Adjoint functors on the derived category of motives

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Voevodsky's derived category of motives is the main arena today for the study of algebraic cycles and motivic cohomology. In this paper we study whether the inclusions of three important subcategories of motives have a left or right adjoint. These adjoint functors are useful constructions when they exist, describing the best approximation to an arbitrary motive by a motive in a given subcategory. We find a fairly complete picture: some adjoint functors exist, including a few which were previously unexplored, while others do not exist because of the failure of finite generation for Chow groups in various situations. For some base fields, we determine exactly which adjoint functors exist.

For a field k and commutative ring R, we consider three subcategories of the derived category of motives, $DM(k;R)$: the category $DMT(k;R)$ of *mixed Tate motives*, the category $DM_{\text{eff}}(k;R)$ of *effective motives*, and the category $D_0(k;R)$ of (non-effective) motives of dimension ≤ 0. Each is a *localizing subcategory* of $DM(k;R)$, meaning a full triangulated subcategory that is closed under arbitrary direct sums in $DM(k;R)$. It is a useful formal property of the category $DM(k;R)$ that it contains the direct sum and the product of an arbitrary set of objects, not necessarily finite.

In these three cases, Neeman's Brown Representability Theorem [15] implies that the inclusion f^* of the subcategory has a right adjoint f_*, and that f_* in turn has a right adjoint $f^{(1)}$:

$$f^* \dashv f_* \dashv f^{(1)}$$

For example, if f^* denotes the inclusion of $DMT(k;R)$ into $DM(k;R)$, the existence of f_* means that for every motive M in $DM(k;R)$, there is a mixed Tate motive $C(M)$ with a map $C(M) \to M$ that induces an isomorphism on motivic homology. This functor has been useful, for example in characterizing mixed Tate motives as the motives which satisfy the motivic Künneth property [25, Theorem 7.2]. The functor $f^{(1)}$ has probably not been considered before.

On the other hand, for many fields k and rings R, and for the three subcategories mentioned above, the sequence of adjoint functors above cannot be extended to the left or right, because of various failures of finite generation for motivic cohomology.

For example, for any algebraically closed field k which is not the algebraic closure of a finite field, we show that the inclusion f^* of $DMT(k;\mathbf{Q})$ into $DM(k;\mathbf{Q})$ does not have a left adjoint, using that the Mordell-Weil group of an elliptic curve over k has infinite rank. In particular, it follows that a product of mixed Tate motives need not be mixed Tate. We deduce that the analogous subcategory of *cellular spectra* in the stable homotopy category $SH(k)$ is not closed under products for some fields k. (The opposite conclusion has been announced at least once.)

By results of Balmer, Dell'Ambrogio, and Sanders, in the case of $DMT(k;R)$ (but not for the other subcategories we consider), f^* has a left adjoint if and only if it has a three-fold right adjoint [3, Theorem 3.3]. So, for many fields k and rings R, the sequence of adjoint functors stops with the three listed above.

By contrast, the Tate-Beilinson conjecture would imply that the inclusion of $DMT(k;\mathbf{Q})$ into $DM(k;\mathbf{Q})$ is a *Frobenius functor* when k is algebraic over a finite field (Theorem 8.1). This is the strong property that the right adjoint to the inclusion is also left adjoint to the inclusion (and so there is an infinite sequence of adjoints). It is not clear what to expect when k is a number field, or when k is replaced by a regular scheme of finite type over $\mathbf{Z}$.

Next, an example by Ayoub, based on Clemens's example of a complex variety with Griffiths group of infinite rank, implies that the inclusion of $DM_{\text{eff}}(\mathbf{C},\mathbf{Q})$ into $DM(\mathbf{C},\mathbf{Q})$ does not have a three-fold right adjoint [11, Proposition A.1]. The same goes for any algebraically closed field of characteristic zero (Theorem 6.1). We also show that for many fields k and rings R, the inclusion of $DM_{\text{eff}}(k;R)$ into $DM(k;R)$ does not have a left adjoint (Theorem 6.2).

An example by Ayoub and Barbieri-Viale, again building on Clemens's example, implies that the inclusion of our third subcategory $D_0(\mathbf{C};\mathbf{Q})$ into $DM(\mathbf{C};\mathbf{Q})$ does not have a left adjoint [2, section 2.5]. This can be viewed as showing that certain generalizations of the Albanese variety do not exist. We give an analogous example with finite coefficients, showing that the inclusion of $D_0(k;\mathbf{F}_p)$ in $DM(k;\mathbf{F}_p)$ does not have a left adjoint in many cases (Theorem 7.1). These results imply that the subcategory $D_0(k;R)$ need not be closed under arbitrary products in $DM(k;R)$, a question that arose during the construction of the motive of a quotient stack [25, after Lemma 8.8]. We also show that for many fields k and rings R, the inclusion of $D_0(k;R)$ into $DM(k;R)$ does not have a three-fold right adjoint (Theorem 7.2).

Finally, we prove a positive result: for any scheme X of finite type over a field k such that the compactly supported motive $M^c(X)$ in $DM(k;R)$ is mixed Tate, the Chow groups $CH_*(X;R)$ are finitely generated R-modules (Theorem 3.1). This helps to clarify what it means for a scheme to be mixed Tate.

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1 Notation

Let k be a field. The *exponential characteristic* of k means 1 if k has characteristic zero, or p if k has characteristic $p>0$. Let R be a commutative ring in which the exponential characteristic of k is invertible. Following Cisinski and Déglise, the derived category $DM(k;R)$ of motives over k with coefficients in R is defined to be the homotopy category of G_m^{tr}-spectra of (unbounded) chain complexes of Nisnevich sheaves with transfers which are A^1-local [18, section 2.3], [5, Example 6.25]. This is a triangulated category with arbitrary direct sums. (Voevodsky originally considered the subcategory $DM_-^{\text{eff}}(k)$ of "bounded above effective motives" [27].) For k perfect, Röndigs and Østvær showed that the category $DM(k;R)$ is equivalent to the homotopy category of modules over the motivic Eilenberg-MacLane spectrum HR in Morel-Voevodsky's stable homotopy category $SH(k)$ [18, Theorem 1].

A separated scheme X of finite type over k determines two motives in $DM(k;R)$, $M(X)$ (called the motive of X) and $M^c(X)$ (called the compactly supported motive of X). These two motives are isomorphic if X is proper over k. Also, there are objects $R(j)$ in $DM(k;R)$ for integers j, called *Tate motives*. Here $DM(k;R)$ is a tensor triangulated category with identity object $R(0)$, and $R(a)\otimes R(b)\cong R(a+b)$ for integers a and b. The motive of projective space is $M(\mathbf{P}^n_k)\cong \oplus_{j=0}^n R(j)[2j]$.

Voevodsky defined *motivic cohomology* and (Borel-Moore) *motivic homology* for any separated scheme X of finite type over k by

$$H^j(X,R(i)) = \mathrm{Hom}(M(X),R(i)[j])$$

and

$$H_j(X,R(i)) = \mathrm{Hom}(R(i)[j],M^c(X))$$

[27, section 2.2]. These include the Chow groups of algebraic cycles with coefficients in R, as $H_{2i}(X,R(i))\cong CH_i(X;R) := CH_i(X)\otimes_{\mathbf{Z}} R$ and $H^{2i}(X,R(i))\cong CH^i(X;R) := CH^i(X)\otimes_{\mathbf{Z}} R$. More generally, the motivic cohomology and motivic homology of any object N in $DM(k;R)$ are defined by $H^j(N,R(i)) = \mathrm{Hom}(N,R(i)[j])$ and $H_j(N,R(i)) = \mathrm{Hom}(R(i)[j],N)$.

