A Few Remarks on Decoupling

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1 Introduction

This note serves as an expository to a few technical remarks in decoupling theory. For simplicity of notation we only consider decoupling for the unit parabola on the plane, but many of the results in this note can be extended naturally to other good curves/surfaces and higher dimensions.

Definition 1 ($l^q(L^p)$-decoupling, extension operator version)**.** *For $g \in L^1([0,1])$ we define the extension operator*

$$Eg(x_1,x_2) = \int_0^1 g(s)e(x_1 s + x_2 s^2)ds,$$

where $e(z) := \exp(2\pi i z)$. For $0 < p, q \le \infty$ and $\delta \in \mathbb{N}^{-2}$, define $D_{p,q}(\delta)$ to be the best constant such that for all $g \in L^1([0,1])$ and all square $B \subseteq \mathbb{R}^2$ of side length δ^{-1}, we have

$$\|Eg\|_{L^p(B)} \le D_{p,q}(\delta) \Big\| \|Eg_j\|_{L^p(w_B)} \Big\|_{l^q(j)},$$

where $g_j = g1_{I_j}$, $I_j := [(j-1)\delta^{1/2}, j\delta^{1/2}]$, $1 \le j \le \delta^{-1/2}$ and $w_B = w_{B,E}$ is a weight function adapted to B (with centre (x_B, y_B) and $E \ge 100$):

$$w_B(x,y) := \left(1 + \delta^{-1}(|x - x_B| + |y - y_B|)\right)^{-E}.$$

We will take for granted the well-known $l^2(L^p)$-decoupling inequality for the unit parabola for $2 \le p \le 6$. Here and henceforth, $\lessapprox$ means $\lesssim$ with possibly an ε-loss: that is, $A(\delta) \lessapprox B(\delta)$ if for every $\varepsilon > 0$ there is some $C_\varepsilon > 0$ such that $A(\delta) \le C_\varepsilon \delta^{-\varepsilon} B(\delta)$ for every $0 < \delta \le 1$. Similar for the notation $\gtrapprox$ and $\approx$.

Theorem 2 (Bourgain-Demeter, [1,2])**.** *For $2 \le p \le 6$ we have $D_{p,2}(\delta) \lessapprox 1$.*

A self-contained proof can be found in [2]. Based on this fact, we will prove the following sharp $l^q(L^p)$-decoupling inequality:

Theorem 3 (Sharp $l^q(L^p)$-decoupling)**.** *For $2 \le p \le \infty$, $2 \le q \le \infty$, we have*

$$D_{p,q}(\delta) \approx \begin{cases} \delta^{-\frac{1}{4}+\frac{1}{2q}} & \text{if } 2 \le p \le 6 \\ \delta^{-\frac{1}{2}+\frac{3}{2p}+\frac{1}{2q}} & \text{if } 6 \le p \le \infty \end{cases}.$$

By taking $q = 2$ and $q = p$, respectively, we arrive at the following sharp $l^2(L^p)$- and $l^p(L^p)$-decoupling for the unit parabola:

Corollary 4. *For $p \geq 2$, we have*

$$D_{p,2}(\delta) \approx \begin{cases} 1 & \text{if } 2 \leq p \leq 6 \\ \delta^{-\frac{1}{4}+\frac{3}{2p}} & \text{if } 6 \leq p \leq \infty \end{cases}.$$

and

$$D_{p,p}(\delta) \approx \begin{cases} \delta^{-\frac{1}{4}+\frac{1}{2p}} & \text{if } 2 \leq p \leq 6 \\ \delta^{-\frac{1}{2}+\frac{2}{p}} & \text{if } 6 \leq p \leq \infty \end{cases}.$$

2 An interpolation theorem for mixed normed spaces

In this section, we aim to prove decoupling for exponents $p > 6$.

Theorem 5 (Bourgain-Demeter, [1])**.** *For $6 \leq p \leq \infty$ we have $D_{p,2}(\delta) \lessapprox \delta^{-\frac{1}{4}+\frac{3}{2p}}$.*

We can use the following interpolation theorem to prove the above theorem.

Theorem 6. *Let X, Y, M, N be σ-finite measure spaces. Let $1 \leq p_i, q_i, r_i, s_i \leq \infty$, $i = 0, 1$. Let $0 < \theta < 1$ and (with the convention that $1/\infty = 0$)*

$$\frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p}, \quad \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1}{q}, \quad \frac{1-\theta}{r_0} + \frac{\theta}{r_1} = \frac{1}{r}, \quad \frac{1-\theta}{s_0} + \frac{\theta}{s_1} = \frac{1}{s}.$$

Assume in addition that either $p, r < \infty$ or $p = r = \infty$.

