Math 115AH Linear Algebra. Homework 8

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Due Monday, November 23.

Problems from Hoffman-Kunze:

Section 8.2: 2, 5, 11, 14, 16.

Section 8.3: 2, 4, 6, 9, 12. (Note: problem 9 may be an unpleasant calculation. Just set the computation up; you don't have to carry it out.)

(1) Let V be the space of all bounded functions f from $[0, 2\pi]$ to the complex numbers such that f has derivatives of all orders except at finitely many points. (Thus f may be discontinuous at finitely many points; we don't care about the value of f at those points.) Define an inner product on V by

$$\langle f, g\rangle = \int_0^{2\pi} f(x)\overline{g(x)}\, dx.$$

You can assume that this definition is OK.

(a) For an integer n, let $f_n(x) = e^{inx}$. Show that the functions f_n are orthogonal, that is, that $\langle f_m, f_n\rangle = 0$ if $m \neq n$.

(b) Let W_N be the subspace of V spanned by

$$f_{-N}, f_{-N+1}, \ldots, f_0, \ldots, f_N.$$

For $0 < \alpha < 2\pi$, let $F(x)$ be 1 on the interval $(0, \alpha)$ and zero otherwise. Let F_N be the orthogonal projection of F to W_N. What is

$$||F_N||^2?$$

The answer will be a sum of $2N + 1$ terms. You should be able to combine terms to get a sum of $N + 1$ real terms.

(c) Show that

$$\pi^2/6 \geq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Hint: Let $G(x) = x$. Calculate the orthogonal projection of G to W_N, use Bessel's inequality, and take a limit.

(d) It's a theorem that if $f \in V$ and f_N denotes the orthogonal projection of f to W_N, then

$$||f - f_N|| \to 0$$

as $N \to \infty$. Deduce that

$$\pi^2/6 = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(e) What infinite sum can you evaluate by taking the orthogonal projections of $H(x) = x(2\pi - x)$?

(2) Let V be a real inner product space. For $x \in V$, define a linear functional $L_v \in V^*$ by

$$L_x(y) = \langle x, y \rangle \in \mathbf{R}$$

for $y \in V$. Define a function $\Phi \colon V \to V^*$ by $\Phi(x) = L_x$.

If V has finite dimension, show that Φ is an isomorphism. What goes wrong if V is a complex inner product space and you try to define an isomorphism $V \to V^*$ by the same method?