For an equidimensional separated scheme X of dimension n over k, motivic homology is isomorphic to Bloch's higher Chow groups:

$$CH^{n-i}(X,j-2i;R)\cong H_j(X,R(i)).$$

It follows that the motivic homology $H_j(X,R(i))$ of a separated k-scheme X is zero unless $j\geq 2i$ and $j\geq i$ and $i\leq \dim(X)$. The isomorphism between motivic homology and higher Chow groups was proved under mild assumptions in [27, Proposition 4.2.9]; see [25, section 5] for references to the full statement.

The triangulated category $DM(k;R)$ is compactly generated [6, Theorem 11.1.13], [13, Proposition 5.5.3]. (For k imperfect, see [7, Proposition 8.1].) A set of compact generators is given by the motives $M(X)(a)$ for smooth projective varieties X over k and integers a. Since $DM(k;R)$ is compactly generated, it contains arbitrary products as well as arbitrary direct sums [16, Proposition 8.4.6].

Define a *thick* subcategory of a triangulated category to be a strictly full triangulated subcategory that is closed under direct summands. We use the following result of Neeman's [15, Theorem 2.1].

Theorem 1.1. *Let $\mathcal{T}$ be a compactly generated triangulated category, and let $\mathcal{P}$ be a set of compact generators. Then any compact object in $\mathcal{T}$ belongs to the smallest thick subcategory of $\mathcal{T}$ that contains $\mathcal{P}$.*

2 Background on triangulated categories

We consider three subcategories of $DM(k;R)$ in this paper. The category $DMT(k;R)$ of *mixed Tate motives* is the smallest localizing subcategory that contains $R(j)$ for all integers j. The category $DM_{\mathrm{eff}}(k;R)$ of *effective motives* is the smallest localizing subcategory that contains $M(X)$ for every smooth projective variety X over k. The category $D_0(k;R)$ of (non-effective) motives of dimension ≤ 0 is the smallest localizing subcategory that contains $M(X)(-b)$ for every smooth projective variety X over k and every integer $b\geq \dim(X)$.

We use the following consequences of Neeman's Brown Representability Theorem [3, Corollary 2.3], [15, Theorem 5.1].

Theorem 2.1. *Let $F\colon \mathcal{S} \to \mathcal{T}$ be a exact functor between triangulated categories, and assume that $\mathcal{S}$ is compactly generated. Then:*

(1) F has a right adjoint if and only if it preserves arbitrary direct sums.

(2) F has a left adjoint if and only if preserves arbitrary products.

Theorem 2.2. *Let $F\colon \mathcal{S} \to \mathcal{T}$ be an exact functor between triangulated categories with right adjoint G, and assume that $\mathcal{S}$ is compactly generated. Then F preserves compact objects if and only if G preserves arbitrary direct sums.*

The following lemma applies to the three subcategories of $DM(k;R)$ considered in this paper: mixed Tate motives, effective motives, and (non-effective) motives of dimension ≤ 0.

Lemma 2.3. *Let $\mathcal{T}$ be a compactly generated triangulated category, and let $\mathcal{S}$ be the smallest localizing subcategory of $\mathcal{T}$ that contains a given set of compact objects in $\mathcal{T}$. Then the inclusion f^* of $\mathcal{S}$ into $\mathcal{T}$ has a right adjoint f_*. Moreover, f_* also has a right adjoint $f^{(1)}\colon \mathcal{S} \to \mathcal{T}$:*

$$f^* \dashv f_* \dashv f^{(1)}$$

The fact that f_* exists means that for every object A of $\mathcal{T}$ there is an object B of $\mathcal{S}$ and a morphism $B \to A$ that is universal for maps from objects of $\mathcal{S}$ to A. This is often a useful construction. In this paper, we ask (in various examples) whether the inclusion f^* of $\mathcal{S}$ into $\mathcal{T}$ also has a left adjoint $f_{(1)}$. Equivalently, for every object A in $\mathcal{T}$, is there an object B of $\mathcal{S}$ with a map $A \to B$ that is universal for maps from A to objects of $\mathcal{S}$?

The notation $f^{(1)}$ was suggested by Balmer, Dell'Ambrogio, and Sanders [3, Corollary 2.14].

Proof. (Lemma 2.3) First, because $\mathcal{S}$ is compactly generated and the inclusion f^* from $\mathcal{S}$ to $\mathcal{T}$ preserves arbitrary direct sums, f^* has a right adjoint, by Theorem 2.1. Next, we use that the given generators for $\mathcal{S}$ are compact in $\mathcal{T}$. It follows that f^* takes compact objects in $\mathcal{S}$ to compact objects in $\mathcal{T}$. Since $\mathcal{S}$ is compactly generated, it follows that f_* preserves arbitrary direct sums, by Theorem 2.2. Since $\mathcal{T}$ is compactly generated, Theorem 2.1 gives that f_* also has a right adjoint $f^{(1)}$. $\square$

The subcategory $DMT(k;R)$ of $DM(k;R)$ is *rigidly-compactly generated*, unlike $DM_{\text{eff}}(k;R)$ and $D_0(k;R)$. This means that $DMT(k;R)$ is a tensor-triangulated category; it has arbitrary direct sums; its compact objects coincide with the rigid objects (also called the strongly dualizable objects); and $DMT(k;R)$ is generated by a set of compact objects. (The key point in checking this is that the duals in $DM(k;R)$ of the given generators $R(j)$ for $DMT(k;R)$, for integers j, are again in $DMT(k;R)$.)

For a tensor exact functor between rigidly-compactly generated categories that preserves arbitrary direct sums, Balmer, Dell'Ambrogio, and Sanders showed that the sequence of adjoint functors in Lemma 2.3 extends one step to the left if and only if it extends one step to the right [3, Theorem 3.3]. In particular:

Theorem 2.4. *Let k be a field and R a commutative ring in which the exponential characteristic of k is invertible. Then the inclusion f^* of $DMT(k;R)$ into $DM(k;R)$ has a left adjoint if and only if it has a three-fold right adjoint (meaning that $f^{(1)}$ above has a right adjoint).*

3 The Chow groups of a mixed Tate scheme

Let X be a scheme of finite type over a field k such that the compactly supported motive $M^c(X)$ is mixed Tate. This implies the *weak Chow Künneth property* that the Chow groups of X do not increase when the base field k is enlarged [25, section 6]. However, that leaves open the question of how big the Chow groups of X are. Note that more general motivic homology groups of a mixed Tate scheme X over k need not be finitely generated abelian groups, as shown by the case $X = \text{Spec}(k)$. (For example, $H_{-1}(k, \mathbf{Z}(-1)) \cong k^*$.)

In this section, we show that for a scheme X of finite type over a field k such that $M^c(X)$ is mixed Tate in $DM(k; R)$, the Chow groups $CH_*(X; R)$ are finitely generated R-modules. This was known for the simplest examples of mixed Tate schemes, *linear schemes* over k in the sense of [24]. On the other hand, there are mixed Tate varieties that are not linear schemes or even rational, for example some Barlow surfaces of general type [1, Proposition 1.9], [25, after Theorem 4.1].

It is natural to ask a stronger question. Let X be a scheme of finite type that has the weak Chow Künneth property with R coefficients, meaning that $CH_*(X; R) \to CH_*(X_E; R)$ is surjective for all finitely generated fields E over k, or equivalently for all fields E over k. Are the Chow groups $CH_*(X; R)$ finitely generated R-modules? The answer is yes for X smooth proper over k [25, Theorem 4.1], but the general question remains open.

