Torsion algebraic cycles and complex cobordism

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Introduction.

Atiyah and Hirzebruch gave the first counterexamples to the Hodge conjecture with integral coefficients [3]. That conjecture predicted that every integral cohomology class of Hodge type (p,p) on a smooth projective variety should be the class of an algebraic cycle, but Atiyah and Hirzebruch found additional topological properties which must be satisfied by the integral cohomology class of an algebraic cycle. Here we provide a more systematic explanation for their results by showing that the classical cycle map, from algebraic cycles modulo algebraic equivalence to integral cohomology, factors naturally through a topologically defined ring which is richer than integral cohomology. The new ring is based on complex cobordism, a well-developed topological theory which has been used only rarely in algebraic geometry since Hirzebruch used it to prove the Riemann-Roch theorem [17].

This factorization of the classical cycle map implies the topological restrictions on algebraic cycles found by Atiyah and Hirzebruch. It goes beyond their work by giving a topological method to show that the classical cycle map can be non-injective, as well as non-surjective. The kernel of the classical cycle map is called the Griffiths group, and the topological proof here that the Griffiths group can be nonzero is the first proof of this fact which does not use Hodge theory. (The proof here gives nonzero torsion elements in the Griffiths group, whereas Griffiths's Hodge-theoretic proof gives non-torsion elements [13].)

This topological argument also gives examples of algebraic cycles in the kernel of various related cycle maps, where few or no examples were known before, thus answering some questions posed by Colliot-Thélène and Schoen ([8], p. 14; [37], p. 13). Colliot-Thélène asked, in particular, whether the map $CH^2(X)/n \to H^4(X, \mathbf{Z}/n)$ is injective for all smooth complex projective varieties X. Here CH^iX is the group of codimension i algebraic cycles modulo rational equivalence. The first examples where Colliot-Thélène's map is not injective were found by Kollár and van Geemen [4], p. 135; very recently, Bloch and Esnault found examples defined over number fields [7]. (Over non-algebraically closed fields k there are other examples of smooth projective varieties X_k with $CH^2(X_k)/n \to H^4_{\text{et}}(X_k, \mathbf{Z}/n)$ not injective, due to Colliot-Thélène and Sansuc as reinterpreted by Salberger (see [9] and [8], Remark 7.6.1), and Parimala and Suresh [31]. These elements of $CH^2(X_k)/n$ are not shown to remain nonzero in $CH^2(X_{\mathbf{C}})/n$, however.) Here our topological method gives examples which can be defined over the rational numbers. The varieties we use, as in Atiyah-Hirzebruch's examples, are quotients of complete intersections by finite groups.

Schoen asked whether the map from the torsion subgroup of $CH^i X$ to Deligne cohomology is injective for all smooth complex projective varieties X [37], p. 13. This was known in many cases: for $i \leq 2$ by Merkur'ev-Suslin [23], p. 338, and for $i = \dim X$ by Roitman [36]. But we show that injectivity can fail for $i = 3$.

Similarly, one can ask whether an algebraic cycle which maps to 0 in Deligne cohomology must be algebraically equivalent to 0, the point being that Griffiths's original examples of cycles which were homologically but not algebraically equivalent to 0 had nonzero image in Deligne cohomology. The answer is no, as Nori showed by a subtle application of Hodge theory [28]. Here we show again that the answer is no, using our topological method. It is interesting that both Nori's examples and ours work in codimension at least 3; Nori suggests that the answer to the question should be yes for codimension 2 cycles.

We now describe our main construction in more detail. If X is a smooth complex algebraic variety, the classical cycle class map sends the ring of algebraic cycles modulo algebraic equivalence to the integral cohomology ring of X. We show that this map factors canonically through the ring $MU^*X \otimes_{MU^*} \mathbf{Z}$, where MU^*X is the complex cobordism ring of the topological space X (see section 1 for definitions). Here MU^*X is a module over the graded ring

$$MU^* = MU^*(\text{point}) = \mathbf{Z}[x_1, x_2, \ldots], \qquad x_i \in MU^{-2i},$$

and we map MU^* to $\mathbf{Z}$ by sending all the generators x_i to 0. The ring $MU^*X \otimes_{MU^*} \mathbf{Z}$ is the same as the integral cohomology ring if the integral cohomology is torsion-free, but in general the map

$$MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X, \mathbf{Z})$$

need not be either injective or surjective, although the kernel and cokernel are torsion. The construction of the new cycle map, from cycles modulo algebraic equivalence to $MU^*X \otimes_{MU^*} \mathbf{Z}$, uses Hironaka's resolution of singularities together with some fundamental results on complex cobordism proved by Quillen and Wilson.

For topological spaces X with torsion in their cohomology, the map $MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X, \mathbf{Z})$ is often not surjective, as one can compute from the fact that all odd-dimensional elements of the Steenrod algebra vanish on the image of MU^*X in $H^*(X, \mathbf{Z}/p)$, for each prime number p. (Equivalently, the two-sided ideal in the Steenrod algebra generated by the Bockstein vanishes on the image of MU^*X.)

The fact that the usual cycle class map goes into the image of MU^*X was essentially observed by Atiyah [2], footnote 1, p. 445; it follows immediately from Hironaka's resolution of singularities. As a result, all odd-dimensional elements of the Steenrod algebra vanish on the image of algebraic cycles in $H^*(X, \mathbf{Z}/p)$. Atiyah and Hirzebruch used a weaker form of this statement to produce their counterexamples to the Hodge conjecture with integral coefficients [3].

Once we have our more refined cycle class, it is natural to try to use it to give a new construction of smooth projective varieties with nonzero Griffiths group. We need to find varieties X such that the map $MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X, \mathbf{Z})$ is not injective, and then we have to hope that some of the elements of the kernel can be represented by algebraic cycles. Unfortunately, it is much harder to find topological spaces X with $MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X, \mathbf{Z})$ not injective than to find non-surjective examples. As far as I know, the only examples in the literature are those produced by Conner and Smith [10], [11], and [39], p. 854. Unfortunately, their examples are defined as cell complexes with explicit attaching maps, and there is no obvious way to approximate such spaces by smooth algebraic varieties.

Fortunately, there are spaces X with $MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X, \mathbf{Z})$ not injective which are more convenient for our purpose: the classifying spaces of some compact Lie groups. We will show that that the map $MU^*BG \otimes_{MU^*} \mathbf{Z} \to H^*(BG, \mathbf{Z})$ is not injective in degree 6 when G is either the Lie group $\mathbf{Z}/2 \times SO(4)$ or the product of $\mathbf{Z}/2$ with a finite Heisenberg group H, a central extension

$$1 \to \mathbf{Z}/2 \to H \to (\mathbf{Z}/2)^4 \to 1.$$

There is a natural way to approximate the classifying space of a Lie group by smooth algebraic varieties, and as a result we manage to produce smooth projective varieties (the quotient of certain complete intersections by the above group $\mathbf{Z}/2 \times H$) for which our cycle map implies that the Griffiths group is not zero.

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1 A quick introduction to complex bordism

A general reference for this section is Stong's book [40]. We begin by defining a weakly complex manifold M to be a smooth real manifold together with a complex vector bundle over M whose underlying real vector bundle is $TM \oplus \mathbf{R}^N$ for some N. Thus, complex manifolds are weakly complex manifolds, but some odd-dimensional manifolds (e.g., S^1, since its tangent bundle is trivial) also admit weakly complex structures. We identify two complex structures on the vector bundle $TM \oplus \mathbf{R}^N$ if they are homotopic, and we also identify a complex structure on the vector bundle $TM \oplus \mathbf{R}^N$ with the obvious complex structure on $TM \oplus \mathbf{R}^N \oplus \mathbf{R}^2 = TM \oplus \mathbf{R}^{N+2}$.

The complex bordism groups MU_iX of a topological space X, $i \geq 0$, are defined as the free abelian group on the set of continuous maps $M \to X$ where M is a closed weakly complex manifold of real dimension i, modulo the relations

$$\begin{aligned} [M_1 \textstyle\coprod M_2 \to X] &= [M_1 \to X] + [M_2 \to X] \\ [\partial W \to X] &= 0, \end{aligned}$$

where W is a compact weakly complex manifold of dimension $i+1$ with boundary together with a continuous map $W \to X$. (The boundary of W inherits a weakly complex structure in a natural way.)

The notion of weakly complex manifold is rather artificial, and one might ask why we don't try to define similar invariants of a topological space X using, say, complex manifolds with continuous maps to X. The justification for the above definition is that the groups MU_*X have excellent formal properties: they form a generalized homology theory [1], which means that they satisfy all the usual formal properties of ordinary homology (Mayer-Vietoris, etc.) except for the dimension axiom: MU_iX can be nonzero for $i > \dim X$. In fact, MU_iX is always nonzero for all even $i \geq 0$, at least, because $MU_i(\text{point})$ is nonzero for all even $i \geq 0$. The groups $MU_* := MU_*(\text{point})$ form a ring, the product corresponding to taking products of weakly complex manifolds, and this ring was computed by Milnor and Novikov [25], [29]:

$$MU_* = \mathbf{Z}[x_1, x_2, x_3, \ldots], \qquad x_i \in MU_{2i}.$$

It happens that all the generators x_i can be represented by complex manifolds. If we tensor the ring with $\mathbf{Q}$, we can take the generators to be $\mathbf{CP}^1$, $\mathbf{CP}^2$, and so on; to get the generators over $\mathbf{Z}$, we have to use certain hypersurfaces, as Milnor showed.

There is a natural map $MU_iX \to H_i(X, \mathbf{Z})$, which sends a bordism class $[M \to X]$ to the image under this map of the fundamental homology class of M (since M has a weakly complex structure, it has a natural orientation). This map clearly has an enormous kernel, but there is a way to define groups related to MU_*X which are much closer to $H_*(X, \mathbf{Z})$. This uses that the groups MU_*X form a module over the ring MU_*. Geometrically, the product $MU_i \otimes_{\mathbf{Z}} MU_jX \to MU_{i+j}X$ sends a weakly complex manifold M^i and a map $M^j \to X$ to the composition $M^i \times M^j \to M^j \to X$, where the first map is the obvious projection. The point is that as long as $i > 0$, the resulting element of $MU_{i+j}X$ maps to 0 in $H_{i+j}(X, \mathbf{Z})$. So we have a natural map

$$MU_*X/(MU_{>0} \cdot MU_*X) \to H_*(X, \mathbf{Z}),$$

or, as I prefer to write it,

$$MU_*X \otimes_{MU_*} \mathbf{Z} \to H_*(X, \mathbf{Z}),$$

where the ring MU_* maps to $\mathbf{Z}$ by sending all the generators x_i, $i \geq 1$, to 0.

If X is a compact complex algebraic scheme, possibly singular, our cycle map will take values in $MU_*X \otimes_{MU_*} \mathbf{Z}$. To include schemes which may be noncompact, we define a variant of the above groups. For any locally compact topological space X, let $MU_i^{\text{BM}}X$ be the free abelian group on the set of proper maps $M \to X$, where M is a weakly complex manifold of real dimension i which may be noncompact, modulo the relations

$$[M_1 \textstyle\coprod M_2 \to X] = [M_1 \to X] + [M_2 \to X]$$
$$[\partial W \to X] = 0,$$

where W is a weakly complex manifold of real dimension $i+1$ with boundary which may be noncompact, together with a proper map $W \to X$. These groups can be identified with bordism groups in the more usual sense for all reasonable spaces X: namely, the groups $MU_*^{\text{BM}}X$ are the reduced bordism groups of the one-point compactification of X. More generally, if $X = \overline{X} - S$ is any compactification, $MU_*^{\text{BM}}X$ is isomorphic to the relative bordism group $MU_*(\overline{X}, S)$, as is defined for any generalized homology theory. The cycle map we will define for an arbitrary complex algebraic scheme X takes values in $MU_*^{\mathrm{BM}}X \otimes_{MU_*} \mathbf{Z}$.

