Rings are understood to be commutative, unless stated otherwise.

(1) Let $R$ be a noetherian ring. We have shown that $X = \text{Spec}(R)$ can be written as the union of finitely many irreducible closed subsets, $X = X_1 \cup X_2 \cup \cdots \cup X_m$, such that $X_i$ is not contained in $X_j$ for any $i \neq j$. Show that such a decomposition of $X$ is unique up to reordering the $X_i$’s.

(2) Write $A^3_C$ for affine 3-space over $C$, meaning $\text{Spec}(C[x, y, z])$. Let $X$ be the closed subset of $A^3_C$ defined by $x^2 = yz$ and $xz = x$. Decompose $X$ into its irreducible components.

(3) Show that the following are equivalent, for a module $M$ over a ring $R$.

(1) $M$ is projective. (2) $\text{Ext}^i_R(M, N) = 0$ for all $R$-modules $N$ and all $i > 0$. (3) $\text{Ext}^1_R(M, N) = 0$ for all $R$-modules $N$. Likewise, show that the following are equivalent, for a module $M$ over a ring $R$. (1) $M$ is flat. (2) $\text{Tor}^i_R(M, N) = 0$ for all $R$-modules $N$ and all $i > 0$. (3) $\text{Tor}^1_R(M, N) = 0$ for all $R$-modules $N$. 