Theorem 3.1. *Let k be a field and R a commutative ring such that the exponential characteristic of k is invertible in R. Let X be a scheme of finite type over k. If $M^c(X)$ is mixed Tate in $DM(k; R)$, then the Chow groups $CH_*(X; R)$ are finitely generated R-modules.*

Proof. The object $M^c(X)$ is compact in $DM(k; R)$. Since we assume that $M^c(X)$ is also mixed Tate (that is, $M^c(X)$ is in the smallest localizing subcategory that contain the objects $R(i)$ for integers i), it is in fact in the smallest thick subcategory of $DM(k; R)$ that contains $R(i)$ for all integers i, by Theorem 1.1. In order to see that X has finitely generated Chow groups, we will analyze which motives $R(i)[j]$ are needed to construct $M^c(X)$.

Let $N_0 = N = M^c(X)$. Consider the following sequence of mixed Tate motives N_a for $a \geq 0$. Given N_a, choose a set of generators for the motivic homology of N_a as a module over the motivic homology of k. Let F_a be the corresponding direct sum (possibly infinite) of objects $R(i)[j]$ together with a map $F_a \to N_a$ that induces a surjection on motivic homology. Let N_{a+1} be a cone of the map $F_a \to N_a$. This defines a sequence of mixed Tate motives $N_0 \to N_1 \to \cdots$.

By construction, the homotopy colimit $\text{hocolim}(N_a)$ has zero motivic homology groups. Since $\text{hocolim}(N_a)$ is a mixed Tate motive, it follows that $\text{hocolim}(N_a) = 0$ (by another of Neeman's results; see [25, Corollary 5.3]). So

$$0 = \text{Hom}(N, \text{hocolim}(N_a)) = \varinjlim \text{Hom}(N, N_a).$$

So there is a natural number a such that the composition $N = N_0 \to N_1 \to \cdots \to N_a$ is zero. By construction, the fiber Y of $N = N_0 \to N_a$ is an iterated extension of direct sums of Tate motives, $F_0, \ldots, F_{a-1}$. Since the map $N \to N_a$ is zero, Y is isomorphic to $N \oplus N_a[-1]$. Thus N is a summand of the extension Y.

The following lemma formalizes an argument by Neeman [14, proof of Lemma 2.3]. We say that an object Y in a triangulated category is an *iterated extension* of objects $F_0, \ldots, F_{a-1}$ if there is a map $f_0 \colon F_0 \to Y$, a map f_1 from F_1 to the cone of f_0, and so on, with the cone of f_{a-1} being zero.

Lemma 3.2. *Let $\mathcal{T}$ be a triangulated category with arbitrary direct sums. Let N be a compact object in $\mathcal{T}$ which is a summand of an iterated extension Y of (possibly infinite) direct sums $F_0, \ldots, F_{a-1}$ of compact objects. Then N is a summand of an iterated extension Y' of objects $F'_0, \ldots, F'_{a-1}$, with each F'_b a finite direct sum of some of the summands of F_b.*

Proof. To make an induction, we prove a more general statement. Let N be a compact object in $\mathcal{T}$ with a morphism to an object Y, and let $Y' \to Y$ be a morphism whose cone is an iterated extension of direct sums $F_0, \ldots, F_{a-1}$ of compact objects in $\mathcal{T}$. Then there is an object N' and a map $N' \to N$ such that the composite $N' \to N \to Y$ factors through Y', and the cone of $N' \to N$ is an iterated extension of objects $F'_0, \ldots, F'_{a-1}$, with each F'_b a finite direct sum of some of the given summands of F_b. For $Y' = 0$, this gives the statement of the lemma.

The proof is by induction on the number a. If $a = 1$, then the cone $F = F_0$ of $Y' \to Y$ is a direct sum of compact objects. Since N is compact, the composition $N \to Y \to F$ factors through a finite direct sum F' of the given summands of F. Then we can complete the commutative square

$$\begin{array}{ccc} N & \longrightarrow & F' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & F \end{array}$$

to a map of triangles

$$\begin{array}{ccccccc} N' & \longrightarrow & N & \longrightarrow & F' & \longrightarrow & N'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & F & \longrightarrow & Y'[1]. \end{array}$$

Thus the cone of $N' \to N$ is a finite direct sum F' of the given summands of $F = F_0$, and the composite $N' \to N \to Y$ factors through Y', as we want.

Now suppose that $a > 1$. Then we can factor the map $Y' \to Y$ (with cone an extension of $F_0, \ldots, F_{a-1}$) as $Y' \to Y'' \to Y$ such that the cone of $Y' \to Y''$ is an extension of $F_0, \ldots, F_{a-2}$ and the cone of $Y'' \to Y$ is F_{a-1}. By the case $a = 1$ of the induction, there is a map $N'' \to N$ with cone a finite subsum F'_{a-1} of F_{a-1} such that $N'' \to N \to Y$ factors through Y''. Then N'' is compact. By induction on a, there is a map $N' \to N''$ with cone an extension of finite subsums $F'_0, \ldots, F'_{a-2}$ of the direct sums $F_0, \ldots, F_{a-2}$ such that $N' \to N'' \to Y''$ factors through Y'. Then we have a commutative diagram

$$\begin{array}{ccccc} N' & \longrightarrow & N'' & \longrightarrow & N \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y'' & \longrightarrow & Y, \end{array}$$

which shows that the composite $N' \to N \to Y$ factors through Y'. Finally, the cone of $N' \to N$ is an extension of $F'_0, \ldots, F'_{a-1}$, by the octahedral axiom. $\square$

We showed above that $N = M^c(X)$ is a summand of an extension $F_0, \ldots, F_{a-1}$ of direct sums of Tate motives. Since N is compact, Lemma 3.2 gives that N is a summand of an extension Y' of *finite* direct sums $F'_0, \ldots, F'_{a-1}$ of Tate motives, where each F'_b is the direct sum of finitely many of the Tate motives that occur in F_b.

We now use that for a scheme X of finite type over k, the motivic homology $H_j(X, R(i))$ vanishes unless $2i \leq j$, by section 1. As a result, we can take F_0 to be a direct sum of objects $R(i)[j]$ with $2i \leq j$. Since N_1 is a cone of the morphism $F_0 \to N_0$ which induces a surjection on motivic homology, we have an exact sequence of motivic homology groups:

$$H_j(N_0, R(i)) \xrightarrow[0]{} H_j(N_1, R(i)) \longrightarrow H_{j-1}(F_0, R(i)).$$

We read off that N_1 has a stronger vanishing property than N_0 does: $H_j(N_1, R(i))$ is zero unless $2i - j \leq -1$. Repeating the argument, we find that each F_b can be chosen to be a direct sum of Tate motives $R(i)[j]$ with $2i - j \leq -b$.

Therefore, each F'_b is a *finite* direct sum of Tate motives $R(i)[j]$ with $2i-j \leq -b$. Since $N = M^c(X)$ is a summand of the extension Y' of $F'_0, \ldots, F'_{a-1}$, we read off that $CH_*(F'_0) \to CH_*(X; R)$ is surjective, and that $CH_*(F'_0)$ is a finitely generated free R-module. Thus the R-module $CH_*(X; R)$ is finitely generated. □

The same argument gives the following variant. The right adjoint f_* to the inclusion of $DMT(k; R)$ into $DM(k; R)$ is also called *colocalization* with respect to mixed Tate motives, $N \mapsto C(N)$.