Also, as for any generalized homology theory, there is a corresponding cohomology theory, complex cobordism MU^*X, which is a ring for any space X. If X is an n-dimensional complex manifold, there is a Poincaré duality isomorphism $MU^iX \cong MU_{2n-i}^{\mathrm{BM}}X$. So, for an n-dimensional complex manifold X, we have a geometric description of the cobordism group MU^iX, as bordism classes of "real codimension i" weakly complex manifolds M (meaning that dim $M = 2n - i$) with proper maps $M \to X$. In a sense this suffices to describe MU^iX for arbitrary spaces X, since at least every finite cell complex is homotopy equivalent to a complex manifold (a regular neighborhood of an embedding in $\mathbf{C}^N$). Still, it may be helpful to mention one other geometric description of cobordism: for any real manifold X, MU^iX is the group of bordism classes of codimension i real manifolds M with a proper map $f\colon M \to X$ and a complex structure on the "stable normal bundle" $f^*TX - TM$ [33].

If X is a compact complex manifold, the above Poincaré duality isomorphism says that $MU^iX = MU_{2n-i}X$. In particular, if X is a point, we have $MU^* := MU^*(\text{point}) = MU_{-*}$. That is, MU^* is a polynomial ring $\mathbf{Z}[x_1, x_2, \ldots]$ with $x_i \in MU^{-2i}$.

The natural way to compute MU^*X, for any CW complex X, is by the Atiyah-Hirzebruch spectral sequence $E_2 = H^*(X, MU^*) \implies MU^*X$. This is a fourth quadrant spectral sequence because the ring MU^* is in dimensions ≤ 0. The E_2 term is the tensor product of $H^*(X,\mathbf{Z})$ on the positive x-axis and MU^* on the negative y-axis. The differentials have the same bidegrees as those in the spectral sequence of a fibration: d_2 has bidegree (2,-1), d_3 has bidegree $(3,-2)$, and so on. The natural map $MU^*X \to H^*(X,\mathbf{Z})$ is the "edge map" corresponding to the top row of the spectral sequence. From the spectral sequence, we can read off several of the basic properties of the complex cobordism ring, say for a finite cell complex X:

If X has real dimension n, MU^iX can be nonzero only for $i \leq n$. It is nonzero for all negative even i, at least.

The differentials are known to be torsion. It follows that $MU^*X \otimes_{\mathbf{Z}} \mathbf{Q}$ is a free $MU^* \otimes_{\mathbf{Z}} \mathbf{Q}$-module, generated by any set of elements of MU^*X which map to a basis for $H^*(X,\mathbf{Q})$. In particular, the natural map $MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X,\mathbf{Z})$ becomes an isomorphism after tensoring with $\mathbf{Q}$.

If the integral cohomology of X has no torsion, then neither does the E_2 term of the spectral sequence. (Here the fact that the ring $MU^* = \mathbf{Z}[x_1, x_2, \ldots]$ has no torsion is crucial.) Since the differentials are always torsion, they must be 0 in this case. Thus, if $H^*(X,\mathbf{Z})$ has no torsion, then MU^*X is a free MU^*-module, and the natural map $MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X,\mathbf{Z})$ is an isomorphism. Likewise, if $H_*^{\mathrm{BM}}(X,\mathbf{Z})$ has no torsion, then the natural map $MU_*^{\mathrm{BM}}X \otimes_{MU_*} \mathbf{Z} \to H_*^{\mathrm{BM}}(X,\mathbf{Z})$ is an isomorphism, by the homology version of the Atiyah-Hirzebruch spectral sequence.

Finally, we need to mention Brown-Peterson cohomology, a simplification of complex cobordism which is more convenient for calculations. Namely, for each prime number p there is a cohomology theory called BP^*X (it is conventional not to indicate p in the notation). Its coefficient ring is the polynomial ring

$$\mathbf{Z}_{(p)}[v_1, v_2, \dots],$$

where $\mathbf{Z}_{(p)}$ is the localization of the ring $\mathbf{Z}$ at the prime p, and $v_1 \in BP^{-2(p-1)}$, $v_2 \in BP^{-2(p^2-1)}$, and so on. Thus the generators of the ring BP^* are much more spread out than those of MU^*, which makes calculations easier. But BP^*X carries all the topological information of MU^*X, because $MU^*X \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ splits as a direct sum of copies of BP^*X in a canonical way [32]. In particular, BP^* is a quotient ring of $MU^* \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ in such a way that $BP^*X = MU^*X \otimes_{MU^*} BP^*$ for all spaces X, and consequently

$$MU^*X \otimes_{MU^*} \mathbf{Z}_{(p)} = BP^*X \otimes_{BP^*} \mathbf{Z}_{(p)}.$$

We will use this to translate results between complex cobordism and Brown-Peterson cohomology as convenient.

Finally, in sections 2 and 5 we will use the related cohomology theories $BP\langle n\rangle^*X$. The basic properties of these theories are stated in section 2. Two useful references are [43] and [19].

2 Quillen's theorem

One of the fundamental facts about complex cobordism is the following classic theorem of Quillen's [33].

Theorem 2.1 *Let X be a finite cell complex. Then the groups $MU^*X \otimes_{MU^*} \mathbf{Z}$ are zero in negative dimensions and equal to $H^0(X, \mathbf{Z})$ in dimension 0.*

Equivalently, MU^*X is generated as an MU^*-module by elements of nonnegative degree.

In fact, Quillen's statement can be improved a little, and we will need part of the improved statement. Namely:

Theorem 2.2 *Let X be a finite cell complex. Then the map*

$$MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X, \mathbf{Z})$$

is an isomorphism in dimensions ≤ 2 and injective in dimensions ≤ 4.

This is best possible. In particular, the map is not surjective in dimension 3 for $X = B(\mathbf{Z}/p)^2$ or a suitable finite skeleton thereof, and it is not injective in dimension 5 for a suitable finite skeleton of $K(\mathbf{Z}, 3) \times B\mathbf{Z}/p$, as one can see by imitating the proof of Corollary 5.3 in this paper (apply it to a finite skeleton of $K(\mathbf{Z}, 3)$ in place of BG). We actually only need injectivity in dimensions ≤ 2 for this paper (in the proofs of Theorem 3.1 and Lemma 4.3), except in Remark 2, section 8.

Proof of Theorem 2.2. This follows by the arguments Wilson used to prove Quillen's theorem [43], as we now explain.

The surjectivity in dimensions ≤ 2 is trivial. In fact, an element of $H^i(X, \mathbf{Z})$ is represented by a map $X \to K(\mathbf{Z}, i)$, and for $i \leq 2$, the Eilenberg-MacLane space $K(\mathbf{Z}, i)$ has torsion-free cohomology (for $i = 0, 1, 2$, respectively, $K(\mathbf{Z}, i)$ is the space $\mathbf{Z}$, S^1, $\mathbf{C}P^\infty$). It follows that $MU^i K(\mathbf{Z}, i)$ maps onto the generator of $H^i(K(\mathbf{Z}, i), \mathbf{Z})$, by the argument near the end of section 1. Pulling back to X proves the desired surjectivity.

It suffices to prove injectivity after tensoring with $\mathbf{Z}_{(p)}$ for each prime number p. As mentioned in section 1, we have

$$MU^*X \otimes_{MU^*} \mathbf{Z}_{(p)} \cong BP^*X \otimes_{BP^*} \mathbf{Z}_{(p)},$$

where BP denotes Brown-Peterson cohomology at the prime p. So it suffices to show that the map

$$BP^*X \otimes_{BP^*} \mathbf{Z}_{(p)} \to H^*(X, \mathbf{Z}_{(p)})$$

is injective in dimensions ≤ 4. We recall from section 1 that the coefficient ring BP^* is a polynomial ring over $\mathbf{Z}_{(p)}$ with generators $v_i \in BP^{-2(p^i-1)}$, $i \geq 1$.

Following Wilson [43], we use the cohomology theories $BP\langle n\rangle$. These are modules over BP (so $BP\langle n\rangle^*X$ is a module over BP^*X), with coefficients $BP\langle n\rangle^* = \mathbf{Z}_{(p)}[v_1, \ldots, v_n]$ as a BP^*-module (all the v_i's for $i > n$ act as 0), and with maps of cohomology theories

$$BP^*X = BP\langle\infty\rangle^*X \to \cdots \to BP\langle 1\rangle^*X \to BP\langle 0\rangle^*X = H^*(X, \mathbf{Z}_{(p)}).$$

There is a long exact sequence

$$BP\langle n\rangle^{k+2(p^n-1)}X \to BP\langle n\rangle^k X \to BP\langle n-1\rangle^k X \to BP\langle n\rangle^{k+2(p^n-1)+1}X,$$

where the first map is multiplication by v_n and the second map is part of the sequence of maps above. Finally, we use Wilson's main theorem [43], p. 118:

Theorem 2.3 *$BP^kX \to BP\langle n\rangle^k X$ is surjective for $k \leq 2(p^n + p^{n-1} + \cdots + 1)$.*

Now we can prove Theorem 2.2. Let $x \in BP^kX$, $k \leq 4$, such that x maps to 0 in $H^k(X, \mathbf{Z}_{(p)})$. We will show that x is a finite sum $x = \sum_{i>0} v_i x_i$, $x_i \in BP^{k+2(p^i-1)}X$; this is equivalent to showing that $x = 0 \in BP^*X \otimes_{BP^*} \mathbf{Z}_{(p)}$.

Consider the maps

$$BP^*X \to BP\langle n\rangle^*X \to BP\langle n-1\rangle^*X.$$

If x is 0 we are done. Otherwise, let n be the positive integer such that x maps to 0 in $BP\langle n-1\rangle^*X$ but not in $BP\langle n\rangle^*X$. Such an n exists because we are assuming that x maps to 0 in $BP\langle 0\rangle^*X = H^*(X, \mathbf{Z}_{(p)})$, while for n large (k being fixed) we have $BP^kX = BP\langle n\rangle^k X$ since X is finite.

We have a commuting diagram, where the first map in each row is multiplication by v_n, and the second row is an exact sequence:

$$\begin{array}{ccccc} BP^{k+2(p^n-1)}X & \longrightarrow & BP^kX & & \\ \downarrow & & \downarrow & & \\ BP\langle n\rangle^{k+2(p^n-1)}X & \longrightarrow & BP\langle n\rangle^k X & \longrightarrow & BP\langle n-1\rangle^k X \end{array}$$

Since the image x' of x in $BP\langle n\rangle^k X$ maps to 0 in $BP\langle n-1\rangle^k X$, we find that x' is equal to v_n times an element x'_n of $BP\langle n\rangle^{k+2(p^n-1)}X$. Now since $n \geq 1$ and $k \leq 4$, we have

$$k + 2(p^n - 1) \leq 4 + 2(p^n - 1) \leq 2(p^n + p^{n-1} + \cdots + 1).$$

This is exactly what we need to apply Wilson's main theorem, Theorem 2.3 above, to show that

$$BP^{k+2(p^n-1)}X \to BP\langle n\rangle^{k+2(p^n-1)}X$$

is surjective. Let x_n be an element of the first group which maps to x'_n. Then $x - v_n x_n$ maps to 0 in $BP\langle n\rangle^k X$. Now repeat this process using $x - v_n x_n$ in place of x. Since $BP^k X = BP\langle N\rangle^k X$ for N sufficiently large as we have said, this process stops after a finite number of steps. Thus x is a finite sum $x = \sum_{i>0} v_i x_i$. QED

3 The new cycle map

For convenience, we write $MU_i^{\text{BM}}X \otimes_{MU_*} \mathbf{Z}$ to denote the degree i subgroup of the graded group $MU_*^{\text{BM}}X \otimes_{MU_*} \mathbf{Z}$.

Theorem 3.1 *Let X be a complex algebraic scheme. We define a homomorphism from the group $Z_i^{alg}X$ of i-dimensional algebraic cycles on X modulo algebraic equivalence to $MU_{2i}^{BM}X \otimes_{MU_*} \mathbf{Z}$ such that the composition*

$$Z_i^{alg}X \to MU_{2i}^{BM}X \otimes_{MU_*} \mathbf{Z} \to H_{2i}^{BM}(X, \mathbf{Z})$$

is the classical cycle class map ([12], chapter 19). This homomorphism is a natural transformation on the category of proper algebraic maps.

