Omega^1_{X/\mathbf{P}^1}$ is isomorphic to $O(2E)$ outside S, using that K_X is trivial. Because $\omega_{X/\mathbf{P}^1}$ is a line bundle on all of X, it follows that $\omega_{X/\mathbf{P}^1} \cong O(2E)$. So the second exact sequence (tensored with B) gives a long exact sequence of cohomology:

$$H^0(X, O(B + 2E)) \to H^0(S, O(B + 2E)) \to H^1(X, \Omega^1_X \otimes B) \to H^1(X, O(B + 2E)).$$

Here $B + 2E$ is nef and big, and so the last cohomology group is zero. We conclude that $H^1(X, \Omega^1_X \otimes B) = 0$ *if and only if the subscheme S imposes linearly independent conditions on sections of the line bundle $B + 2E$.* Thus for elliptic K3s, Bott vanishing reduces to a question about sections of a line bundle, which is much easier to analyze.

In the case at hand, we can describe all sections of $B + 2E = B_0 + (m + 2)E$ explicitly. We have $h^0(L) = (L^2 + 4)/2$ for L nef and big on a K3 surface X, and so $h^0(B+2E) = m+3$. But we get an $(m+3)$-dimensional space of sections of $O(B+2E) = O(B_0) \otimes \pi^* O(m+2)$ by pulling back sections of $O(m+2)$ on $\mathbf{P}^1$, and so those are all the sections. In other words, the linear system of $B+2E$ is exactly the set of divisors $B_0 + E_1 + \cdots + E_{m+2}$ for some fibers $E_1, \ldots, E_{m+2}$ of π.

If π has a fiber E_0 with a cusp p, then the subscheme S has degree 2 at p (and is contained in E_0), as shown above; so S does not impose linearly independent conditions on sections of $B+2E$ in this case. Otherwise, all singular fibers of π have a single node, and so S consists of 24 points in distinct fibers of π. It follows that S imposes linearly independent conditions on sections of $B+2E$ if and only if $m+2 \geq 23$, that is, $B^2 \geq 40$. □

We now address the cases where $B \cdot E$ is 2, 3, or 4. The K3 surface (X, B) is said to be *hyperelliptic*, *trigonal*, or *tetragonal*, respectively (because all smooth curves in the linear system of B have the given gonality).

Before proving Theorem 6.1, we use it to give examples such that $B \cdot E$ is 1, 2, or 3 and $H^1(X, \Omega^1_X \otimes B) \neq 0$ for arbitrarily large values of B^2, in contrast to Theorem 5.1. (This was done above when $B \cdot E = 1$.) When $B \cdot E = 4$, by contrast, the theorem says that $H^1(X, \Omega^1_X \otimes B) = 0$ whenever $B^2 \geq 194$.

Example 6.3. There are polarized K3 surfaces (X, B) with B^2 arbitrarily large such that there is a curve E in X with $E^2 = 0$ and $B \cdot E = 2$, while $H^1(X, \Omega^1_X \otimes B) \neq 0$.

Let X be the double cover of $Y = \mathbf{P}^1 \times \mathbf{P}^1$ ramified along a smooth curve D in the linear system of $-2K_Y = O(4,4)$. Then X is a K3 surface, with two elliptic fibrations defined by the two compositions $X \to Y \to \mathbf{P}^1$. Write $\pi \colon X \to \mathbf{P}^1$ for the first fibration, E for a fiber of π, and C_0 for a fiber of the second fibration; then $C_0 \cdot E = 2$. Let S be the non-smooth locus of π, a closed subscheme of degree 24 in X. By choosing D to have intersection with one curve $p \times \mathbf{P}^1$ equal to a single point with multiplicity 4, we can arrange that the corresponding fiber of π is of type III (two $\mathbf{P}^1$s tangent at one point). Let $B = C_0 + mE$. By Theorem 6.1, $H^1(X, \Omega^1_X \otimes B) \neq 0$, while $B^2 = 4m$ can be arbitrarily large. (One can give a similar example with $B^2 \equiv 2 \pmod 4$ by taking X to be a double cover of $P(O \oplus O(1)) \to \mathbf{P}^1$, rather than of $\mathbf{P}^1 \times \mathbf{P}^1$.)

