# Math 115AH Linear Algebra. Sample midterm 2

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This is a sample midterm 2 (the exam from Spring 2014). Note that this may not include all the topics (including definitions and theorems) that you need to know. The exam will cover the material from homeworks 1 to 7, that is (in the book, Hoffman-Kunze): chapter 1 as background, and then sections 2.1–2.4, 3.1–3.5, 3.7, 6.2, and 8.1.

In the following questions, $V$ and $W$ are understood to be finite-dimensional vector spaces over a subfield of $\mathbf{C}$ (sometimes a specific field).

(1) Let $\langle x, y\rangle$ be an inner product on a real vector space $V$ with norm $||x||$. Find an expression for

$$||x+y||^2 + ||x-y||^2$$

in terms of $||x||$ and $||y||$.

(2) Let $T\colon V \to V$ be linear, and suppose that $T^3 = 0$.

(a) What are the possible eigenvalues of $T$? Hint: if $Tx = \lambda x$, what are $T^2 x$ and $T^3 x$?

(b) Give an example of such a $T$ with $T \neq 0$.

(3) Let $W_1$ and $W_2$ be two subspaces of a finite-dimensional vector space $V$. Prove that

$$(W_1 + W_2)^0 = W_1^0 \cap W_2^0.$$

(Here $W^0$ is the annihilator of $W$ in $V^*$.

(4) Let $T\colon V \to W$ and $S\colon W \to V$ be linear maps.

(a) Show that if $\lambda$ is a nonzero eigenvalue of $TS$, then $\lambda$ is a nonzero eigenvalue of $ST$. (Hint: if $STx = \lambda x$, show that $y = Tx$ is an eigenvector of $TS$.) Explain where $\lambda \neq 0$ is used in your proof.

(b) Give an example where $TS$ and $ST$ do not have the same eigenvalues.

(5) Let $T\colon V \to V$ be a linear map, and let $v_1 \in V$. Let $v_2 = Tv_1$ and $v_3 = T^2 v_1$. Finally, suppose that

$$\mathcal{B} = \{v_1, v_2, v_3\}$$

is a basis for $V$. In particular, we can write

$$T^3 v_1 = c_1 v_1 + c_2 T v_1 + c_2 T^2 v_1.$$

(a) What is

$$[T]_{\mathcal{B}},$$

the matrix of $T$ with respect to the basis $\mathcal{B}$?

(b) What is the characteristic polynomial of $T$ in terms of $c_1, c_2, c_3$? (Hint: it might help to expand down the last column of $[T]_{\mathcal{B}}$.)

Extra Credit: In problem 5, prove that $T^3$ is a linear combination of 1, $T$, and $T^2$. (Here 1 is the identity map.)