Proof. The map is defined to send an irreducible i-dimensional subvariety $Z \subset X$ to the class of the map $[\tilde{Z} \to Z \subset X]$ in $MU_{2i}^{\text{BM}}X \otimes_{MU_*} \mathbf{Z}$, where $\tilde{Z} \to Z$ is any resolution of singularities of Z, that is, a proper birational map with $\tilde{Z}$ smooth. Such resolutions exist, by Hironaka [16]. The first step is to show that the various elements of $MU_{2i}^{\text{BM}}X$ that can arise from different resolutions $\tilde{Z}$ of a fixed variety Z are all equal in $MU_{2i}^{\text{BM}}X \otimes_{MU_*} \mathbf{Z}$.

Let $\tilde{Z}_1$ and $\tilde{Z}_2$ be any two resolutions of Z. By Hironaka [16], it is possible to blow up $\tilde{Z}_1$ repeatedly along smooth subvarieties to get a variety $\tilde{Z}_1{}'$ which maps to Z_2, giving a commutative diagram.

$$\begin{array}{ccc} \tilde{Z}_1{}' & \longrightarrow & \tilde{Z}_1 \\ \downarrow & & \downarrow \\ \tilde{Z}_2 & \longrightarrow & Z \end{array}$$

By Quillen's theorem (Theorem 2.1 above), for any finite complex X, the group $MU^j X \otimes_{MU^*} \mathbf{Z}$ is 0 for $j < 0$ and equals $H^0(X, Z)$ for $j = 0$. By Poincaré duality for complex cobordism (see section 1), it follows that, for any smooth complex i-manifold, the group $MU_j^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$ is 0 for $j > 2i$ and equals $H_{2i}^{\mathrm{BM}}(X, \mathbf{Z})$ for $j = 2i$. In particular, if $X \to Y$ is a proper birational morphism of smooth i-dimensional complex varieties, we have

$$[X \to Y] = [Y \to Y] \in MU_{2i}^{\mathrm{BM}} Y \otimes_{MU_*} \mathbf{Z},$$

because this is true in $H_{2i}^{\mathrm{BM}}(Y, \mathbf{Z}) = \mathbf{Z}$. Thus, in the situation of the previous paragraph, we have

$$[\tilde{Z}_1{}' \to \tilde{Z}_1] = [\tilde{Z}_1 \to \tilde{Z}_1] \in MU_{2i}^{\mathrm{BM}} \tilde{Z}_1 \otimes_{MU_*} \mathbf{Z}$$

and

$$[\tilde{Z}_1{}' \to \tilde{Z}_2] = [\tilde{Z}_2 \to \tilde{Z}_2] \in MU_{2i}^{\mathrm{BM}} \tilde{Z}_2 \otimes_{MU_*} \mathbf{Z}.$$

It follows that

$$[\tilde{Z}_1 \to X] = [\tilde{Z}_1{}' \to X] = [\tilde{Z}_2 \to X]$$

in $MU_{2i}^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$. Thus, any two resolutions of a subvariety $Z \subset X$ define the same element of $MU_{2i}^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$, which we are now justified in calling the class $[Z]$ of Z in $MU_{2i}^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$.

Thus we have a natural map $Z_i X \to MU_{2i}^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$ for any complex algebraic scheme X, where $Z_i X$ is the group of algebraic i-cycles on X, that is, the free abelian group on the set of closed i-dimensional irreducible subvarieties of X. For a cycle α, we write $[\alpha]$ for its class in $MU_*^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$. Our next step will be to check that this map is a natural transformation on the category of proper algebraic maps.

We recall the definition of the map $f_* : Z_i X \to Z_i Y$ associated to a proper algebraic map $f : X \to Y$ [12]. For a closed subvariety $Z \subset X$ of dimension i, $f(Z)$ is a closed subvariety of Y, and we define

$$f_*(Z) = \begin{cases} \deg\,(f : Z \to f(Z)) f(Z) & \text{if } \dim f(Z) = i \\ 0 & \text{if } \dim f(Z) < i. \end{cases}$$

So let $X \to Y$ be a proper algebraic map, and Z an i-dimensional subvariety of X. To show that the map $Z_i X \to MU_{2i}^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$ is a natural transformation means to show that the class in $MU_{2i}^{\mathrm{BM}} Y \otimes_{MU_*} \mathbf{Z}$ of the cycle $f_*(Z)$ is equal to the image of the class $[Z] \in MU_{2i}^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$ under the natural map

$$f_* : MU_*^{\mathrm{BM}} X \to MU_*^{\mathrm{BM}} Y.$$

There are two cases, depending on whether the dimension of $f(Z)$ is i or less than i.

If $f(Z)$ has dimension i, let $\tilde{Z}_2$ be a resolution of singularities of $f(Z)$, and let $\tilde{Z}_1$ be a resolution of singularities of Z such that the rational map from Z to $\tilde{Z}_2$ becomes well-defined on $\tilde{Z}_1$.

$$\begin{array}{ccccc} \tilde{Z}_1 & \longrightarrow & Z & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{Z}_2 & \longrightarrow & f(Z) & \longrightarrow & Y \end{array}$$

Clearly the map $\tilde{Z}_1 \to \tilde{Z}_2$ has the same degree d as the map $f : Z \to f(Z)$. By Quillen's theorem, Theorem 2.1 above, we have

$$[\tilde{Z}_1 \to \tilde{Z}_2] = d[\tilde{Z}_2 \to \tilde{Z}_2] \in MU_{2i}^{\mathrm{BM}} \tilde{Z}_2 \otimes_{MU_*} \mathbf{Z},$$

since $MU_{2i}^{\mathrm{BM}} \tilde{Z}_2 \otimes_{MU_*} \mathbf{Z} \cong H_{2i}^{\mathrm{BM}}(\tilde{Z}_2, \mathbf{Z}) = \mathbf{Z}$. Now the image under $f_* : MU_*^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z} \to MU_*^{\mathrm{BM}} Y \otimes_{MU_*} \mathbf{Z}$ of the class $[Z]$ is by definition equal to the class of the map $\tilde{Z}_1 \to Y$, and since this map factors through $\tilde{Z}_2$, the above equality means that this class is equal to d times $[\tilde{Z}_2 \to Y]$, that is, to d times $[f(Z)]$. This proves functoriality of our map in this case.

The argument is similar if $f(Z)$ has dimension less than i. In this case we use Quillen's theorem to prove that a proper holomorphic map $X \to Y$ between complex manifolds with dim $X = i >$ dim Y has $[X \to Y] = 0 \in MU_{2i}^{\mathrm{BM}} Y \otimes_{MU_*} \mathbf{Z}$, because this group is 0. As in the previous case, we apply this result to a resolution $\tilde{Z}_2$ of $f(Z)$ and a resolution $\tilde{Z}_1$ of Z which maps to $\tilde{Z}_2$, and we find that $f_*[Z] = 0 \in MU_*^{\mathrm{BM}} Y \otimes_{MU_*} \mathbf{Z}$.

Thus we have defined a natural transformation $Z_* X \to MU_*^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$ on the category of complex algebraic schemes and proper algebraic maps. To finish the proof of the theorem, we have to show that this map is well defined on cycles modulo algebraic equivalence. That is, we have to show that for every smooth compact connected curve C and every $(i+1)$-dimensional subvariety $W \subset X \times C$ with the second projection $f : W \to C$ not constant, we have

$$[(p_1)_* f^*(a)] = [(p_1)_* f^*(b)] \in MU_{2i}^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$$

for every pair of points a, $b \in C$, where $p_1 : W \to X$ is the first projection. (The cycles $f^*(a)$ and $f^*(b)$ in W are defined in [12], chapter 2, as the cycles associated to the subschemes $f^{-1}(a)$ and $f^{-1}(b)$ of W.) In view of the naturality we have proved, it suffices to prove that

$$[f^*(a)] = [f^*(b)] \in MU_{2i}^{\mathrm{BM}} W \otimes_{MU_*} \mathbf{Z}.$$

Let $\pi : \tilde{W} \to W$ be a resolution of singularities of W. The pushforwards of the cycles $(f\pi)^*(a)$, $(f\pi)^*(b)$ on $\tilde{W}$ to W are the cycles $f^*(a)$, $f^*(b)$ on W, by Fulton [12], p. 34, proof of (c). So it suffices to prove that

$$[(f\pi)^*(a)] = [(f\pi)^*(b)] \in MU_{2i}^{\mathrm{BM}} \tilde{W} \otimes_{MU_*} \mathbf{Z}.$$

But we know that algebraically equivalent cycles are homologous, so that these two cycles are equal in $H_{2i}^{\mathrm{BM}}(\tilde{W}, \mathbf{Z})$. And the extension of Quillen's theorem given in Theorem 2.2 shows that $MU^2 \tilde{W} \otimes_{MU^*} \mathbf{Z} \cong H^2(\tilde{W}, \mathbf{Z})$; that is, by Poincaré duality on the smooth $(i+1)$-dimensional variety $\tilde{W}$, we have $MU_{2i}^{\mathrm{BM}} \tilde{W} \otimes_{MU_*} \mathbf{Z} = H_{2i}^{\mathrm{BM}}(\tilde{W}, \mathbf{Z})$. Thus

$$[(f\pi)^*(a)] = [(f\pi)^*(b)] \in MU_{2i}^{\mathrm{BM}} \tilde{W} \otimes_{MU_*} \mathbf{Z}.$$

This proves that the cycle map is well-defined on algebraic equivalence classes. QED

Remark. The cycle class map is in fact well-defined on a slightly weaker equivalence relation than algebraic equivalence, as explained in Remark 2, section 8.

4 Products

If X is a smooth complex algebraic variety of dimension n, then cycles modulo algebraic equivalence, graded by $Z^i_{\text{alg}}X = Z^{\text{alg}}_{n-i}X$, form a ring, as do the groups $MU^iX \otimes_{MU^*} \mathbf{Z} = MU^{\text{BM}}_{2n-i}X \otimes_{MU_*} \mathbf{Z}$.

Theorem 4.1 *If X is a smooth variety, then the cycle map $Z^i_{alg}X \to MU^{2i}X \otimes_{MU^*} \mathbf{Z}$ is a ring homomorphism.*

The proof follows the outline of Fulton's proof that the usual cycle map $Z^*_{\text{alg}}X \to H^*(X, \mathbf{Z})$ is a ring homomorphism [12], chapter 19.

Proof. It is equivalent to check that the map from cycles modulo rational equivalence to $MU^*X \otimes_{MU^*} \mathbf{Z}$ is a ring homomorphism. (We do things this way because most of Fulton's book is written in terms of cycles modulo rational equivalence; the same arguments would apply to cycles modulo algebraic equivalence.)

We recall the construction of the intersection product on cycles modulo rational equivalence given by Fulton and MacPherson [12]. Given cycles α and β on a smooth variety X, there is a product cycle $\alpha \times \beta$ on $X \times X$, and the product $\alpha\beta \in CH_*X$ is defined as the pullback of $\alpha \times \beta$ to the diagonal by a map $CH_i(X \times X) \to CH_{i-n}X$. This pullback map is defined, more generally, for any regular embedding: if X is any local complete intersection subscheme of codimension d in a scheme Y, then there is a pullback map $CH_iY \to CH_{i-d}X$. For the fundamental example of a regular embedding, the inclusion of the zero-section of a vector bundle into the total space of the vector bundle, $X \hookrightarrow E$, the pullback map $CH_iE \to CH_{i-d}X$ ($d = \text{rank } E$) is defined to be the inverse of the natural map $CH_{i-d}X \to CH_iE$, sending a subvariety $Z \subset X$ to $E|_Z \subset E$, which one proves to be an isomorphism. For an arbitrary regular embedding $X \to Y$ of codimension d, the pullback map sends a subvariety $V \subset Y$ to the pullback under the zero-section inclusion $X \to N_{X/Y}$ of the normal cone C to $V \cap X$ in V. Here the normal cone C (defined as Spec $(\oplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1})$, where $\mathcal{I}$ is the ideal sheaf defining $V \cap X$ in V) is a subscheme of the normal bundle $N_{X/Y}$ of the same dimension, i, as V, so C gives an i-dimensional cycle in $N_{X/Y}$, which pulls back to an element of $CH_{i-d}X$ by the map we have already defined.