Example 6.4. There are polarized K3 surfaces (X, B) with B^2 arbitrarily large such that there is a curve E in X with $E^2 = 0$ and $B \cdot E = 3$, while $H^1(X, \Omega^1_X \otimes B) \neq 0$.

To see this, let X be a smooth anticanonical divisor in $\mathbf{P}^1 \times \mathbf{P}^2$ such that one fiber E_0 of the elliptic fibration $\pi \colon X \to \mathbf{P}^1$ consists of three lines through a point (thus, a fiber of type IV). Let A be the pullback to X of $O(1)$ on $\mathbf{P}^2$, and let E be the pullback of $O(1)$ on $\mathbf{P}^1$; then $A^2 = 2$, $A \cdot E = 3$, and $E^2 = 0$. Let $B = A + mE$. By Theorem 6.1, $H^1(X, \Omega^1_X \otimes B) \neq 0$, while $B^2 = 6m + 2$ can be arbitrarily large.

Proof. (Theorem 6.1) Let $r = B \cdot E$. For $r = 1$, the theorem follows from Theorem 6.2. From now on, assume that $2 \leq r \leq 4$.

We use the following analysis of K3 surfaces of low Clifford genus, due to Reid, Brawner, and Stevens [31, section 2.11], [6, Tables A.1-A.4], [34, table in section 1].

Proposition 6.5. *Let X be a complex K3 surface with a line bundle L. Suppose that there is a curve E in X with $E^2 = 0$ and $r := L \cdot E$ between 1 and 4. Suppose that $L + sE$ is ample for some integer s. Finally, suppose that $r = 1$ and $L^2 \geq 2$,* *or $r = 2$ and $L^2 \geq 8$, or $r = 3$ and $L^2 \geq 14$, or $r = 4$ and $L^2 \geq 26$. Then L is nef, and $h^0(L) - h^0(L - E) = r$.*

Proof. The references cited determine the possible values of the sequence of integers $h^0(L + mE)$. (For $r \geq 2$, that sequence describes the scroll $P(O(e_1) \oplus \cdots O(e_r)) \to \mathbf{P}^1$ that contains the image of X under the morphism to projective space given by $L + mE$ for m large.) In particular, these results say that $h^0(L) - h^0(L - E) = r$ under our assumption on L^2. It follows that $h^0(L - E)$ is given by Riemann-Roch and hence that $h^1(L - E) = 0$ (because $h^2(L - E)$ is easily seen to be zero). By Knutsen and Lopez's characterization of line bundles with vanishing cohomology on a K3 surface, it follows that $L - E$ has degree at least -1 on any (-2)-curve in X [21, Theorem]. Using that plus the fact that $L + mE$ is ample for m large, we deduce that L is nef. $\square$

As in the proof of Theorem 6.2, let S be the non-smooth locus of π, viewed as a closed subscheme of degree 24 in X, supported at the singular points of fibers of π. We computed that S has degree 1 at nodes, 2 at cusps (on fibers of type II), 3 at type III, and 4 at type IV. Moreover, each connected component of S is contained (as a scheme) in a fiber of π.

By the proof of Theorem 6.2, if $H^1(X, \Omega^1_X \otimes B)$ is zero, then S imposes linearly independent conditions on sections of $B + 2E$. Moreover, the converse holds if $B - 2E$ is nef and big. Suppose that $r = 2$ and π has a fiber of type III, or $r = 3$ and π has a fiber of type IV. (We are assuming $r \geq 2$ now, but the argument would be the same in the case where $r = 1$ and π has a fiber of type II.) Let S_0 be the connected component of S at the given singular point. Then S_0 has degree $r + 1$. On the other hand, the line bundle $B + 2E$ is ample and has degree r on the given fiber E_0 (which has r irreducible components), and so it has degree only 1 on each component. It follows that $h^0(E_0, B + 2E)$ is only r. So the restriction map $H^0(X, B + 2E) \to H^0(S_0, B + 2E) = \mathbf{C}^{r+1}$ is not surjective. So S does not impose independent conditions on sections of $B + 2E$, and hence $H^1(X, \Omega^1_X \otimes B)$ is not zero. The first part of the theorem is proved.

