Due on Monday, June 2.

In the following problems, varieties are over an algebraically closed field \( k \), unless stated otherwise. A “curve of genus \( g \)” is usually understood to be smooth and projective over \( k \).

(1) Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Show that the morphism \( f : \mathbb{A}^1_k \rightarrow \mathbb{A}^1_k \) defined by \( x \mapsto x^p \) is a bijective morphism but not an isomorphism. Where is the derivative of \( f \) zero? Is \( f : \mathbb{A}^1 \rightarrow \mathbb{A}^1 \) birational?

(2) Show that any curve of genus zero is isomorphic to \( \mathbb{P}^1 \).

(3) Show that any curve \( X \) of genus 1 can be written as a degree-2 ramified covering of \( \mathbb{P}^1 \) (meaning that there is a morphism \( X \rightarrow \mathbb{P}^1 \) of degree 2). Curves of genus 1 are called elliptic curves. Show that an elliptic curve is not rational (that is, it is not birational to \( \mathbb{P}^1 \)).

(4) For an effective divisor \( D \) on a curve \( X \) of genus \( g \), show that \( h^0(X, O(D)) \leq \deg(D) + 1 \). Show that equality holds if and only if \( D = 0 \) or \( g = 0 \).

(5) Let \( X \) be a smooth projective curve, \( p \) a point in \( X \). Show that there is a nonconstant rational function on \( X \) which is regular outside \( p \).

(6) A curve \( X \) is called hyperelliptic if it has genus \( g \geq 2 \) and there is a morphism \( X \rightarrow \mathbb{P}^1 \) of degree 2.
   
   (a) If \( X \) is a curve of genus 2, show that the canonical bundle \( K_X \) defines a morphism \( X \rightarrow \mathbb{P}^1 \) of degree 2. Thus every curve of genus 2 is hyperelliptic.
   
   (b) For \( g(X) \geq 2 \), show that the canonical bundle \( K_X \) defines a morphism \( X \rightarrow \mathbb{P}^{g-1} \). (The main point here is to check that \( K_X \) is basepoint-free.) If \( X \) is not hyperelliptic, show that the canonical bundle defines an embedding of \( X \) in \( \mathbb{P}^{g-1} \), the canonical embedding.

   (c) Compute the genus of a smooth plane quartic curve \( X \) (“quartic” means degree 4), by describing the canonical bundle of \( X \). Show that \( X \) is not hyperelliptic. (You may use that if a curve \( X \) of any genus \( g \geq 2 \) is hyperelliptic, then the canonical map \( X \rightarrow \mathbb{P}^{g-1} \) is a double cover of its image, which is a rational normal curve.)

(7) Show that any elliptic curve \( X \) can be embedded as a smooth curve of degree \( d \) in \( \mathbb{P}^{d-1} \) for any \( d \geq 3 \). Show that a transverse intersection of two smooth quadrics in \( \mathbb{P}^3 \) is indeed an elliptic curve of degree 4. But show that an elliptic curve of degree \( d \) in \( \mathbb{P}^{d-1} \) is not a complete intersection for \( d \geq 5 \). (Hint: from Homework 2, you know the canonical bundle of any smooth complete intersection curve in any \( \mathbb{P}^n \).)