The product on MU^{BM}_*X, for a complex n-manifold X, can be defined similarly. There is an external product $MU^{\text{BM}}_*X \otimes_{MU^*} MU^{\text{BM}}_*X \to MU^{\text{BM}}_*(X \times X)$, and the internal product is defined by composing that with a pullback map $MU^{\text{BM}}_i(X \times X) \to MU^{\text{BM}}_{i-2n}X$. The pullback map is defined more generally: for any codimension-d complex submanifold X of a complex manifold Y, there is a pullback map $MU^{\text{BM}}_iY \to MU^{\text{BM}}_{i-2d}X$. It can be defined as cap product with an "orientation class" $u_{XY} \in MU^{2d}(Y, Y - X)$. To define u_{XY}, identify $MU^{2d}(Y, Y - X)$ with $MU^{2d}(N_{X/Y}, N_{X/Y} - X)$ by excision (where X is included in the normal bundle $N_{X/Y}$ as the zero-section); then u_{XY} is the Thom class of the complex vector bundle $N_{X/Y}$. (This pullback map is easier to define than the one on Chow groups, because a tubular neighborhood of $X \subset Y$ is diffeomorphic to the normal bundle $N_{X/Y}$, whereas in algebraic geometry there is typically no neighborhood of $X \subset Y$ which is algebraically or even analytically isomorphic to the normal bundle of X.)

Comparing these constructions, we see that the theorem would follow from the commutativity of the diagram of pullback maps,

$$\begin{array}{ccc} CH_iY & \longrightarrow & CH_{i-d}X \\ \downarrow & & \downarrow \\ MU_{2i}^{\mathrm{BM}}Y \otimes_{MU_*} \mathbf{Z} & \longrightarrow & MU_{2(i-d)}^{\mathrm{BM}}X \otimes_{MU_*} \mathbf{Z}, \end{array}$$

for any codimension-d smooth algebraic subvariety X of a smooth algebraic variety Y. It seems natural to prove this in somewhat greater generality. Namely, for any codimension-d regular embedding (that is, local complete intersection) $X \to Y$ of complex algebraic schemes, Baum, Fulton, and MacPherson define an orientation class $u_{XY} \in MU^{2d}(Y, Y - X)$ which agrees with that defined above when X and Y are smooth [5], p. 137. (In fact, they define such a class in any complex-oriented cohomology theory, that is, any generalized cohomology theory with Thom classes for complex vector bundles. Complex cobordism is complex-oriented; in fact, it is the universal such theory, so that the Baum-Fulton-MacPherson orientation class in complex cobordism maps to the corresponding orientation class in any other complex-oriented cohomology theory.) The construction uses that the normal cone of a regularly embedded subscheme $X \to Y$ is a vector bundle over X, called the normal bundle $N_{X/Y}$. Briefly, they extend the normal bundle $N_{X/Y}$ to a topological complex vector bundle Q on a tubular neighborhood N of X, and they construct a continuous section of Q which vanishes on X "as a scheme"; then this section gives a map $(N, N - X) \to (Q, Q - N)$, and u_{XY} is the pullback of the Thom class in $MU^{2d}(Q, Q - N)$ to $MU^{2d}(N, N - X) = MU^{2d}(Y, Y - X)$. Cap product with this class u_{XY} defines a pullback map $MU_i^{\mathrm{BM}}Y \to MU_{i-2d}^{\mathrm{BM}}X$ of MU_*-modules.

The theorem follows from the following statement about that pullback map.

Lemma 4.2 *For any codimension-d regular embedding $X \to Y$ of complex algebraic schemes, the following diagram of pullback maps commutes.*

$$\begin{array}{ccc} CH_iY & \longrightarrow & CH_{i-d}X \\ \downarrow & & \downarrow \\ MU_{2i}^{BM}Y \otimes_{MU_*} \mathbf{Z} & \longrightarrow & MU_{2(i-d)}^{BM}X \otimes_{MU_*} \mathbf{Z} \end{array}$$

Proof. This is easy to check for the fundamental example of regular embeddings, the inclusion $X \to E$ of the zero-section of a vector bundle over a scheme X. In this case, the pullback maps $CH_iE \to CH_{i-d}X$ and $MU_i^{\mathrm{BM}}E \to MU_{i-2d}^{\mathrm{BM}}X$ are both isomorphisms, so it suffices to prove that the inverse maps commute:

$$\begin{array}{ccc} CH_{i-d}X & \longrightarrow & CH_iE \\ \downarrow & & \downarrow \\ MU_{2(i-d)}^{\mathrm{BM}}X \otimes_{MU_*} \mathbf{Z} & \longrightarrow & MU_{2i}^{\mathrm{BM}}E \otimes_{MU_*} \mathbf{Z}. \end{array}$$

The map on the top row sends a subvariety $Z \subset X$ to the subvariety $E|_Z \subset E$, and the map on the bottom row sends a proper map $\tilde{Z} \to X$, for a weakly complex manifold $\tilde{Z}$, to the obvious proper map $E|_{\tilde{Z}} \to E$. The cycle map sends $Z \subset X$ to the map $\tilde{Z} \to X$ for any resolution $\tilde{Z}$ of X; since $E|_{\tilde{Z}}$ is a resolution of $E|_Z$, the diagram commutes.

Also, we need to check the lemma for regular embeddings of codimension one, at least in the following situation.

Lemma 4.3 *Let X be an n-dimensional variety, T a smooth curve, $f : X \to T$ a non-constant map, $t \in T$. Then the inclusion of the subscheme $f^{-1}(t) \subset X$ is a codimension-one regular embedding. Moreover, the class in $MU^{BM}_{2n-2}(f^{-1}(t)) \otimes_{MU_*} \mathbf{Z}$ of the cycle associated to the scheme $f^{-1}(t)$ is equal to the Baum-Fulton-MacPherson pullback of the class of X from X to the subscheme $f^{-1}(t)$.*

Proof. This is similar to the proof that the cycle map is well-defined on algebraic equivalence (Theorem 3.1). To begin, observe that the Baum-Fulton-MacPherson orientation class in $MU^2(X, X - f^{-1}(t))$ for the subscheme $f^{-1}(t) \subset X$ is the pullback $f^* u_{t,T}$ of the orientation class $u_{t,T} \in MU^2(T, T - t)$. So it suffices to show that

$$f^* u_{t,T} \cap [X] = [f^{-1}(t)] \in MU^{\mathrm{BM}}_{2n-2}(f^{-1}(t)) \otimes_{MU_*} \mathbf{Z},$$

where $[f^{-1}(t)]$ denotes the class of the cycle associated to the scheme $f^{-1}(t)$.

To prove this equality, let $\pi : \tilde{X} \to X$ be a resolution. By Fulton [12], p. 34, proof of (c), we have

$$\pi_*((f\pi)^{-1}(t)) = f^{-1}(t)$$

as cycles on the scheme $f^{-1}(t)$. Also, in $MU^{\mathrm{BM}}_{2n-2}(f^{-1}(t)) \otimes_{MU_*} \mathbf{Z}$, we have

$$\begin{aligned} \pi_*((f\pi)^* u_{t,T} \cap [\tilde{X}]) &= \pi_*(\pi^* f^* u_{t,T} \cap [\tilde{X}]) \\ &= f^* u_{t,T} \cap \pi_*[\tilde{X}] \\ &= f^* u_{t,T} \cap [X]. \end{aligned}$$

As a result, it suffices to show that

$$(f\pi)^* u_{t,T} \cap [X] = [(f\pi)^{-1}(t)] \in MU^{\mathrm{BM}}_{2n-2}((f\pi)^{-1}(t)) \otimes_{MU_*} \mathbf{Z}.$$

In other words, replacing f by $f\pi$, it suffices to prove that

$$f^* u_{t,T} \cap [X] = [f^{-1}(t)] \in MU^{\mathrm{BM}}_{2n-2}(f^{-1}(t)) \otimes_{MU_*} \mathbf{Z}$$

for X smooth of dimension n.

For X smooth, we can identify $MU^{\mathrm{BM}}_{2n-2}(f^{-1}(t)) \otimes_{MU_*} \mathbf{Z}$ with $MU^2(X, X - f^{-1}(t)) \otimes_{MU^*} \mathbf{Z}$. This is isomorphic to $H^2(X, X - f^{-1}(t), \mathbf{Z})$ by the extension of Quillen's theorem given in Theorem 2.2. (The theorem is stated in terms of the cohomology of a space, rather than a pair of spaces, but for any generalized cohomology theory h^* we can identify $h^i(X, X - S)$ with reduced h^i of a pointed space (the mapping cylinder modulo $X - S$), and the theorem clearly applies to the reduced cohomology of a pointed space.) So it suffices to prove the above equality in $H^2(X, X - f^{-1}(t), \mathbf{Z}) = H^{\mathrm{BM}}_{2n-2}(f^{-1}(t), \mathbf{Z})$, as is done in [12], p. 373. QED

Now we can prove Lemma 4.2 for a general regular embedding $X \to Y$ of complex algebraic schemes. As mentioned at the beginning of the proof of Theorem 4.1, the pullback map $CH_iY \to CH_{i-d}X$ sends a subvariety $V \subset Y$ to the pullback of $C_{V\cap X}V \subset N_{X/Y}$ to X. Since we have checked the lemma for the inclusion of the scheme X into the vector bundle $N_{X/Y}$, the lemma in general reduces to the statement that the pullback of $[V] \in MU^{\mathrm{BM}}_{2i}Y \otimes_{MU_*} \mathbf{Z}$ to X is equal to the pullback of the class of the normal cone $[C_{V\cap X}V] \in MU^{\mathrm{BM}}_{2i}N_{X/Y} \otimes_{MU_*} \mathbf{Z}$ to X.

To prove this, we use that every embedding of one scheme in another has a natural "deformation to the normal cone" [12], chapter 5. That is, for a closed subscheme $X \subset Y$, there is a scheme M_XY and a $\mathbf{P}^1$-family of embeddings of X, $X \times \mathbf{P}^1 \subset M_XY$, with a commutative diagram

$$\begin{array}{ccc} X \times \mathbf{P}^1 & \longrightarrow & M_XY \\ & \searrow & \downarrow{\scriptstyle \rho} \\ & & \mathbf{P}^1, \end{array}$$

such that for $t \in \mathbf{P}^1 - \{\infty\} = A^1$, $\rho^{-1}(t) \cong Y$ and the embedding $X \subset \rho^{-1}(t)$ is the given embedding $X \subset Y$, and over ∞, the embedding is the embedding of X in the normal cone C_XY of X in Y. The map $M_XY \to \mathbf{P}^1$ is flat, and so the inverse image of each point of $\mathbf{P}^1$ is a regularly embedded subscheme of codimension 1. Explicitly, M_XY is the blow-up of $Y \times \mathbf{P}^1$ along $X \times \infty$, with the proper transform of $Y \times \infty$ omitted.

To prove the lemma, let $X \to Y$ be a regular embedding and let $V \subset Y$ be a subvariety. Then $M' = M_{V\cap X}V$ is a subvariety of the scheme $M = M_XY$. Here M' is a variety (irreducible and reduced), not just a scheme, because V is a variety: the blow-up of a variety (here $V \times \mathbf{P}^1$) along a subscheme is always a variety.

$$\begin{array}{ccccc} V\cap X & \longrightarrow & C & \longrightarrow & \{\infty\} \\ \downarrow & & \downarrow & & \downarrow \\ (V\cap X)\times \mathbf{P}^1 & \longrightarrow & M' & \longrightarrow & \mathbf{P}^1 \\ \uparrow & & \uparrow & & \uparrow \\ V\cap X & \longrightarrow & V & \longrightarrow & \{0\} \end{array} \quad \subset \quad \begin{array}{ccccc} X & \longrightarrow & N_{X/Y} & \longrightarrow & \{\infty\} \\ \downarrow & & \downarrow & & \downarrow \\ X\times \mathbf{P}^1 & \xrightarrow[F]{} & M & \longrightarrow & \mathbf{P}^1 \\ \uparrow & & \uparrow & & \uparrow \\ X & \longrightarrow & Y & \longrightarrow & \{0\} \end{array}$$

The variety V and the normal cone $C = C_{V\cap X}V$ are the fibers over 0 and ∞, respectively, of the map $M' \to \mathbf{P}^1$, so Lemma 4.3 gives that the fundamental class in $MU^{\mathrm{BM}}_{2i}V \otimes_{MU_*} \mathbf{Z}$ of the subvariety $V \subset Y$ is the Baum-Fulton-MacPherson pullback of the fundamental class of M', by the codimension-one regular embedding $V \subset M'$. (To apply Lemma 4.3, we use that M' is a variety, not just a scheme.) It follows that that the class $[V] \in MU^{\mathrm{BM}}_{2i}Y \otimes_{MU_*} \mathbf{Z}$ is the pullback of the class

$[M'] \in MU^{\mathrm{BM}}_{2(i+1)} M \otimes_{MU_*} \mathbf{Z}$ by the codimension-one regular embedding $Y \subset M$. Likewise, the class in $MU^{\mathrm{BM}}_{2i} N_{X/Y} \otimes_{MU_*} \mathbf{Z}$ of the normal cone $C = C_{V\cap X}V$, viewed as a scheme and thus as a cycle with multiplicities, is the pullback of the class of $[M']$ on M by the codimension-one regular embedding $N_{X/Y} \subset M$.