For the converse, suppose that $r = 2$ and $B^2 \geq 92$, or $r = 3$ and $B^2 \geq 140$, or $r = 4$ and $B^2 \geq 194$. Also, if $r = 2$, assume that π has no fiber of type III, and if $r = 3$, assume that π has no fiber of type IV. We want to deduce that $H^1(X, \Omega^1_X \otimes B) = 0$.

Let $L = B - 21E$, so that $L^2 = B^2 - 42r$. Thus either $r = 2$ and $L^2 \geq 8$, or $r = 3$ and $L^2 \geq 14$, or $r = 4$ and $L^2 \geq 26$. By Proposition 6.5 (using that $L + 21E$ is ample), L is nef, and $h^0(L) - h^0(L - E) = r$. So, for each fiber E_0 of π, the image of the restriction $H^0(X, L) \to H^0(E_0, L)$ has dimension r. Using again that $L + 21E$ is ample, the line bundle L is ample on E_0, with degree $r \leq 4$. It follows that E_0 has at most r irreducible components. So E_0 has type I_n for $n \leq r$ or II or III (with r equal to 3 or 4) or IV (with $r = 4$). By Riemann-Roch for 1-dimensional schemes [33, Tag 0BS6] plus Serre duality, $H^0(E_0, L)$ has dimension r. (Use that E_0 is Gorenstein, with trivial canonical bundle.) So $H^0(X, L) \to H^0(E_0, L)$ is surjective, for each fiber E_0 of π.

We have shown that $L = B - 21E$ is nef, and it is big since $L^2 > 0$. So $B - 2E$ is also nef and big. Using that, the proof of Theorem 6.2 shows that

$H^1(X, \Omega^1_X \otimes B) = 0$ (as we want) if and only if S imposes independent conditions on sections of $B + 2E$. Here $B + 2E = L + 23E$.

Let E_0 be any singular fiber of π, and let $S_0 = S \cap E_0$, which is an open subscheme of S. We showed above that $H^0(X, L) \to H^0(E_0, L)$ is surjective. Also, L is ample on E_0. It follows that $H^0(E_0, L) \to H^0(S_0, L)$ is surjective, by inspection of the possible types of singular fibers (since we have excluded the case where $r = 2$ and E_0 is of type III, or $r = 3$ and E_0 is of type IV). Therefore, $H^0(X, L) \to H^0(S_0, L)$ is surjective. It is then clear that $H^0(X, L + 23E) \to H^0(S, L + 23E)$ is surjective, using sections of $O(23E)$ that vanish on all singular fibers of π except one. (We are using that the number of singular fibers is at most 24.) Since $B + 2E = L + 23E$, this completes the proof that $H^1(X, \Omega^1_X \otimes B) = 0$. $\square$

References

[1] P. Achinger, J. Witaszek, and M. Zdanowicz. Liftability of the Frobenius morphism and images of toric varieties. `arXiv:1708.03777` 1

[2] E. Arbarello, A. Bruno, and E. Sernesi. Mukai's program for curves on a K3 surface. *Algebraic Geometry* **1** (2014), 532–557. 7

[3] E. Arbarello, A. Bruno, and E. Sernesi. On hyperplane sections of K3 surfaces. *Algebraic Geometry* **4** (2017), 562–596. 2, 9

[4] V. Batyrev and D. Cox. On the Hodge structure of projective hypersurfaces in toric varieties. *Duke Math. J.* **75** (1994), 293–338. 1

[5] A. Beauville. Fano threefolds and K3 surfaces. *The Fano conference*, 175–184. University of Torino (2004). 2, 6, 7, 9

[6] J. Brawner. Tetragonal curves, scrolls and K3 surfaces. *Trans. Amer. Math. Soc.* **349** (1997), 3075–3091. 15

[7] A. Buch, J. Thomsen, N. Lauritzen, and V. Mehta. The Frobenius morphism on a toric variety. *Tohoku Math. J.* **49** (1997), 355–366. 1

[8] C. Ciliberto, T. Dedieu, C. Galati, and A. Knutsen. Moduli of curves on Enriques surfaces. `arXiv:1902.07142` 6