Theorem 3.3. *Let k be a field and R a commutative ring such that the exponential characteristic of k is invertible in R. Let X be a scheme of finite type over k. If the colocalization $C(M^c(X))$ in $DMT(k; R)$ is compact, then the Chow groups $CH_*(X; R)$ are finitely generated R-modules.*

4 Products of mixed Tate motives

Theorem 4.1. *Let k be a field and R a commutative ring. If the product $\prod_{m=1}^{\infty} R(0)$ in $DM(k; R)$ is mixed Tate, then the R-module $CH_i(Y; R)$ is finitely generated for every smooth projective variety Y over k and every integer i.*

Proof. Suppose that $P := \prod_{m=1}^{\infty} R(0)$ in $DM(k; R)$ is mixed Tate. That implies that for every smooth projective variety Y over k, Dugger-Isaksen's Künneth spectral sequence

$$E_2^{pq} = \mathrm{Tor}_{-p,-q,j}^{H_*(k,R(*))}(H_*(P, R(*)), H_*(Y, R(*))) \Rightarrow H_{-p-q}(P \otimes M(Y), R(j))$$

converges to the motivic homology of $P \otimes M(Y)$ [8, Proposition 7.10]. Here, for bigraded modules M and N over a bigraded ring S, $\mathrm{Tor}^S_{a,i,j}(M, N)$ denotes the (i, j)th bigraded piece of $\mathrm{Tor}^S_a(M, N)$. For this purpose, the group $H^M_i(X, R(j))$ has bigrading (i, j).

Next, $P \otimes M(Y)$ is isomorphic to $\prod_{m=1}^{\infty} M(Y)$. (To prove that, use that $M(Y)$ is strongly dualizable in $DM(k; R)$ (a reference is [25, Lemma 5.5]), and check that the abelian group of maps from any object W in $DM(k;R)$ to $P \otimes M(Y)$ can be identified with the group of maps from W to $\prod_{m=1}^{\infty} M(Y)$.)

The motivic homology of P is (trivially) the product of infinitely many copies of the motivic homology of $R(0)$. (In particular, $H_i(P, R(j)) = 0$ unless $i \geq 2j$ and $i \geq j$ and $j \leq 0$, just as we would have for a 0-dimensional variety.) As a result, the Künneth spectral sequence with $R(j)$ coefficients is concentrated in columns ≤ 0 and rows $\leq -2j$. If we write $H_*(P)$ for the bigraded group $H_*(P, R(*))$, the E_2 term looks like:

$$\begin{tikzcd}
0 & 0 & 0 & 0 \\
{[\mathrm{Tor}_2^{H_*k}(H_*P, H_*Y)]_{2j,j}} \arrow[drr] & {[\mathrm{Tor}_1^{H_*k}(H_*P, H_*Y)]_{2j,j}} & {[H_*P \otimes_{H_*k} H_*Y]_{2j,j}} & 0 \\
{[\mathrm{Tor}_2^{H_*k}(H_*P, H_*Y)]_{2j+1,j}} & {[\mathrm{Tor}_1^{H_*k}(H_*P, H_*Y)]_{2j+1,j}} & {[H_*P \otimes_{H_*k} H_*Y]_{2j+1,j}} & 0
\end{tikzcd}$$

So there are no differentials into or out of the upper right group, $E_2^{0,-2j}$. We deduce that the homomorphism

$$CH_*(P) \otimes_R CH_*(Y;R) \to CH_*(P \otimes M(Y)) = CH_*\bigg(\prod_{m=1}^{\infty} M(Y)\bigg)$$

is an isomorphism. In particular, it is surjective.

That is,

$$\bigg(\prod_{m=1}^{\infty} R\bigg) \otimes_R CH_*(Y;R) \to \prod_{m=1}^{\infty} CH_*(Y;R)$$

is surjective. But (by definition of the tensor product of R-modules) any element of the tensor product on the left maps to a sequence $(a_1, a_2, \ldots)$ in $\prod_m CH_*(Y;R)$ such that $a_1, a_2, \ldots$ all lie in some finitely generated R-submodule of $CH_*(Y;R)$. So we get a contradiction if $CH_*(Y;R)$ is not finitely generated as an R-module. $\square$

Another proof that $DMT(k;R)$ is not closed under products in $DM(k;R)$, when there is a k-variety whose Chow groups are not finitely generated, can be given as follows. By Theorem 2.1, $DMT(k;R)$ is closed under products in $DM(k;R)$ if and only if the inclusion f^* of $DMT(k;R)$ into $DM(k;R)$ has a left adjoint. By Balmer, Dell'Ambrogio, and Sanders, that holds if and only if f^* has a three-fold right adjoint (Theorem 2.4 above). This in turn is equivalent to $f^{(1)}$ preserving arbitrary direct sums (Theorem 2.1), or again to f_* (also called $N \mapsto C(N)$) preserving compact objects (Theorem 2.2). By Theorem 3.3, if that holds, then $CH_*(X;R)$ is a finitely generated R-module for every smooth projective k-variety X.

Theorem 4.1 implies that the subcategory of mixed Tate motives is not closed under products in $DM(k;R)$, in many cases. For example:

Corollary 4.2. *Let k be an algebraically closed field. Let p be the exponential characteristic of k, and write $R = \mathbf{Z}[1/p]$. Then the product $\prod_{m=1}^{\infty} R(0)$ in $DM(k;R)$ is not mixed Tate. In particular, the subcategory of mixed Tate motives is not closed under products in $DM(k;R)$, and the inclusion $DMT(k;R) \to DM(k;R)$ does not have a left adjoint or a three-fold right adjoint.*

Proof. By Theorem 4.1, to show that $\prod_{m=1}^{\infty} R(0)$ in $DM(k;R)$ is not mixed Tate, it suffices to give an example of a smooth projective variety Y over k such that $CH_0(Y)[1/p]$ is not finitely generated as an R-module. Since k is algebraically closed, we can take Y to be any elliptic curve over k. Then we have an exact sequence

$$0 \to Y(k) \to CH_0(Y) \to \mathbf{Z} \to 0.$$

The group of points $Y(k)$ (with p inverted) is not finitely generated, because it has prime-to-p torsion of arbitrarily large order. Since $DMT(k;R)$ is not closed under products in $DM(k;R)$, the inclusion does not have a left adjoint. By Balmer, Dell'Ambrogio, and Sanders, since $DMT(k;R)$ is rigidly-compactly generated, it follows that the inclusion does not have a three-fold right adjoint (Theorem 2.4). □

We can also consider motives with rational coefficients:

Corollary 4.3. *Let k be an algebraically closed field which is not the algebraic closure of a finite field. Then the product $\prod_{m=1}^{\infty} \mathbf{Q}(0)$ in $DM(k;\mathbf{Q})$ is not mixed Tate. So the subcategory of mixed Tate motives is not closed under products in $DM(k;\mathbf{Q})$, and the inclusion $DMT(k;\mathbf{Q}) \to DM(k;\mathbf{Q})$ does not have a left adjoint or a three-fold right adjoint.*

Proof. By Theorem 4.1, to show that $\prod_{m=1}^{\infty} \mathbf{Q}(0)$ in $DM(k;\mathbf{Q})$ is not mixed Tate, it suffices to find a smooth projective variety X over k such that $CH_0(X;\mathbf{Q})$ has infinite dimension as a $\mathbf{Q}$-vector space. Since k is not the algebraic closure of a finite field, this holds for any elliptic curve X over k, by Frey and Jarden [9, Theorem 9.1]. The other statements follow as in the proof of Corollary 4.2. □

By contrast, Theorem 8.1 shows, under the Tate-Beilinson conjecture, that for k algebraic over a finite field, the subcategory of mixed Tate motives *is* closed under products in $DM(k;\mathbf{Q})$, and the inclusion $DMT(k;\mathbf{Q}) \to DM(k;\mathbf{Q})$ has a left adjoint.