As a result, the pullback of the class $[V]$ on Y to $MU^{\mathrm{BM}}_{2(i-d)} X \otimes_{MU_*} \mathbf{Z}$ is the pullback of $[M']$ on M first by the codimension-one regular embedding $Y \subset M$, then by the codimension-d regular embedding $X \times 0 \subset Y$. Likewise, the pullback of the class $[C]$ on $N_{X/Y}$ to X is the pullback of the same class $[M']$ first by the codimension-one regular embedding $N_{X/Y} \subset M$, then by the codimension-d regular embedding $X \subset N_{X/Y}$. By the above commutative diagrams, using the naturality of Baum-Fulton-MacPherson's pullback maps on bordism groups, we find that these two elements of $MU^{\mathrm{BM}}_{2(i-d)} X \otimes_{MU_*} \mathbf{Z}$ are the pullbacks of the same class $[M']$ on M by the codimension-d regular embedding $F : X \times \mathbf{P}^1 \to M$, followed by the pullback to $X \times 0$ or $X \times \infty$ respectively. Since these last two pullbacks are equal on $MU^{\mathrm{BM}}_*(X \times \mathbf{P}^1)$, the two elements of $MU^{\mathrm{BM}}_{2(i-d)} X \otimes_{MU_*} \mathbf{Z}$ are equal. QED

5 Non-injectivity of the map $MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X, \mathbf{Z})$

In this section we construct some topological spaces X for which the map $MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X, \mathbf{Z})$ is not injective, and such that there is a natural way to approximate the homotopy type of X by smooth algebraic varieties. Namely, X will be the classifying space BG of a compact Lie group G (we will explain the relation to algebraic geometry in section 7).

We need to take several precautions when talking about the complex cobordism of an infinite CW complex such as BG. The point is that $MU^*(\text{point})$ is nonzero in all even dimensions ≤ 0, so that MU^iX is affected by all the cells in X of dimension $\geq i$. One phenomenon here is described by Milnor's exact sequence, which holds for all generalized cohomology theories h^* and all infinite CW complexes X [26]:

$$0 \to \varprojlim{}^1_n h^{i-1}(X_n) \to h^iX \to \varprojlim{}_n h^i(X_n) \to 0.$$

Here X_n denotes the n-skeleton of X. The Atiyah-Hirzebruch spectral sequence $H^*(X, h^*) \Longrightarrow h^*(X)$ actually converges to $\varprojlim h^*(X_n)$, not to h^*X.

Fortunately, the only infinite complexes we will need to consider are classifying spaces of compact Lie groups BG, in which case the $\varprojlim^1$ term for MU-theory and the other cohomology theories we consider is 0, by Landweber [22]. For such spaces, Landweber also proves a strong Mittag-Leffler statement about the inverse limit in $MU^*X = \varprojlim MU^*X_n$: namely, for each n there is an $m \geq n$ such that (in all dimensions at once) we have im $(MU^*X \to MU^*X_n) =$ im $(MU^*X_m \to MU^*X_n)$. To give a little context for these statements: for the space $X = K(\mathbf{Z}, 3)$, which is outside the class we consider, the $\varprojlim^1$ group is nonzero, and the Mittag-Leffler statement fails, because im $(MU^3K(\mathbf{Z}, 3)_n \to H^3(K(\mathbf{Z}, 3)_n, \mathbf{Z}) = \mathbf{Z})$ is a subgroup of finite index which decreases to 0 as n goes to infinity. See [35] for some clarification of this phenomenon.

Since each group MU^iBG is an inverse limit, we have to view it as a topological abelian group. In particular, tensor products involving MU^*BG will always mean *completed* tensor products. For example,

$$MU^*BG \otimes_{MU^*} \mathbf{Z} := \varprojlim \quad [(\text{im } MU^*BG \to MU^*(BG)_n) \otimes_{MU^*} \mathbf{Z}].$$

Likewise, following Kono and Yagita [20], we define

$$MU^*BG \otimes_{MU^*} MU^*BH := \varprojlim \quad (\text{im } MU^*BG \to MU^*(BG)_n) \otimes_{MU^*} (\text{im } MU^*BH \to MU^*(BH)_n).$$

Now we can state the main result of this section.

Theorem 5.1 *Let G be either $SO(4)$ or the central extension $1 \to \mathbf{Z}/2 \to G \to (\mathbf{Z}/2)^4 \to 1$ contained in $SO(4)$. Then the maps*

$$MU^4BG \otimes_{MU^*} \mathbf{Z}/2 \to H^4(BG, \mathbf{Z}/2)$$

and

$$MU^6(BG \times B\mathbf{Z}/2) \otimes_{MU^*} \mathbf{Z} \to H^6(BG \times B\mathbf{Z}/2, \mathbf{Z})$$

are not injective.

We begin by explaining why product groups are convenient for this question. In the simplest examples, $MU^*BG \otimes_{MU^*} \mathbf{Z} \to H^*(BG, \mathbf{Z})$ tends to be injective but not surjective. It happens, however, that even if this map is injective for two groups G_1 and G_2, it need not be injective for $G_1 \times G_2$. Specifically, if the map $MU^*BG \otimes_{MU^*} \mathbf{Z} \to H^*(BG, \mathbf{Z})$ is injective but not split injective, as a map of abelian groups, then the map $MU^*(BG \times B\mathbf{Z}/p) \otimes_{MU^*} \mathbf{Z} \to H^*(BG \times B\mathbf{Z}/p, \mathbf{Z})$ is not injective for some prime p. The following lemma expresses this idea more precisely.

Lemma 5.2 *Let G be a compact Lie group. If the map $MU^kBG \otimes_{MU^*} \mathbf{Z}/p \to H^k(BG, \mathbf{Z}/p)$ is not injective, then $MU^{k+2}(BG \times B\mathbf{Z}/p) \otimes_{MU^*} \mathbf{Z} \to H^{k+2}(BG \times B\mathbf{Z}/p, \mathbf{Z})$ is not injective.*

Proof. By Landweber [21], Theorem 2′, $MU^*B\mathbf{Z}/p$ is flat in the sense that we have an isomorphism of topological abelian groups

$$MU^*(X \times B\mathbf{Z}/p) = MU^*X \otimes_{MU^*} MU^*B\mathbf{Z}/p$$

for all finite complexes X. (Kono and Yagita [20] conjecture the same statement for all compact Lie groups in place of $\mathbf{Z}/p$.) As a consequence, we have an isomorphism of topological abelian groups

$$MU^*(BG \times B\mathbf{Z}/p) = MU^*BG \otimes_{MU^*} MU^*B\mathbf{Z}/p$$

for all compact Lie groups G. The right side is a completed tensor product, as explained above.

It follows that

$$MU^*(BG \times B\mathbf{Z}/p) \otimes_{MU^*} \mathbf{Z} = (MU^*BG \otimes_{MU^*} \mathbf{Z}) \otimes_{\mathbf{Z}} (MU^*B\mathbf{Z}/p \otimes_{MU^*} \mathbf{Z}),$$

where the right side is again a completed tensor product. Here $MU^*B\mathbf{Z}/p \otimes_{MU^*} \mathbf{Z}$ maps isomorphically to $H^*(B\mathbf{Z}/p, \mathbf{Z}) = \mathbf{Z}[[c_1]]/(pc_1)$, where c_1 is in dimension 2. So

$$MU^*(BG \times B\mathbf{Z}/p) \otimes_{MU^*} \mathbf{Z} = MU^*BG \otimes_{MU^*} \mathbf{Z} \oplus \prod_{i \geq 1} c_1^i \; MU^*BG \otimes_{MU^*} \mathbf{Z}/p,$$

and by our assumption in this lemma, the group $c_1 \cdot MU^kBG \otimes_{MU^*} \mathbf{Z}/p$ maps non-injectively to the corresponding group in

$$\begin{aligned} & H^*(BG, \mathbf{Z}) \oplus \prod_{i \geq 1} c_1^i \cdot H^*(BG, \mathbf{Z}/p) \\ & = H^*(BG, \mathbf{Z}) \otimes_{\mathbf{Z}} H^*(B\mathbf{Z}/p, \mathbf{Z}) \\ & \subset H^*(BG \times B\mathbf{Z}/p, \mathbf{Z}). \end{aligned}$$

Thus $MU^{k+2}(BG \times B\mathbf{Z}/p) \otimes_{MU^*} \mathbf{Z} \to H^{k+2}(BG \times B\mathbf{Z}/p, \mathbf{Z})$ is not injective. QED

Corollary 5.3 *Let G be a compact Lie group. If the image of the map $MU^kBG \to H^k(BG, \mathbf{Z})$ contains p times an element $x \in H^k(BG, \mathbf{Z})$, for some prime number p, but does not contain x itself or x plus any element killed by p in $H^k(BG, \mathbf{Z})$, then*

$$MU^kBG \otimes_{MU^*} \mathbf{Z}/p \to H^*(BG, \mathbf{Z}/p)$$

and

$$MU^{k+2}(BG \times B\mathbf{Z}/p) \otimes_{MU^*} \mathbf{Z} \to H^{k+2}(BG \times B\mathbf{Z}/p, \mathbf{Z})$$

are not injective.

Proof. The hypothesis implies that an element of MU^kBG which maps to $px \in H^k(BG, \mathbf{Z})$ is nonzero in $MU^k(BG) \otimes_{MU^*} \mathbf{Z}/p$, and it clearly maps to 0 in $H^k(BG, \mathbf{Z}/p)$. Thus Lemma 5.2 applies. QED

Thus, to prove Theorem 5.1, it suffices to prove that the image of $MU^4BG \to H^4(BG, \mathbf{Z})$ contains 2 times some element $x \in H^4(BG, \mathbf{Z})$ but not x plus any 2-torsion element, when G is $SO(4)$ or the group of order 32 mentioned in the theorem. For $G = SO(4)$, Kono and Yagita [20] computed $MU^*BSO(4)$, and we can read this off from their calculation. For clarity, here is a direct proof. The point is that there is a class $\chi \in H^4(BSO(4), \mathbf{Z})$, the Euler class, such that 2χ is in the image of $MU^4BSO(4)$ but χ is not. See Milnor-Stasheff [27] for the cohomology of $BSO(n)$ and in particular the definition of the Euler class. In fact, in $H^4(BSO(4), \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$, we can compute that

$$2\chi = c_2A - c_2B,$$

where $B : SO(4) \to GL(4, \mathbf{C})$ is the obvious representation and A is the representation given by thinking of $SO(4)$ as a double cover of $SO(3) \times SO(3)$, and then projecting to the first $SO(3)$:

$$A : SO(4) \to SO(3) \times SO(3) \to SO(3) \to GL(3, \mathbf{C}).$$

(To check that $2\chi = c_2 A - c_2 B$, use that $SO(4)$ is doubly covered by $SU(2) \times SU(2)$, and $H^4(BSO(4), \mathbf{Z})$ injects into $H^4(BSU(2) \times BSU(2), \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$, where the equality is easy to check.) A complex vector bundle has Chern classes in complex cobordism which map to the usual Chern classes in ordinary cohomology [1], so this equality means that 2χ is in the image of $MU^4 BSO(4)$. For convenience, let C be the element $c_2 A - c_2 B$ in $MU^4 BSO(4)$, so that C maps to 2χ in $H^4(BSO(4), \mathbf{Z})$.