[9] C. Ciliberto, T. Dedieu, and E. Sernesi. Wahl maps and extensions of canonical curves and K3 surfaces. *J. reine angew. Math.*, to appear. `https://doi.org/10.1515/crelle-2018-0016` 1, 2, 6, 7, 9, 10, 11

[10] C. Ciliberto, A. Lopez, and R. Miranda. Classification of varieties with canonical curve section via Gaussian maps on canonical curves. *Amer. J. Math.* **120** (1998), 1–21. 12

[11] F. Cossec and I. Dolgachev. *Enriques surfaces.* Birkhäuser (1989). 12

[12] O. Debarre. *Higher-dimensional algebraic geometry.* Springer (2001). 3

[13] P. Deligne and L. Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.* **89** (1987), 247–270. 3

[14] S. Feyzbakhsh. Mukai's program (reconstructing a K3 surface from a curve) via wall-crossing. *J. Reine Angew. Math.,* to appear. `https://doi.org/10.1515/crelle-2019-0025` 1, 6, 7

[15] O. Fujino. Multiplication maps and vanishing theorems for toric varieties. *Math. Z.* **257** (2007), 631–641. 1

[16] W. Fulton. *Introduction to toric varieties.* Princeton (1993). 4

[17] M. Green and R. Lazarsfeld. Special divisors on curves on a K3 surface. *Invent. Math.* **89** (1987), 357–370. 10

[18] R. Hartshorne. *Algebraic geometry.* Springer (1977). 5, 10

[19] D. Huybrechts. *Lectures on K3 surfaces.* Cambridge (2016). 6

[20] A. Knutsen. On kth order embeddings of K3 surfaces and Enriques surfaces. *Manu. Math.* **104** (2001), 211–237. 10

[21] A. Knutsen and A. Lopez. A sharp vanishing theorem for line bundles on K3 or Enriques surfaces. *Proc. Amer. Math. Soc.* **135** (2007), 3495–3498. 16

[22] R. Lazarsfeld. *Positivity in algebraic geometry*, 2 vols. Springer (2004). 3, 5

[23] Y. Manin. *Cubic forms,* 2nd ed. North-Holland (1986). 3

[24] G. Martens. On curves on K3 surfaces. *Algebraic curves and projective geometry* (Trento, 1988), 174–182. Lecture Notes in Math. 1387, Springer (1989). 10

[25] S. Mori. On degrees and genera of curves on smooth quartic surfaces in $\mathbf{P}^3$. *Nagoya Math. J.* **96** (1984), 127–132. 8

[26] S. Mori and S. Mukai. The uniruledness of the moduli space of curves of genus 11. *Algebraic geometry* (Tokyo/Kyoto, 1982), 334–353. Lecture Notes in Math. 1016, Springer (1983). 2, 6, 7

[27] S. Mukai. Fano 3-folds. *Complex projective geometry* (Trieste–Bergen, 1989), 255–263. London Math. Soc. Lecture Note Ser. 179, Cambridge (1992). 2, 6, 7

[28] M. Mustaţă. Vanishing theorems on toric varieties. *Tohoku Math. J.* **54** (2002), 451–470. 1

[29] Y. Prokhorov. On the degree of Fano threefolds with canonical Gorenstein singularities. *Mat. Sb.* **196** (2005), 81–122; *Sb. Math.* **196** (2005), 77–114. 11

[30] Y. Prokhorov. On Fano-Enriques threefolds. *Mat. Sb.* **198** (2007), 117–134; *Sb. Math.* **198** (2007), 559–574. 11

[31] M. Reid. Chapters on algebraic surfaces. *Complex algebraic geometry* (Park City, 1993), 3–159. Amer. Math. Soc. (1997). 15

[32] B. Saint-Donat. Projective models of K-3 surfaces. *Amer. J. Math.* **96** (1974), 602–639. 8

[33] The Stacks Project Authors. *Stacks Project* (2018). `http://stacks.math.columbia.edu` 5, 16

[34] J. Stevens. Rolling factors deformations and extensions of canonical curves. *Doc. Math.* **7** (2002), 185–226. 15

[35] J. Wahl. The Jacobian algebra of a graded Gorenstein singularity. *Duke Math. J.* **55** (1987), 843–871. 10

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