Finally, we can say something with finite coefficients:

Theorem 4.4. *Let p be a prime number. Then the product $\prod_{m=1}^{\infty} \mathbf{F}_p(0)$ in $DM(\mathbf{C};\mathbf{F}_p)$ is not mixed Tate. So the subcategory of mixed Tate motives is not closed under products in $DM(\mathbf{C};\mathbf{F}_p)$, and the inclusion $DMT(\mathbf{C};\mathbf{F}_p) \to DM(\mathbf{C};\mathbf{F}_p)$ does not have a left adjoint or a three-fold right adjoint.*

For any algebraically closed field k of characteristic zero in place of $\mathbf{C}$, these results hold for all prime numbers p congruent to 1 modulo 3.

Proof. By Theorem 4.1, to show that $DMT(k;\mathbf{F}_p)$ is not closed under products in $DM(k;\mathbf{F}_p)$, it suffices to exhibit a smooth projective variety X over k with $CH_i(X;\mathbf{F}_p) = CH_i(X)/p$ infinite for some i. For k algebraically closed, $CH_0(X;\mathbf{F}_p) = CH_0(X)/p$ is finite for every smooth projective variety X over k, and so the proof has to be slightly different from the previous cases. We can instead use Schoen's theorem that, for k algebraically closed of characteristic zero and $p \equiv 1 \pmod 3$, the product X of three copies of the Fermat cubic curve $x^3+y^3+z^3=0$ over k has $CH_1(X)/p$ infinite [21, Theorem 0.2]. (Schoen proves this for $k = \overline{\mathbf{Q}}$, and then we can use Suslin's theorem that $CH_i(X;\mathbf{F}_p) \to CH_i(X_F,\mathbf{F}_p)$ is an isomorphism for every algebraically closed field $F/\overline{\mathbf{Q}}$ [22, Corollary 2.3.3].)

Strengthening a result by Rosenschon and Srinivas [19], I showed that $CH_1(X)/p$ is infinite for X a very general principally polarized abelian 3-fold over $\mathbf{C}$ and *all* prime numbers p [26]. This yields the result we want over $\mathbf{C}$. The statements about adjoint functors follow as in the proof of Corollary 4.2. $\square$

5 Products of cellular spectra

Let k be a field. Following Dugger-Isaksen, the subcategory of *cellular spectra* in the stable homotopy category $SH(k)$ is the smallest localizing subcategory that contains the spheres $S^{a,b}$ for all integers a and b [8]. Here $S^{1,1}$ is the class of the pointed curve $(A^1 - 0, 1)$ over k, and $S^{1,0}$ is the circle as a simplicial set. We have $S^{a+1,b} = S^{a,b}[1]$, in terms of the structure of $SH(k)$ as a triangulated category. The natural functor from $SH(k)$ to $DM(k; R)$ takes $S^{a,b}$ to $R(b)[a]$.

Corollary 5.1. *Let k be an algebraically closed field which is not the algebraic closure of a finite field. Then $S^{0,0}_{\mathbf{Q}}$ is cellular in $SH(k)$, but the product $\prod_{m=1}^{\infty} S^{0,0}_{\mathbf{Q}}$ in $SH(k)$ is not cellular. So the subcategory of cellular spectra is not closed under products in $SH(k)$, and the inclusion of this subcategory into $SH(k)$ does not have a left adjoint. It also does not have a three-fold right adjoint.*

Proof. Following Bökstedt and Neeman, the *homotopy colimit* $X_\infty = \text{hocolim}(X_0 \to X_1 \to \cdots)$ in a triangulated category with arbitrary direct sums is defined as a cone of the morphism

$$1 - s\colon \oplus_{i\geq 0} X_i \to \oplus_{i\geq 0} X_i,$$

where s is the given map from each X_i to X_{i+1} [4]. The spectrum $S^{0,0}_{\mathbf{Q}}$ is cellular in $SH(k)$, because it can be defined as the homotopy colimit of the sequence

$$S^{0,0} \xrightarrow[2]{} S^{0,0} \xrightarrow[3]{} \cdots .$$

We can think of $SH(k; \mathbf{Q})$ as a full subcategory of $SH(k)$, with the rationalization of a spectrum X defined as $X \wedge S^{0,0}_{\mathbf{Q}}$, or equivalently as the homotopy colimit of

$$X \xrightarrow[2]{} X \xrightarrow[3]{} \cdots .$$

It is clear that rationalization $SH(k) \to SH(k; \mathbf{Q})$ takes cellular objects in $SH(k)$ to cellular objects in $SH(k; \mathbf{Q})$ (meaning objects in the smallest localizing subcategory of $SH(k; \mathbf{Q})$ that contains all rational spheres $S^{a,b}_{\mathbf{Q}}$).

Suppose that $\prod_{m=1}^{\infty} S^{0,0}_{\mathbf{Q}}$ is cellular in $SH(k)$. Then the rationalization $(\prod_{m=1}^{\infty} S^{0,0}_{\mathbf{Q}})_{\mathbf{Q}}$ is cellular in $SH(k; \mathbf{Q})$. From the definition of the rationalization as a homotopy colimit, we see that this rationalization is simply $\prod_{m=1}^{\infty} S^{0,0}_{\mathbf{Q}}$. We conclude that $\prod_{m=1}^{\infty} S^{0,0}_{\mathbf{Q}}$ is cellular in $SH(k; \mathbf{Q})$.

Since k is algebraically closed, -1 is a sum of squares in k. Under that assumption, Cisinski and Déglise deduced from Morel's work that $SH(k; \mathbf{Q})$ is equivalent to the derived category of motives, $DM(k; \mathbf{Q})$ [6, Corollary 16.2.14]. So $\prod_{m=1}^{\infty} \mathbf{Q}(0)$ is a mixed Tate motive in $DM(k; \mathbf{Q})$, contradicting Corollary 4.3. So in fact $\prod_{m=1}^{\infty} S^{0,0}_{\mathbf{Q}}$ in $SH(k)$ is not cellular. As a result, the subcategory of cellular spectra is not closed under products in $SH(k)$.

As a result, the inclusion f^* of cellular spectra into $SH(k)$ does not have a left adjoint. The inclusion does have a right adjoint f_*, which in turn has a right adjoint $f^{(1)}$, by Theorem 2.3. Since the subcategory of cellular spectra is rigidly-compactly generated and f^* does not have a left adjoint, it follows from Balmer, Dell'Ambrogio, and Sanders that $f^{(1)}$ does not have a right adjoint [3, Theorem 3.3]. $\square$

6 Effective motives

Here we show that the inclusion from the subcategory of effective motives $DM_{\text{eff}}(k;R)$ to $DM(k;R)$ does not have a left adjoint or a three-fold right adjoint, in many cases. For the three-fold right adjoint, this is a reformulation of an example by Ayoub. The right adjoint f_* of the inclusion f^* has been used by Huber and Kahn under the name $\nu_{\leq 0}$ (or step 0 of the *slice filtration*) [12].