To prove that χ itself is not in the image of $MU^4 BSO(4)$, we show that a certain odd-dimensional Steenrod operation, Sq^3, is nonzero on the image of χ in

$$H^*(BSO(4), \mathbf{Z}/2) = \mathbf{Z}/2[w_2, w_3, w_4].$$

We use Wu's formula for the Steenrod operations in $H^*(BSO(n), \mathbf{Z}/2)$ [27], p. 94:

$$Sq^3 \chi = Sq^3 w_4 = w_3 w_4 \neq 0.$$

Thus $\chi \in H^4(BSO(4), \mathbf{Z})$ is not in the image of $MU^4 BSO(4)$. Since $H^4(BSO(4), \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$ has no torsion, Corollary 5.3 applies, proving Theorem 5.1 for the case of $SO(4)$. (By the way, the same thing happens for $SO(2n)$ for all $n \geq 2$: $2^{n-1}\chi \in H^{2n}(BSO(2n), \mathbf{Z})$ is a polynomial in Chern classes of representations of $SO(2n)$, so it is in the image of $MU^{2n} BSO(2n)$, but χ itself is not in the image. It is plausible that $2^{n-1}\chi$ should be the smallest multiple of χ which is in the image of $MU^{2n} BSO(2n)$, as the above calculation shows for $SO(4)$ and as Inoue [18] showed for $SO(6)$.)

Now we turn to the construction of a similar example among finite groups. The idea is to use a finite subgroup $G \subset SO(4)$ and the restriction of the Euler class χ to $H^4(BG, \mathbf{Z})$. Then it is automatic that 2χ is in the image of $MU^4 BG$, and we just have to choose G so that $\chi +$ (2-torsion in $H^4(BG, \mathbf{Z})$) does not intersect the image of $MU^4 BG$. (Throughout this paper, a 2-torsion element of an abelian group will mean an element x with $2x = 0$, not just an element killed by some power of 2.) Since the phenomenon we are considering is 2-local, it is natural to take G to be a reasonably big 2-subgroup of $SO(4)$. In general, $SO(n)$ contains a fairly big abelian 2-subgroup, the group $(\mathbf{Z}/2)^{n-1}$ of diagonal matrices with entries ± 1, but it turns out that abelian subgroups of $SO(4)$ do not have the property we want. Fortunately, we get a more interesting subgroup of $SO(4)$ by defining G to be the inverse image of the subgroup $(\mathbf{Z}/2)^2 \times (\mathbf{Z}/2)^2 \subset SO(3) \times SO(3)$ under the double cover $SO(4) \to SO(3) \times SO(3)$. Thus G is an extra-special group of order 32, that is, a central extension

$$1 \to \mathbf{Z}/2 \to G \to (\mathbf{Z}/2)^4 \to 1$$

with center exactly $\mathbf{Z}/2$. In different terminology, G is a Heisenberg group and the embedding $G \subset SO(4)$, the unique irreducible representation of G of dimension greater than one, is the Schrödinger representation of G.

In what is probably the most beautiful calculation in the cohomology of groups, Quillen [34] computed the $\mathbf{Z}/2$-cohomology of the extra-special 2-groups. For the above group G, which Quillen calls "the real case," where the central extension is classified by the quadratic form $x_1x_2 + x_3x_4$ over $\mathbf{Z}/2$, Corollary 5.12 in Quillen's paper says that the embedding $G \subset SO(4)$ makes $H^*(BG, \mathbf{Z}/2)$ a free module over $H^*(BSO(4), \mathbf{Z}/2) = \mathbf{Z}/2[w_2, w_3, w_4]$. It follows that

$$Sq^3\chi = Sq^3 w_4 = w_3w_4 \neq 0 \in H^*(BG, \mathbf{Z}/2),$$

since the same is true in $H^*(BSO(4), \mathbf{Z}/2)$. We need to show a little more than this, namely that if $y \in H^4(BG, \mathbf{Z})$ is killed by 2, then $Sq^3(\chi+y) \neq 0$. To see this, we have to use a second description of $H^*(BG, \mathbf{Z}/2)$ from Quillen's paper: $H^*(BG, \mathbf{Z}/2)$ is the tensor product of a quotient ring of $H^*(B(\mathbf{Z}/2)^4, \mathbf{Z}/2)$ with the polynomial ring $\mathbf{Z}/2[w_4]$. (The homomorphism $H^*(B(\mathbf{Z}/2)^4, \mathbf{Z}/2) \to H^*(BG, \mathbf{Z}/2)$ comes from the abelianization map $G \to (\mathbf{Z}/2)^4$.) Now if y is an element of $H^4(BG, \mathbf{Z})$ killed by 2, then y is the Bockstein of an element of $H^3(BG, \mathbf{Z}/2)$, and by Quillen's second description of $H^*(BG, \mathbf{Z}/2)$, all of $H^3(BG, \mathbf{Z}/2)$ is in the image of $H^*(B(\mathbf{Z}/2)^4, \mathbf{Z}/2)$. It follows that $Sq^3 y$ is also in the image of $H^*(B(\mathbf{Z}/2)^4, \mathbf{Z}/2)$. Since $Sq^3\chi = w_3w_4$ is not in that subring, we have that $Sq^3(\chi + y) \neq 0$ for all $y \in H^4(BG, \mathbf{Z})$ killed by 2. This is what we need for Corollary 5.3 to apply. Thus the proof of Theorem 5.1 is complete. In particular, the map $MU^6(BG \times B\mathbf{Z}/2) \otimes_{MU^*} \mathbf{Z} \to H^6(BG \times B\mathbf{Z}/2, \mathbf{Z})$ is not injective. QED

For the next section, we need the following strengthening of the first part of Theorem 5.1. Let $BP\langle 1\rangle$ be the cohomology theory mentioned in section 2, for the prime 2: thus $BP\langle 1\rangle$ has coefficient ring $\mathbf{Z}_{(2)}[v_1]$, $v_1 \in BP\langle 1\rangle^{-2}$. In other words, $BP\langle 1\rangle$ is the localization of connective K-theory at the prime 2, which gives the ring structure on $BP\langle 1\rangle^* X$ in a natural way.

Lemma 5.4 *Let G be either $SO(4)$ or the central extension $1 \to \mathbf{Z}/2 \to G \to (\mathbf{Z}/2)^4 \to 1$ contained in $SO(4)$. Then the map*

$$BP\langle 1\rangle^4 BG \otimes_{BP\langle 1\rangle^*} \mathbf{Z}/2 \to H^4(BG, \mathbf{Z}/2)$$

is not injective.

Proof. There are maps of cohomology theories

$$MU^*X \to BP^*X \to BP\langle 1\rangle^*X \to BP\langle 0\rangle^*X = H^*(X, \mathbf{Z}_{(2)}) \to H^*(X, \mathbf{Z}/2)$$

(see section 2). So it suffices to show that the element $C \in MU^4 BG \otimes_{MU^*} \mathbf{Z}/2$ remains nonzero in $BP\langle 1\rangle^4 BG \otimes_{BP\langle 1\rangle^*} \mathbf{Z}/2$. The same proof works, because the Steenrod operation Sq^3 on the $\mathbf{Z}/2$-cohomology of any space vanishes on the image of $BP\langle 1\rangle$, not just on the image of MU. (Equivalently, Milnor's operation Q_1 vanishes on the image of $BP\langle 1\rangle$; see the proof of Lemma 6.3 for a more general statement.) QED

6 Finite complexes with $MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X, \mathbf{Z})$ not injective

In section 5, we gave examples of compact Lie groups G such that the maps $MU^*BG \otimes_{MU^*} \mathbf{Z}/2 \to H^*(BG, \mathbf{Z}/2)$ or $MU^*BG \otimes_{MU^*} \mathbf{Z} \to H^*(BG, \mathbf{Z})$ are not injective. In this section, we show that the elements we construct in the kernel remain nonzero when restricted from BG to its n-skeleton, for some finite dimension n. This is not hard to prove if we don't need to know exactly what dimension is necessary, but we prefer to prove this for the smallest possible dimension, although this seems to require a long calculation. The result will be used in section 7 to construct our examples in algebraic geometry.

Let G be the central extension

$$1 \to \mathbf{Z}/2 \to G \to (\mathbf{Z}/2)^4 \to 1$$

considered in section 5. We defined an element $C \in MU^4BG$ which is nonzero in $MU^4BG \otimes_{MU^*} \mathbf{Z}/2$ but which maps to 0 in $H^4(BG, \mathbf{Z}/2)$.

Proposition 6.1 *The element $C \in MU^4BG \otimes_{MU^*} \mathbf{Z}/2$ remains nonzero in $MU^4X_7 \otimes_{MU^*} \mathbf{Z}/2$, where X_7 denotes the 7-skeleton of (any cell decomposition of) BG. In fact, C remains nonzero in $BP\langle 1\rangle^4 X_7 \otimes_{BP\langle 1\rangle^*} \mathbf{Z}/2$, where $BP\langle 1\rangle$ is the cohomology theory discussed in section 2.*

Proof. In view of the map of cohomology theories $MU \to BP\langle 1\rangle$, it suffices to prove that C is nonzero in $BP\langle 1\rangle^4 X_7 \otimes_{BP\langle 1\rangle^*} \mathbf{Z}/2$. We proved in Lemma 5.4 that $C \in BP\langle 1\rangle^4 BG$ is nonzero in $BP\langle 1\rangle^4 BG \otimes_{BP\langle 1\rangle^*} \mathbf{Z}/2$ by computing that C maps to $2\chi \in H^4(BG, \mathbf{Z})$, where $Sq^3(\chi + (\text{any 2-torsion element in } H^4(BG, \mathbf{Z})))$ is nonzero in $H^7(BG, \mathbf{Z}/2)$. The same calculations apply to the 7-skeleton of BG. QED

We now consider the element $C \otimes c_1 \in MU^6(BG \times B\mathbf{Z}/2) \otimes_{MU^*} \mathbf{Z}$, for G the central extension mentioned above, which maps to 0 in $H^6(BG \times B\mathbf{Z}/2, \mathbf{Z})$ by section 5.

Proposition 6.2 *The element $C \otimes c_1 \in MU^6(BG \times B\mathbf{Z}/2) \otimes_{MU^*} \mathbf{Z}$ remains nonzero in $MU^6(BG \times B\mathbf{Z}/2)_{15} \otimes_{MU^*} \mathbf{Z}$, where $(BG \times B\mathbf{Z}/2)_{15}$ denotes the 15-skeleton of $BG \times B\mathbf{Z}/2$.*

The proof occupies the rest of this section.

Proof. Slightly more precisely, we will show that $C \otimes c_1$ is nonzero in $MU^6(X_7 \times Y_8) \otimes_{MU^*} \mathbf{Z}$, where X_7 is the 7-skeleton of BG and Y_8 is the 8-skeleton of $B\mathbf{Z}/2$. Let BP^* denote the Brown-Peterson cohomology theory associated to the prime number 2 (see the end of section 1). As explained at the end of section 1, we have

$$MU^*X \otimes_{MU^*} \mathbf{Z}_{(2)} = BP^*X \otimes_{BP^*} \mathbf{Z}_{(2)}$$

for any space X, so it suffices to prove that $C \otimes c_1$ is nonzero in $BP^6(X_7 \times Y_8) \otimes_{BP^*} \mathbf{Z}_{(2)}$.

The proof is suggested by looking at the proof that $BP^*X \otimes_{BP^*} \mathbf{Z}_{(2)} \to H^*(X, \mathbf{Z}_{(2)})$ is injective for any finite complex X in degrees ≤ 4 (Theorem 2.2), and seeing how it can fail in degree 6. Namely, if one tries to use that proof to show that an element y in BP^6X which maps to 0 in $H^6(X, \mathbf{Z}_{(2)})$ must be 0 in $BP^6X \otimes_{BP^*} \mathbf{Z}_{(2)}$, the first step works: by the exact sequence

$$BP\langle 1\rangle^8 X @> v_1 >> BP\langle 1\rangle^6 X @>>> H^6(X, \mathbf{Z}_{(2)}),$$

we can write $y = v_1 x$ for some $x \in BP\langle 1\rangle^8 X$. But the next step in that proof will fail: x need not in general lift to $BP\langle 2\rangle^8 X$, since the degree of the class x is too large for Wilson's theorem, Theorem 2.3, to apply.

