Math 115AH Linear Algebra. Homework 1

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Due Friday, October 2.

(1) In set theory, $A - B$ means the set of elements of A which are not in B. I write $A \subset B$ to mean that A is a subset of B (possibly equal to B). That is, every element of A is an element of B. Writing $A \not\subset B$ means that A is not a subset of B.

The following statements are understood to begin with "for all sets A, B, C". Prove the correct ones and find counterexamples to the incorrect ones. You can use Venn diagrams for the counterexamples.

a. $(A \cap B) \cup C = A \cap (B \cup C)$
b. $(A \cup B) \cap C = A \cup (B \cap C)$
c. $(A - B) \cap (C - B) = (A \cap C) - B$
d. $(A - B) \cap (C - B) = A - (B \cup C)$
e. If $A \cap C = B \cap C$, then $A = B$.
f. If $A \cup C = B \cup C$, then $A = B$.
g. If $A \subset B$, then $A \cap C \subset B \cap C$.
h. If $A \not\subset B$ and $B \not\subset C$, then $A \not\subset C$.
i. If $A \subset C$ and $B \subset C$, then $A \cup B \subset C$.
j. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

(2) Prove that $\sqrt{21} \notin \mathbf{Q}$. [You can use the usual properties of factorization of integers into primes. For instance, if a and b are integers and a prime number p divides ab, then p divides a or b. In proofs, write in complete sentences and explain what you are doing. A sequence of equations by itself is almost never a satisfactory proof.]

(3) Let $F \subset \mathbf{C}$ be a field and let S be a set. Let V be an F-vector space. Let W be the set of functions from S to V. We can add two elements f and g of W by

$$(f + g)(s) = f(s) + g(s).$$

Define scalar multiplication by an element c in F by:

$$(cf)(s) = c \cdot f(s).$$

Show that these definitions make W an F-vector space. You should check the axioms on page 28 until you get bored.

Homework from the text (Hoffman-Kunze):

Section 2.1: 2, 4, 5, 6, 7. Of course, in 7, either show an explicit example of the failure of the vector space axioms or check all the axioms. In problem 2, you should state what axiom you are using in each step and (where needed) how you are using it.