Theorem 6.1. *Let k be an algebraically closed field of characteristic zero. Let f^* be the inclusion of $DM_{\text{eff}}(k,\mathbf{Q})$ into $DM(k,\mathbf{Q})$. Then the right adjoint f_* of f^* does not preserve compact objects; the right adjoint $f^{(1)}$ of f_* does not preserve arbitrary direct sums; and $f^{(1)}$ does not have a right adjoint:*

$$f^* \dashv f_* \dashv f^{(1)}$$

Proof. Ayoub showed that $f_*\colon DM(k,\mathbf{Q}) \to DM_{\text{eff}}(k,\mathbf{Q})$ does not preserve compact objects, for k algebraically closed of characteristic zero with sufficiently large transcendence degree. He used Clemens's example of a complex 3-fold X whose Griffith group has infinite rank [11, Proposition A.1]. The argument works for any algebraically closed field of characteristic zero by using instead Schoen's example of a 3-fold over $\overline{\mathbf{Q}}$ whose Griffiths group has infinite rank [20]. It follows that the right adjoint $f^{(1)}$ of f_* does not preserve arbitrary direct sums, by Theorem 2.2. Therefore, $f^{(1)}$ does not have a right adjoint. $\square$

A simpler argument shows that the inclusion f^* from $DM_{\text{eff}}(k;R)$ to $DM(k;R)$ does not have a left adjoint in most cases:

Theorem 6.2. *Let k be a field, and let R be a commutative noetherian ring in which the exponential characteristic of k is invertible. If the inclusion from $DM_{\text{eff}}(k;R)$ to $DM(k;R)$ has a left adjoint, then every motivic cohomology group $H^j(X,R(i))$ is a finitely generated R-module for every smooth projective variety X over k. This fails, for example, if $R=\mathbf{Q}$ and k is not an algebraic extension of a finite field; or if $R=\mathbf{Z}$ and k is an infinite field; or if $R=\mathbf{F}_p$ for a prime number congruent to 1 modulo 3 and k is an algebraically closed field of characteristic zero; or if $R=\mathbf{F}_p$ for any prime number p and $k=\mathbf{C}$.*

Proof. Suppose that the inclusion f^* from $DM_{\text{eff}}(k;R)$ to $DM(k;R)$ has a left adjoint $f_{(1)}$. Since f^* preserves arbitrary direct sums, $f_{(1)}$ must preserve compact objects, by Theorem 2.2.

Let X be a smooth projective variety over k. Let j be an integer. By the isomorphism between motivic cohomology and higher Chow groups, $H^j(X,R(0))$ is isomorphic to $CH^0(X,-j;R)$, which is R if $j=0$ and zero otherwise. Let N be a compact object in $DM_{\text{eff}}(k;R)$. By Theorem 1.1, N belongs to the smallest thick subcategory of $DM(k;R)$ that contains $M(X)$ for all smooth projective varieties X over k. Since R is noetherian, the exact sequences for Hom in a triangulated category yield that $\text{Hom}(N,R(0)) = H^0(N,R(0))$ is a finitely generated R-module.

For every object A in $DM(k;R)$, the definition of $f_{(1)}$ gives a map $A \to f_{(1)}(A)$ which is universal for maps from A into $DM_{\text{eff}}(k;R)$. In particular, $H^0(f_{(1)}(A),R(0))$ maps isomorphically to $H^0(A,R(0))$. Let A be compact in $DM(k;R)$; then $f_{(1)}(M)$ is compact in $DM_{\text{eff}}(k;R)$. So $H^0(M,R(0))$ is a finitely generated R-module. Since $H^j(X,R(i)) \cong H^0(M(X)(-i)[-j],R(0))$ for any smooth projective variety X over k and integers i and j, it follows that all motivic cohomology groups of smooth projective varieties with R coefficients are finitely generated.

It remains to show that this conclusion fails for the pairs (k,R) mentioned in the theorem. First, if $R = \mathbf{Q}$, then the $\mathbf{Q}$-vector space $H^1(k,\mathbf{Q}(1)) = k^* \otimes \mathbf{Q}$ has infinite dimension if the field k is not an algebraic extension of a finite field. Next, if $R = \mathbf{Z}$, then the abelian group $H^1(k,\mathbf{Z}(1)) = k^*$ is not finitely generated if k is an infinite field. Finally, if $R = \mathbf{F}_p$ for a prime number $p \equiv 1 \pmod 3$ and k is algebraically closed of characteristic zero, then Schoen found a smooth projective 3-fold X over k with $CH^2(X)/p$ infinite [21, Theorem 0.2]. If $R = \mathbf{F}_p$ for any prime number p, I exhibited a smooth complex projective 3-fold X with $CH^2(X)/p$ infinite [26]. $\square$

7 The dimension filtration on motives

Let $D_0(k;R)$ (also called $d_{\leq 0}DM(k;R)$ by analogy with Voevodsky's notation [27, section 3.4]) be the smallest localizing subcategory of $DM(k;R)$ that contains $M(X)(-b)$ for all smooth projective varieties X over k and all integers b such that $b \geq \dim(X)$. The subcategory $D_0(k;R)$ was useful for constructing and studying the compactly supported motive of a quotient stack over k, for example of a classifying stack BG [25, section 8].

In this section we show that the inclusion of $D_0(k;R)$ into $DM(k;R)$ does not have a left adjoint or a three-fold right adjoint, in many cases. Ayoub and Barbieri-Viale gave the first example where the left adjoint does not exist [2, section 2.5]. These examples imply that the subcategory $D_0(k;R)$ need not be closed under products in $DM(k;R)$, which answers a question in [25], after Lemma 8.8.

One can think of the nonexistence of a left adjoint as meaning that certain generalizations of the Albanese variety do not exist. Indeed, Ayoub and Barbieri-Viale, generalizing an earlier result by Barbieri-Viale and Kahn, showed that for a field k, the inclusion

$$d_{\leq 1}DM_{\text{eff}}(k;\mathbf{Q}) \to DM_{\text{eff}}(k;\mathbf{Q})$$

has a left adjoint LAlb, related to the Albanese variety of a smooth projective variety [2, Theorem 2.4.1].

Theorem 7.1. *(1) The subcategory $D_0(\mathbf{C};\mathbf{Q})$ is not closed under products in $DM(\mathbf{C},\mathbf{Q})$, and the inclusion functor from $D_0(\mathbf{C};\mathbf{Q})$ to $DM(\mathbf{C};\mathbf{Q})$ does not have a left adjoint.*

(2) Let k be an algebraically closed field of characteristic zero, and let p be a prime number congruent to 1 modulo 3. Then the subcategory $D_0(k;\mathbf{F}_p)$ is not closed under products in $DM(k;\mathbf{F}_p)$, and the inclusion functor from $D_0(k;\mathbf{F}_p)$ to

$DM(k;\mathbf{F}_p)$ does not have a left adjoint. If $k = \mathbf{C}$, then this holds for any prime number p.

It would be interesting to find out whether the inclusion of $D_0(k;R)$ into $DM(k;R)$ has a left adjoint for other fields k and commutative rings R.

Proof. (1) Ayoub and Barbieri-Viale showed that the inclusion

$$d_{\leq 2}DM_{\text{eff}}(\mathbf{C};\mathbf{Q}) \to DM_{\text{eff}}(\mathbf{C};\mathbf{Q})$$

does not have a left adjoint, using Clemens's example of a complex variety with Griffiths group of infinite rank [2, section 2.5]. (In contrast to Theorem 6.1, it is not clear how to generalize Ayoub and Barbieri-Viale's argument to arbitrary algebraically closed fields of characteristic zero.) The same argument gives that the inclusion of $D_0(\mathbf{C};\mathbf{Q})$ into $DM(\mathbf{C};\mathbf{Q})$ does not have a left adjoint. Equivalently, by Theorem 2.1, the subcategory $D_0(\mathbf{C};\mathbf{Q})$ is not closed under products in $DM(\mathbf{C};\mathbf{Q})$.