This suggests the method we will use to show that $C \otimes c_1 \in BP^6(X_7 \times Y_8)$ is nonzero in $BP^6(X \times Y_8) \otimes_{BP^*} \mathbf{Z}_{(2)}$: we will prove the following statement.

Lemma 6.3 *If x is an element of $BP\langle 1\rangle^8(X_7 \times Y_8)$ such that*

$$C \otimes c_1 = v_1 x \in BP\langle 1\rangle^6(X_7 \times Y_8),$$

then x is not in the image of $BP\langle 2\rangle^8(X_7 \times Y_8)$.

To see that this implies Proposition 6.2, suppose that the Proposition is false, that is, that $C \otimes c_1$ is 0 in $BP^6(X_7 \times Y_8) \otimes_{BP^*} \mathbf{Z}_{(2)}$. This means that we can write

$$C \otimes c_1 = \sum_{i \geq 1} v_i x_i \in BP^6(X_7 \times Y_8)$$

for some elements $x_i \in BP^{6+2(2^i-1)}(X_7 \times Y_8)$. We can then apply the map of cohomology theories

$$BP = BP\langle\infty\rangle \to \cdots \to BP\langle 2\rangle \to BP\langle 1\rangle,$$

which sends multiples of v_i to 0 for $i \geq 2$, to get an equality

$$C \otimes c_1 = v_1 x_1$$

in $BP\langle 1\rangle^6(X_7 \times Y_8)$. Here x_1 lifts to BP and hence to $BP\langle 2\rangle$, so this cannot happen by Lemma 6.3.

The benefit of this proof of Proposition 6.2 is that instead of trying to compute Brown-Peterson cohomology, with its big coefficient ring $BP^* = \mathbf{Z}_{(2)}[v_1, v_2, \ldots]$, we can spend most of our time computing in the much simpler cohomology theory $BP\langle 1\rangle$, with coefficient ring $\mathbf{Z}_{(2)}[v_1]$. The theory $BP\langle 1\rangle$ can be described as the localization of connective K-theory at the prime 2.

Proof of Lemma 6.3. We are given an equation $C \otimes c_1 = v_1 x \in BP\langle 1\rangle^6(X_7 \times Y_8)$ for an element $x \in BP\langle 1\rangle^8(X_7 \times Y_8)$. To show that x is not in the image of $BP\langle 2\rangle$, we can use the long exact sequence mentioned in section 2:

$$BP\langle 2\rangle^8(X_7 \times Y_8) \to BP\langle 1\rangle^8(X_7 \times Y_8) \to BP\langle 2\rangle^{15}(X_7 \times Y_8).$$

Thus, we want to show that x has nonzero image in $BP\langle 2\rangle^{15}(X_7 \times Y_8)$; so we need to understand the map from $BP\langle 1\rangle^i X$ to $BP\langle 2\rangle^{i+7}X$, for any space X, which gives the obstruction to lifting an element of $BP\langle 1\rangle^i X$ to $BP\langle 2\rangle$.

It turns out to be enough to know that this map corresponds to Milnor's operation Q_2 on $\mathbf{Z}/2$-cohomology, in the sense that there is a commutative diagram,

$$\begin{array}{ccc} BP\langle 1\rangle^i X & \longrightarrow & BP\langle 2\rangle^{i+7}X \\ \downarrow & & \downarrow \\ H^i(X,\mathbf{Z}/2) & \xrightarrow[Q_2]{} & H^{i+7}(X,\mathbf{Z}/2). \end{array}$$

More generally, the obstruction to lifting an element of $BP\langle k-1\rangle$ to $BP\langle k\rangle$ corresponds in the same way to the operation Q_k on $\mathbf{Z}/2$-cohomology ([43], proof of Proposition 1.7). Here Milnor's operations

$$Q_k : H^i(X,\mathbf{Z}/2) \to H^{i+2^{k+1}-1}(X,\mathbf{Z}/2)$$

are defined in terms of Steenrod squares by

$$Q_0 = Sq^1, \qquad Q_{k+1} = Sq^{2^{k+1}}Q_k + Q_k Sq^{2^{k+1}}$$

[24]. Notice that the operations Q_k are all odd-dimensional, so the results of this paragraph refine the statement that all odd-dimensional Steenrod operations vanish on the image of BP^*X (or, equivalently, the image of MU^*X) in $H^*(X,\mathbf{Z}/2)$. Namely, the map from BP^*X to $\mathbf{Z}/2$-cohomology factors,

$$BP^*X = BP\langle\infty\rangle^*X \to \cdots \to BP\langle 1\rangle^*X \to BP\langle 0\rangle^*X = H^*(X,\mathbf{Z}_{(2)}) \to H^*(X,\mathbf{Z}/2),$$

and the results of this paragraph imply that $\mathbf{Z}/2$-cohomology classes in the image of $BP\langle k\rangle^*X$ are killed by the odd-dimensional operations $Q_0,\ldots,Q_k$.

Recall that we are given an element $x \in BP\langle 1\rangle^8(X_7\times Y_8)$ such that $C\otimes c_1 = v_1 x$, and we want to show that x does not lift to $BP\langle 2\rangle$. By the above remarks, it suffices to show that the image of x in $H^8(X_7\times Y_8,\mathbf{Z}/2)$ has $Q_2 x \neq 0$. Now let us use a convenient property of Milnor's operations Q_k: they are primitive elements in the Steenrod algebra, which means that

$$Q_k(xy) = Q_k(x)y + xQ_k(y)$$

for $\mathbf{Z}/2$-cohomology classes x, y on any space [24]. As a result, remembering that Q_2 raises dimension by 7, we see for dimensional reasons that Q_2 is 0 on all the Künneth components $H^i(X_7,\mathbf{Z}/2)\otimes H^{8-i}(Y_8,\mathbf{Z}/2)$ of $H^8(X_7\times Y_8,\mathbf{Z}/2)$ except one, the subgroup $H^7(X_7,\mathbf{Z}/2)\otimes H^1(Y_8,\mathbf{Z}/2)$. And Q_2 sends that subgroup injectively into

$$H^{15}(X_7\times Y_8,\mathbf{Z}/2) = H^7(X_7,\mathbf{Z}/2)\otimes H^8(Y_8,\mathbf{Z}/2),$$

because the map

$$Q_2 : \mathbf{Z}/2 = H^1(Y_8,\mathbf{Z}/2) \to H^8(Y_8,\mathbf{Z}/2)$$

is injective. (Remember that Y_8 is the 8-skeleton of $B\mathbf{Z}/2 = \mathbf{R}P^\infty$, which has $\mathbf{Z}/2$-cohomology ring $\mathbf{Z}/2[u]$, $u \in H^1$. So

$$\begin{aligned} Q_2 u &= Sq^4 Sq^2 Sq^1 u \\ &= ((u^2)^2)^2 \\ &= u^8. \end{aligned}$$

This is the only place where it matters that we have taken at least the 8-skeleton of $B\mathbf{Z}/2$.) Putting all this together, if we can show that the image of x in $H^8(X_7 \times Y_8, \mathbf{Z}/2)$ has nonzero Künneth component in $H^7(X_7, \mathbf{Z}/2) \otimes H^1(Y_8, \mathbf{Z}/2)$, then x does not lift to $BP\langle 2\rangle$, and Lemma 6.3 will be proved.

Thus we want to show that if $x \in BP\langle 1\rangle^8(X_7 \times Y_8)$ satisfies $C \otimes c_1 = v_1 x$, then the image of x in $H^8(X_7 \times Y_8, \mathbf{Z}/2)$ has nonzero Künneth component in $H^7 \otimes H^1$. By restricting such an element x to the smaller space $X_7 \times Y_2$, where $Y_2 = \mathbf{R}P^2$ is the 2-skeleton of $B\mathbf{Z}/2$, we see that it is enough to prove the following lemma. (Calculations will be easier on this smaller space.)

Lemma 6.4 *If x is an element of $BP\langle 1\rangle^8(X_7 \times Y_2)$ such that $C \otimes c_1 = v_1 x$, then the image of x in $H^8(X_7 \times Y_2, \mathbf{Z}/2)$ has nonzero Künneth component in $H^7(X_7) \otimes H^1(Y_2)$.*

Proof of Lemma 6.4. We need to analyze $BP\langle 1\rangle^*(X_7 \times Y_2)$ as a module over $BP\langle 1\rangle^* = \mathbf{Z}_{(2)}[v_1]$. There are various spectral sequences for computing these groups. In particular, for a fibration $F \to E \to B$ and any generalized cohomology theory h^*, there is a spectral sequence

$$H^*(B, h^* F) \Longrightarrow h^*(E).$$

For h^* = ordinary cohomology, this is the usual spectral sequence of a fibration, and for F = point, this is the Atiyah-Hirzebruch spectral sequence. The spectral sequence that seems to compute most directly what we want to know about $BP\langle 1\rangle^*(X_7 \times Y_2)$ is the spectral sequence of the above type,

$$H^*(Y_2, BP\langle 1\rangle^* X_7) \Longrightarrow BP\langle 1\rangle^*(Y_2 \times X_7).$$

Notice that $BP\langle 1\rangle^* X_7$ is only nonzero in degrees ≤ 7. So this spectral sequence converging to $BP\langle 1\rangle^*(Y_2 \times X_7)$ looks like:

$$\begin{array}{ccc}
H^0(Y_2, BP\langle 1\rangle^7 X_7) & H^1(Y_2, BP\langle 1\rangle^7 X_7) & H^2(Y_2, BP\langle 1\rangle^7 X_7) \\
 & \searrow & \\
H^0(Y_2, BP\langle 1\rangle^6 X_7) & H^1(Y_2, BP\langle 1\rangle^6 X_7) & H^2(Y_2, BP\langle 1\rangle^6 X_7) \\
 & \searrow & \\
H^0(Y_2, BP\langle 1\rangle^5 X_7) & H^1(Y_2, BP\langle 1\rangle^5 X_7) & H^2(Y_2, BP\langle 1\rangle^5 X_7) \\
 & \searrow & \\
H^0(Y_2, BP\langle 1\rangle^4 X_7) & H^1(Y_2, BP\langle 1\rangle^4 X_7) & H^2(Y_2, BP\langle 1\rangle^4 X_7) \\
 & & \\
\vdots & &
\end{array}$$

Here the only possible differential is d_2, shown above, and in fact this is 0 because the restriction map $BP\langle 1\rangle^*(Y_2 \times X_7) \to BP\langle 1\rangle^*(X_7)$ (corresponding to the left column in the spectral sequence) is surjective. So each group $BP\langle 1\rangle^*(Y_2 \times X_7)$ is filtered into pieces which are exactly the groups in the above diagram.

The hypothesis of Lemma 6.4 is the equality $c_1 \otimes C = v_1 x$ in $BP\langle 1\rangle^6(Y_2 \times X_7)$. Here $c_1 \otimes C$ belongs to the bottom piece of the filtration of this group, the subgroup $H^2(Y_2, BP\langle 1\rangle^4(X_7))$. I claim that x does not belong to the bottom piece of $BP\langle 1\rangle^8(Y_2 \times X_7)$, that is, to $H^2(Y_2, BP\langle 1\rangle^6(X_7))$. Suppose that it does. Since $Y_2 = \mathbf{R}P^2$, we have $H^2(Y_2, A) \cong A/2$ for all abelian groups A; so the identity $c_1 \otimes C = v_1 x$ would say, under our assumption, that the element C in $BP\langle 1\rangle^4(X_7)/2$ is equal to v_1 times an element of $BP\langle 1\rangle^6(X_7)/2$. But that is false since we know that

$$C \neq 0 \in BP\langle 1\rangle^4 X_7 \otimes_{BP\langle 1\rangle^*} \mathbf{Z}/2$$

(Proposition 6.1). So this paragraph's claim is proved: the element $x \in BP\langle 1\rangle^8(Y_2 \times X_7)$ must have nonzero image in the top piece of the filtration of this group, that is, in $H^1(Y_2, BP\langle 1\rangle^7(X_7))$.

