Math 115AH Linear Algebra. Homework 3

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Due Friday, October 16.
Problems from Hoffman-Kunze:
Section 2.3: 14.
Section 2.4: 1, 2, 4, 6, 7.

Hint on one of the problems: A complex number c is defined to be *algebraic* if there is a positive integer n and rational numbers $a_0, \ldots, a_{n-1}$ such that

$$c^n + a_{n-1}c^{n-1} + \cdots + a_0 = 0.$$

A complex number which is not algebraic is called *transcendental*. For this homework, you can use the fact that there are real numbers which are transcendental. (A famous example: Lindemann showed in 1882 that the number $\pi = 3.14159\ldots$, the area of a circle of radius 1, is transcendental.)

(1) Let V be a vector space. Let S be a subset of V such that

$$S = S_1 \cup S_2$$

and

$$S_1 \cap S_2 = \emptyset.$$

Suppose that the set S is linearly independent in V. Prove:

$$\text{span}(S_1) \cap \text{span}(S_2) = \{0\}.$$

You can assume that S is finite.

(2) Let V be the vector space of functions $f\colon \mathbf{R} \to \mathbf{R}$. Let

$$S = \{\ldots, e^{-2x}, e^{-x}, 1, e^x, e^{2x}, \ldots\} = \{e^{nx} : x \in \mathbf{Z}\}.$$

Show that the set S is linearly independent.

To see how to attack a problem like this, you could start by trying a few cases, such as $\{1, e^x, e^{2x}\}$ or $\{1, e^x, e^{2x}, e^{3x}\}$. Try differentiating the supposed linear dependence relation.