(2) Let $R = \mathbf{F}_p$. Let $f^*\colon D_0 \to DM(k;R)$ be the inclusion. Since $D_0(k;R)$ is the smallest localizing subcategory containing a certain set of compact objects, the inclusion f^* has a right adjoint f_*. Suppose that f^* also has a left adjoint $f_{(1)}$. Since f^* preserves arbitrary direct sums, $f_{(1)}$ must take compact objects in $DM(k;R)$ to compact objects in D_0, by Theorem 2.2.

Let X be a smooth projective 3-fold over k. Then $M(X)(-2)$ is compact in $DM(k;R)$, and so $f_{(1)}(M(X)(-2))$ is compact in D_0. By section 1,

$$\begin{aligned} CH^2(X;R) &= H^4(X,R(2)) \\ &= \mathrm{Hom}_{DM}(M(X),R(2)[4]) \\ &\cong \mathrm{Hom}_{DM}(M(X)(-2)[-4],f^*(R)) \end{aligned}$$

(which makes sense because the object R is in D_0)

$$\cong \mathrm{Hom}_{D_0}(f_{(1)}(M(X)(-2)[-4]),R).$$

I claim that $\mathrm{Hom}_{D_0}(N,R)$ is finite for every compact object N in D_0. We know that N can be obtained from the objects $M(Y)(j)[b]$ with Y smooth projective over k, $b \in \mathbf{Z}$ and $j + \dim(Y) \leq 0$ by finitely many cones and taking a summand. So it suffices to show that $\mathrm{Hom}_{D_0}(M(Y)(j)[b],R)$ is finite for every smooth projective variety Y over k, $b \in \mathbf{Z}$, and $j+\dim(Y) \leq 0$. Equivalently, we want to show that the motivic cohomology group $H^b(Y,R(a))$ is finite for all smooth projective varieties Y over k, all $b \in \mathbf{Z}$, and all $a \geq \dim(Y)$. This was proved by Suslin: the group mentioned is isomorphic to etale cohomology $H^b_{\text{et}}(Y,\mathbf{Z}/p(a))$ and hence is finite, using that k is algebraically closed [23, Corollary 4.3].

Thus, by two paragraphs back, $CH^2(X)/p$ is finite for every smooth projective 3-fold X over k. This contradicts the fact that there is a smooth projective 3-fold X over k with $CH^2(X)/p$ infinite, under our assumptions on k and p [21, Theorem 0.2], [26]. We conclude that the inclusion of $D_0(k;R)$ into $DM(k;R)$ does not have a left adjoint. ∎

A simpler argument shows that the inclusion f^* from $D_0(k;R)$ to $DM(k;R)$ does not have a three-fold right adjoint in most cases:

Theorem 7.2. *Let k be a field, and let R be a commutative noetherian ring in which the exponential characteristic of k is invertible. Suppose that there is a smooth projective k-variety such that some motivic cohomology group $H^j(X, R(i))$ is not finitely generated as an R–module. Let f^* be the inclusion of $D_0(k, R)$ into $DM(k, R)$. Then the right adjoint f_* of f^* does not preserve compact objects; the right adjoint $f^{(1)}$ of f_* does not preserve arbitrary direct sums; and $f^{(1)}$ does not have a right adjoint:*

$$f^* \dashv f_* \dashv f^{(1)}$$

These negative results hold, for example, if $R = \mathbf{Q}$ and k is not an algebraic extension of a finite field; or $R = \mathbf{Z}$ and k is an infinite field; or $R = \mathbf{F}_p$ for any prime number p and $k = \mathbf{C}$; or $R = \mathbf{F}_p$ with p a prime number congruent to 1 modulo 3 and k is an algebraically closed field of characteristic zero.

Proof. Suppose that there is a smooth projective variety X over k such that some motivic cohomology group $H^j(X, R(i))$ is not finitely generated as an R-module. We will show that the right adjoint $f_* \colon DM(k; R) \to D_0(k; R)$ does not preserve compact objects. Given that, the right adjoint $f^{(1)}$ of f_* does not preserve arbitrary direct sums, by Theorem 2.2. Therefore, $f^{(1)}$ does not have a right adjoint.

If f_* preserves compact objects, then for every compact object M in $DM(k; R)$, we have a compact object f_*M in $D_0(k; R)$ and a map $f_*M \to M$ which is universal for maps from $D_0(k; R)$ to M. In particular, since $R(0)$ is in $D_0(k; R)$, $H_0(f_*M, R(0)) \to H_0(M, R(0))$ is a bijection.

Let X be a smooth projective variety of dimension n, and let b be an integer such that $b \geq n$. (The objects $N = M(X)(-b)$ of this form generate $D_0(k; R)$.) I claim that the R-module $H_0(N[-j]; R(0))$ is finitely generated for all integers j. This group is $H_j(X, R(b))$. By the isomorphism of motivic homology with higher Chow groups (see section 1), this group is zero if $b > n$, and

$$\begin{aligned} H_j(X, R(n)) &\cong CH^0(X, j - 2n; R) \\ &\cong \begin{cases} R & \text{if } j = 2n \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus $H_0(N[-j]; R(0))$ is either 0 or R, and hence is a finitely generated R-module.

Every compact object in $D_0(k; R)$ belongs to the smallest thick subcategory that contains $M(X)(-b)$ for all smooth projective varieties X over k and all $b \geq \dim(X)$ (Theorem 1.1). Therefore, the long exact sequences for Hom in a triangulated category, plus the fact that R is noetherian, yield that the R-module $H_0(N, R(0))$ is finitely generated for all compact objects N in $D_0(k; R)$. If $f^{(1)}$ has a right adjoint, then (as explained above) it would follow that the R-module $H_0(N, R(0))$ is finitely generated for all compact objects N in $DM(k; R)$. In particular, all motivic homology groups of smooth projective k-varieties with R coefficients would be finitely generated, as we want.

Finite generation of motivic cohomology fails for the pairs (k, R) mentioned in the theorem, by the proof of Theorem 6.2. □

8 Mixed Tate motives over finite fields

We now show that some of the questions in this paper would have a different answer for k algebraic over a finite field, assuming the Tate-Beilinson conjecture. I do not know what to expect over number fields k, or with k replaced by a regular scheme of finite type over $\mathbf{Z}$.

Let p be a prime number. The *strong Tate conjecture* over $\mathbf{F}_p$ says that for smooth projective varieties X over $\mathbf{F}_p$ and a prime number $l \neq p$, the generalized eigenspace for the eigenvalue 1 of Frobenius on $H^{2i}(X_{\overline{\mathbf{F}_p}}, \mathbf{Q}_l(i))$ is spanned by codimension-i algebraic cycles on X with $\mathbf{Q}_l$ coefficients. The *Tate-Beilinson conjecture* over $\mathbf{F}_p$ is the combination of the strong Tate conjecture over $\mathbf{F}_p$ with the conjecture that rational and numerical equivalence coincide, for algebraic cycles with $\mathbf{Q}$ coefficients on smooth projective varieties over $\mathbf{F}_p$.

