Homework 7 for Math 131C Topics in Analysis

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Spring 2018

Due Tuesday, June 5, 2018.

(1) Show that $L^2([0,1]) \subset L^1([0,1])$ (for example, using the Cauchy-Schwarz inequality). Give an example to show that this inclusion is not an equality. Show that neither of $L^1(\mathbf{R})$ or $L^2(\mathbf{R})$ contains the other.

(2) Show that the space of smooth (C^∞) functions with compact support is dense in $L^2(\mathbf{R})$. (Hint: Reduce to showing that step functions can be approximated in the L^2 norm by smooth functions with compact support. Reduce further to the case of the characteristic function of an interval. Graph the function

$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

which is known to be smooth. Use g to construct the approximating functions you need.)

(3) Show that the functions x^n for $n \geq 0$ span $L^2([-1,1])$. (Hint: reduce to the properties of Fourier series in $L^2([0,1])$, which you can easily translate to $L^2([-1,1])$.) By the Gram-Schmidt process, these functions determine an orthonormal basis $e_0(x), e_1(x), \ldots$ for $L^2([-1,1])$ (which are polynomials, clearly). Compute $e_0, \ldots, e_3$ explicitly. (For example, $e_0 = 1/\sqrt{2}$.)

(4) Show that the "Haar functions"

$$e_0^0(x) = 1 \text{ for } 0 \leq x \leq 1,$$

$$e_n^k(x) = \begin{cases} 2^{n/2} & \text{for } \frac{k-1}{2^n} \leq x \leq \frac{k-1/2}{2^n}, \\ -2^{n/2} & \text{for } \frac{k-1/2}{2^n} \leq x \leq \frac{k}{2^n}, \\ 0 & \text{otherwise,} \end{cases}$$

defined for $1 \leq k \leq 2^n$ and $n \geq 1$ (and $k = n = 0$), form an orthonormal basis for $L^2([0,1])$. Graph the first few functions. (Hint: to show that these functions span, show that $\int_0^x f = 0$ for every x of the form $k/2^n$ with $1 \leq k \leq 2^n$ and $n \geq 1$, and deduce that $\int_E f - 0$ for every measurable subset E of $[0,1]$.)