¿From the Atiyah-Hirzebruch spectral sequence for $BP\langle 1\rangle$-cohomology, it is immediate that the top-degree group $BP\langle 1\rangle^7(X_7)$ maps isomorphically to $H^7(X_7, \mathbf{Z}_{(2)})$. Also, since $Y_2 = \mathbf{R}P^2$, the group $H^1(Y_2, A)$ is equal to the subgroup of elements killed by 2 in A, for any abelian group A. So the group $H^1(Y_2, BP\langle 1\rangle^7(X_7))$ is isomorphic to the 2-torsion subgroup of $H^7(X_7, \mathbf{Z}_{(2)})$.

Now at last we need some specific information about the space X_7, not just that it has dimension 7: we need to know that there is no 4-torsion in $H^7(X_7, \mathbf{Z}_{(2)})$. Indeed, X_7 is the 7-skeleton of BG, where G is the Heisenberg group

$$1 \to \mathbf{Z}/2 \to G \to (\mathbf{Z}/2)^4 \to 1.$$

So

$$H^7(X_7, \mathbf{Z}_{(2)}) = H^7(BG, \mathbf{Z}_{(2)}) \oplus (\text{free abelian group}),$$

and Harada and Kono showed that the integral cohomology of G or any other extraspecial 2-group is a $\mathbf{Z}/2$-vector space in degrees $\not\equiv 0 \pmod 4$ [14].

It follows that the 2-torsion subgroup of $H^7(X_7, \mathbf{Z}_{(2)})$ injects into $H^7(X_7, \mathbf{Z}/2)$. Equivalently, the map

$$H^1(Y_2, BP\langle 1\rangle^7(X_7)) \to H^1(Y_2, H^7(X_7, \mathbf{Z}/2))$$

is injective. Since our element $x \in BP\langle 1\rangle^8(Y_2 \times X_7)$ has nonzero image in the first group, it has nonzero image in the second group. Equivalently, the image of x in $H^8(Y_2 \times X_7, \mathbf{Z}/2)$ has nonzero Künneth component in

$$H^1(Y_2, \mathbf{Z}/2) \otimes H^7(X_7, \mathbf{Z}/2).$$

Thus Lemma 6.4 is proved. QED

7 Non-injectivity of the classical cycle maps in algebraic geometry

Theorem 7.1 *There is a smooth complex projective variety X of dimension 7 such that the map $CH^2X/2 \to H^4(X, \mathbf{Z}/2)$ is not injective. The variety X and the element of $CH^2X/2$ in the kernel which we construct can be defined over $\mathbf{Q}$.*

Proof. Let G be the Heisenberg group

$$1 \to \mathbf{Z}/2 \to G \to (\mathbf{Z}/2)^4 \to 1,$$

as in section 5. Let $X = Y/G$ be the quotient of a complete intersection $Y \subset P(V)$ by a free action of G, where G acts linearly on the vector space V. Such varieties exist for any finite group G, over any infinite field, and for X of any dimension $r \geq 1$, by Godeaux and Serre; see [38], section 20. Since X is the quotient of Y by a free G-action, there is a natural homotopy class of maps $X \to BG$, and since the G-action on Y is "linearized" we get a natural homotopy class of maps $X \to \mathbf{CP}^\infty$ (or equivalently, a natural element of $H^2(X, \mathbf{Z})$). By Atiyah and Hirzebruch [3], p. 42, the product map $X \to BG \times \mathbf{CP}^\infty$ is r-connected, where r is the complex dimension of X. In particular, X contains the r-skeleton of BG up to homotopy.

In section 5, we defined two complex representations A and B of $SO(4)$, of dimensions 3 and 4. We restrict these to $G \subset SO(4)$, and we define $C = c_2A - c_2B \in MU^4BG$. We proved that C maps to 0 in $H^4(BG, \mathbf{Z}/2)$, but C remains nonzero in $MU^4X_7 \otimes_{MU^*} \mathbf{Z}/2$, where X_7 denotes the 7-skeleton of BG, by Proposition 6.1. Let $X = Y/G$ be a Godeaux-Serre variety for this group with complex dimension at least 7. Since X contains the 7-skeleton of BG up to homotopy, the class $C \in MU^4BG$ pulls back to a nonzero element of $MU^4X \otimes_{MU^*} \mathbf{Z}/2$, and it clearly maps to 0 in $H^4(X, \mathbf{Z}/2)$.

Moreover, the complex representations A and B of G give algebraic vector bundles over $X = Y/G$, and we can consider the algebraic cycle $C := c_2A - c_2B \in CH^2X$. It maps to the above class $C \in MU^4X \otimes_{MU^*} \mathbf{Z}/2$, which implies that C is nonzero in $CH^2X/2$ but maps to 0 in $H^4(X, \mathbf{Z}/2)$.

The variety X can be defined over $\mathbf{Q}$ (or any infinite field) by Serre's construction, and the cycle $C \in CH^2X$ can be defined over $\mathbf{Q}$ because the representations A and B of the group G can be defined over $\mathbf{Q}$. QED

Theorem 7.2 *There is a smooth complex projective variety X of dimension 15 and an element $\alpha \in CH^3X$ with the following properties:*

$2\alpha = 0 \in CH^3X$;

α maps to 0 in $H^6(X, \mathbf{Z})$ and also in the intermediate Jacobian $H^5(X, \mathbf{C})/(F^3H^5(X, \mathbf{C}) + H^5(X, \mathbf{Z}))$;

α is not algebraically equivalent to 0.

Proof. Let G be the Heisenberg group

$$1 \to \mathbf{Z}/2 \to G \to (\mathbf{Z}/2)^4 \to 1$$

as above, and let X be a Godeaux-Serre variety $X = Y/(G \times \mathbf{Z}/2)$ of dimension at least 15. Let α be the class $Cc_1 \in CH^3X$. Here $C = c_2A - c_2B$, where A and B are the 3- and 4-dimensional representations of G considered above, and c_1 denotes the first Chern class of the nontrivial character of $\mathbf{Z}/2$. Clearly $2\alpha = 0$, since $2c_1 = 0$.

The image of α in $MU^6X \otimes_{MU^*} \mathbf{Z}$ is the pullback to X of the element $C \otimes c_1 \in MU^6(BG \times B\mathbf{Z}/2) \otimes_{MU^*} \mathbf{Z}$ considered in Proposition 6.2. Since X contains the 15-skeleton of $BG \times B\mathbf{Z}/2$ up to homotopy, that lemma implies that α is nonzero in $MU^6X \otimes_{MU^*} \mathbf{Z}$ but maps to 0 in $H^6(X, \mathbf{Z})$.

Since α is nonzero in $MU^6X \otimes_{MU^*} \mathbf{Z}$, it is not algebraically equivalent to 0. The intermediate Jacobian for codimension-3 cycles on X, $H^5(X, \mathbf{C})/(F^3H^5(X, \mathbf{C}) + H^5(X, \mathbf{Z}))$, is actually 0, since $H^5(X, \mathbf{C}) \subset H^5(Y, \mathbf{C})$ and Y is a complete intersection of dimension ≥ 15. Thus α is 0 in the intermediate Jacobian as well as in ordinary cohomology. QED

8 Further comments

Remark 1. Our main example is a codimension-3 cycle on a smooth projective variety of rather large dimension, 15. It is worth mentioning that in a sense this large dimension should be inessential. Namely, if X is a smooth complex projective variety and Y is a smooth ample hypersurface in X, then the restriction map $CH^iX \to CH^iY$ is conjectured to be an isomorphism for $i <$ dim $Y/2$, which would be a version of the Lefschetz hyperplane theorem for Chow groups [15], [30], p. 643. Moreover if Y is a very general smooth hypersurface whose class in CH^1X is a sufficiently high multiple of an ample class, then Nori conjectured that much more should be true: $CH^iX \to CH^iY$ should be an isomorphism for all $i <$ dim Y [28], p. 368, [30], p. 644. Actually Nori and Paranjape only state these conjectures after tensoring with $\mathbf{Q}$, but they seem plausible integrally in view of Kollár and van Geemen's Trento examples [4], p. 135. (Nori also conjectured that for Y a very general high-degree complete intersection in a smooth projective variety X, $CH^iX \otimes \mathbf{Q} \to CH^iY \otimes \mathbf{Q}$ should be injective for $i =$ dim Y. Here we cannot expect to have the corresponding integral statement: if $\alpha \in CH^3X$ is the cycle in Theorem 7.2, then α restricts to 0 in CH^3Y for every complete intersection 3-fold $Y \subset X$, by Roitman's theorem.)

These conjectures would imply the corresponding isomorphisms for cycles modulo algebraic equivalence in place of Chow groups. In particular, the integral version of Nori's conjecture would imply that our nonzero element of the Griffiths group ker $(Z^3_{\text{alg}}X \to H^6(X, \mathbf{Z}))$, for dim $X = 15$, remains nonzero on a very general high-degree complete intersection $Y \subset X$ of dimension as small as 4.

But we could not expect to prove that our cycle remains nonzero in $Z^3_{\text{alg}}Y$ for such a small-dimensional variety Y just using the cycle class defined in this paper. In fact, on a variety Y of such small dimension, our cycle would be 0 in $MU^6Y \otimes_{MU^*} \mathbf{Z}$ as well as in $H^6(Y, \mathbf{Z})$, because for any finite cell complex X of real dimension at most $14 = 2(2^2 + 2 + 1)$, the map $MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X, \mathbf{Z})$ is injective. (Proof: By Johnson and Wilson [19], Proposition 4.1 and Theorem 1.1, dim $X \leq 14$ implies that hom $\dim_{MU^*} MU^*X \leq 2$, and by Conner and Smith [10], that implies that $MU^*X \otimes_{MU^*} \mathbf{Z} \to H^*(X, \mathbf{Z})$ is injective.)

Remark 2. Bloch defined an interesting filtration of the group of algebraic cycles homologically equivalent to 0, with the smallest subgroup being the cycles algebraically equivalent to 0 [6], p. 380. Namely, one says that a k-dimensional cycle α on a variety X is r-equivalent to 0 if α is contained in some $(k+r)$-dimensional algebraic subset $S \subset X$ such that $[\alpha] = 0 \in H_{2k}^{\mathrm{BM}}(S, \mathbf{Z})$. Then 1-equivalence is the same thing as algebraic equivalence by Bloch [6], Lemma 1.1 (stated $\otimes\mathbf{Q}$, but the proof works integrally), and r-equivalence is the same as homological equivalence for $r \geq \dim X - \dim \alpha$. The cycle map $Z_*^{\mathrm{alg}} X \to MU_*^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$ is well-defined on 1-equivalence by Theorem 3.1, and it is not well-defined on 3-equivalence by Theorem 7.2, since that result gives a codimension-3 cycle which is homologically equivalent to 0 but nonzero in $MU_*^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$. It seems possible that the cycle map is well-defined on 2-equivalence, but I can only prove a weaker statement, as follows.

Define a k-dimensional cycle α on a variety X to be strongly r-equivalent to 0 if α is the pushforward, as a cycle, of a cycle α' on some *smooth* scheme S' of dimension $\leq k + r$ with a proper map $S' \to X$, such that $[\alpha'] = 0 \in H_{2k}^{\mathrm{BM}}(S', \mathbf{Z})$. Then strong 1-equivalence is the same as 1-equivalence, i.e., algebraic equivalence, by the proof of Lemma 1.1 in [6]; for $r \geq 2$ r-equivalence is in general different from strong r-equivalence, although they are the same $\otimes\mathbf{Q}$ under the assumption of the Hodge conjecture (and 2-equivalence $\otimes\mathbf{Q}$ is the same as strong 2-equivalence $\otimes\mathbf{Q}$ without any conjecture). Anyway, the point of this definition is that we can prove that the cycle map is well-defined on strong 2-equivalence, as follows. If α is the pushforward of a cycle α' which is homologically equivalent to 0 on a smooth $(k+2)$-dimensional scheme S', then $[\alpha'] = 0$ in $MU_{2k}^{\mathrm{BM}} S \otimes_{MU_*} \mathbf{Z}$ by the extension of Quillen's theorem given in Theorem 2.2, where we use injectivity in degree 4. It follows that $[\alpha] = 0 \in MU_{2k}^{\mathrm{BM}} X \otimes_{MU_*} \mathbf{Z}$. That is, the cycle map is well-defined on strong 2-equivalence.

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