Theorem 8.1. *Let k be an algebraic extension field of $\mathbf{F}_p$. Assume the Tate-Beilinson conjecture. Then the inclusion f^* of the subcategory $DMT(k;\mathbf{Q})$ into $DM(k;\mathbf{Q})$ is a Frobenius functor. That is, the right adjoint functor f_* from $DM(k;\mathbf{Q})$ to $DMT(k;\mathbf{Q})$ is also left adjoint to f^*. It follows that the subcategory $DMT(k;\mathbf{Q})$ is closed under both direct sums and products in $DM(k;\mathbf{Q})$.*

Thus, given Tate-Beilinson, there is an infinite sequence of adjoint functors, consisting of f^* and f_* in turn:

$$\cdots \dashv f^* \dashv f_* \dashv f^* \dashv f_* \dashv \cdots .$$

As far as I know, the Bass conjecture (that K-groups of smooth varieties over $\mathbf{F}_p$ are finitely generated) would not be enough to imply that f^* has a left adjoint. In particular, Bruno Kahn explained to me that the Bass conjecture is not known to imply Parshin's conjecture, which is needed for the following argument. By contrast, the analog of the Bass conjecture for etale motivic cohomology would imply Parshin's conjecture.

Proof. Let k be an algebraic extension field of $\mathbf{F}_p$, and let X be a smooth projective variety over k. Given the Tate-Beilinson conjecture, the Chow groups $CH^*(X,\mathbf{Q})$ are finite-dimensional $\mathbf{Q}$-vector spaces (and in fact $\dim_{\mathbf{Q}} CH^i(X,\mathbf{Q}) \leq \dim_{\mathbf{Q}_l} H^{2i}_{\text{et}}(X_{\overline{k}}, \mathbf{Q}_l)$). Also, Geisser showed that the Tate-Beilinson conjecture implies Parshin's conjecture that $K_i(X)\otimes\mathbf{Q} = 0$ for $i > 0$ [10, Theorem 1.2]. Equivalently, $H^j(X,\mathbf{Q}(i)) = 0$ for $j \neq 2i$.

Let $f^*\colon DMT(k;\mathbf{Q}) \to DM(k;\mathbf{Q})$ be the inclusion. Since $DMT(k;\mathbf{Q})$ is the smallest localizing subcategory containing a certain set of compact objects, the inclusion f^* has a right adjoint f_* (by Lemma 2.3). We also write $N \mapsto C(N)$ for f_*. To prove that f^* also has a left adjoint $f_{(1)}$, it suffices to show that f^* has a three-fold right adjoint, by Theorem 2.4. Equivalently, we have to show that $f^{(1)}$ preserves arbitrary direct sums (Theorem 2.1), or again that f_* (also called $N \mapsto C(N)$) preserves compact objects (Theorem 2.2).

The subcategory of compact objects in $DM(k;\mathbf{Q})$ is the smallest thick subcategory that contains $M(X)(b)$ for all smooth projective varieties X over k and all integers b. So it suffices to show that $C(M(X)(b))$ is compact under these assumptions. Since $DMT(k;\mathbf{Q})$ is closed under tensoring with $\mathbf{Q}(b)$, it suffices to show that $C(M(X))$ is compact for every smooth projective k-variety X.

As discussed above, our assumptions give that the $\mathbf{Q}$-vector space $H^j(X, \mathbf{Q}(i))$ is zero if $j \neq 2i$ and finite-dimensional if $j = 2i$. Also, Quillen's calculation of the K-theory of finite fields [17, Theorem 8] gives that $\mathrm{Hom}_{DM(k;\mathbf{Q})}(\mathbf{Q}(0), \mathbf{Q}(i)[j])$ is $\mathbf{Q}$ if $i = j = 0$ and zero otherwise. It follows that there is a finite direct sum N of Tate motives $\mathbf{Q}(i)[2i]$ and a morphism $N \to M(X)$ that induces an isomorphism on motivic homology groups. So N is isomorphic to $C(M(X))$, and we have shown that $C(M(X))$ is compact. This completes the proof that the inclusion of $DMT(k;\mathbf{Q})$ into $DM(k;\mathbf{Q})$ has a left adjoint as well as a right adjoint.

Finally, we want to show that f^* is a Frobenius functor, that is, that the right adjoint f_* to the inclusion f^* is also left adjoint to f^*. We know from Lemma 2.3 that f_* has a right adjoint $f^{(1)}$. Recall that we use the notation $N \mapsto C(N)$ for f_*. By Balmer, Dell'Ambrogio, and Sanders, the object $\omega_f = f^{(1)}(\mathbf{Q}(0))$ (the *relative dualizing object* for f^*) is characterized by the existence of a natural bijection

$$\mathrm{Hom}_{DMT}(C(N), \mathbf{Q}(0)) \cong \mathrm{Hom}_{DM}(N, \omega_f)$$

for all N in $DM(k;\mathbf{Q})$ [3, Definition 1.4]. Given that f^* has a left adjoint $f_{(1)}$, f^* is a Frobenius functor if and only if $\omega_f \cong \mathbf{Q}(0)$ [3, Remark 1.15].

Thus, it suffices to show that for N in $DM(k;\mathbf{Q})$, the map $C(N) \to N$ induces a bijection $H^0(N, \mathbf{Q}(0)) \to H^0(C(N), \mathbf{Q}(0))$. Let $\mathcal{S}$ be the full subcategory of objects N such that $H^0(C(N)[j], \mathbf{Q}(0)) \to H^0(N[j], \mathbf{Q}(0))$ is a bijection for all integers j. Clearly $\mathcal{S}$ is a triangulated subcategory. Also, $N \mapsto C(N)$ preserves arbitrary direct sums, by Theorems 2.1 and 2.3. It follows that $\mathcal{S}$ is a localizing subcategory, using that $H^0(\oplus N_\alpha, \mathbf{Q}(0)) \cong \prod H^0(N_\alpha, \mathbf{Q}(0))$ for any set of objects N_α. So $\mathcal{S}$ is equal to $DM(k;\mathbf{Q})$ as we want if $\mathcal{S}$ contains $M(X)(-b)$ for all smooth projective varieties X and all integers $-b$.

To prove this, we use that, by the analysis of $C(M(X))$ above, the motive $N = M(X)(-b)[-c]$ for integers b and c satisfies

$$C(N) \cong \oplus_j \mathbf{Q}(j-b)[2j-c] \otimes CH_j(X, \mathbf{Q}).$$

We have

$$\begin{aligned} H^0(N, \mathbf{Q}(0)) &\cong H^c(X, \mathbf{Q}(b)) \\ &\cong \begin{cases} 0 & \text{if } c \neq 2b \\ CH^b(X;\mathbf{Q}) & \text{if } c = 2b. \end{cases} \end{aligned}$$

On the other hand, by the description of $C(N)$ above,

$$H^0(C(N), \mathbf{Q}(0)) \cong \begin{cases} 0 & \text{if } c \neq 2b \\ CH_b(X;\mathbf{Q})^* & \text{if } c = 2b. \end{cases}$$

Since rational and numerical equivalence coincide (by the Tate-Beilinson conjecture), the natural map $CH^b(X;\mathbf{Q}) \to CH_b(X;\mathbf{Q})^*$ is a bijection. This shows that $M(X)(-b)$ is in the subcategory $\mathcal{S}$ for all smooth projective varieties X over k and all integers b. As a result, $\mathcal{S}$ is equal to $DM(k;\mathbf{Q})$. That is, the inclusion from $DMT(k;\mathbf{Q})$ into $DM(k;\mathbf{Q})$ is a Frobenius functor. $\square$

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