Assume T is a linear operator from $L^{p_i}(X, L^{r_i}(M))$ to $L^{q_i}(Y, L^{s_i}(N))$ with operator norms A_i, $i = 0, 1$. Then T maps $L^p(X, L^r(M))$ to $L^q(Y, L^s(N))$, with operator norm bounded above by $A_0^{1-\theta} A_1^{\theta}$.

Remark 1. There is still a tiny gap when $r < p = \infty$ or $p < r = \infty$, which we fail to prove due to technicality. But this case is rarely used in applications, so we leave it open.

Remark 2. The theorem actually holds for exponents below 1. This will not be used for our purpose now, but it will be useful in some multilinear inequalities where the Lebesgue exponents could go below 1. An idea of the proof can be found in Question 10 of [6].

2.1 Proof of the interpolation theorem

Proof. We first assume $p, r < \infty$. The proof will be given in the following steps.

Step 1. Applying duality. We will be using the following easy consequence of the duality of L^p norms:

Lemma 7 (Duality for mixed L^p-norms)**.** *Let X, Y be σ-finite measure spaces, and let $1 \leq p, q \leq \infty$. Then*

$$\|f\|_{L^p(X, L^q(Y))} = \sup\left\{ \left| \iint f(x,y) g(x,y) dy dx \right| : \|g\|_{L^{p'}(X, L^{q'}(Y))} = 1 \right\}.$$

If, furthermore, $p, q > 1$, then

$$\|f\|_{L^p(X,L^q(Y))} = \sup\left\{\left|\iint f(x,y)g(x,y)dydx\right| : g \text{ simple}, \|g\|_{L^{p'}(X,L^{q'}(Y))} = 1\right\}.$$

Here, a simple function is of the form $\sum_{n=1}^{N} a_n 1_{A_n}$, where $a_n \in \mathbb{C}$ and A_n's are disjoint measurable sets in $X \times Y$ of finite measure.

Proof of lemma. For the first equality, the ($\geq$) side is by applying Hölder's inequality twice. The ($\leq$) side follows by applying the duality for a single norm twice. (Note that in any case, the absolute value on the right hand side can be placed inside or outside the integral without changing the equality. Note also that the inner test function can be taken to be a measurable function of both x and y, as can be seen from the proof of the duality of a single norm.)

The second equality follows from the density of simple functions in mixed norms, since X and Y are σ-finite. □

Using the lemma, it suffices to show that for all f, g simple with $\|g\|_{L^q(Y,L^s(N))} = 1$, we have

$$A_0^{\theta-1}A_1^{-\theta}\left|\iint Tf(n,y)g(n,y)dndy\right| \leq \|f\|_{L^p(X,L^r(M))}. \tag{1}$$

Step 2. Introducing a complex parameter. Let S denote the open strip $\mathrm{Re}(z) \in (0,1)$. For $z \in \overline{S}$, we define $p_z, r_z \in (0,\infty]$ such that

$$\frac{1}{p_z} = \frac{1-z}{p_0} + \frac{z}{p_1} \quad , \frac{1}{r_z} = \frac{1-z}{r_0} + \frac{z}{r_1}$$

Note that $p_\theta = p$ and $r_z = r$.

Define $f_z(m,x) = 0$ if $f(m,x) = 0$, and for $f_z(m,x) \neq 0$, define

$$f_z(m,x) = \frac{f(m,x)}{|f(m,x)|}\frac{|f(m,x)|^{\frac{r}{r_z}}}{\|f(\cdot,x)\|_{L^r(M)}^{\frac{r}{r_z}}}\frac{\|f(\cdot,x)\|_{L^r(M)}^{\frac{p}{p_z}}}{\|f\|_{L^p(X,L^r(M))}^{\frac{p}{p_z}}}\|f\|_{L^p(X,L^r(M))},$$

so that

$$\|f_z\|_{L^{p_i}(X,L^{r_i}(M))} = \|f\|_{L^p(X,L^r(M))} \tag{2}$$

whenever $\mathrm{Re}(z) = i$, $i = 0,1$, and $f_\theta = f$.

Since f is simple and T is linear, we may write

$$Tf_z = \sum_{j=1}^{J} a_{j,z}T(1_{A_j}),$$

where A_j is a measurable set in $M \times X$ and $a_{j,z} \in \mathbb{C}$ is analytic and has order of growth 1 in $z \in S$.

Similarly, define $q_z, s_z \in (0,\infty]$ such that

$$\frac{1}{q_z} = \frac{1-z}{q_0} + \frac{z}{q_1} \quad \frac{1}{s_z} = \frac{1-z}{s_0} + \frac{z}{s_1},$$

Note that $q_\theta = q$ and $s_\theta = s$.

Define $g_z(n,y) = 0$ if $g(n,y) = 0$, and for $g_z(n,y) \neq 0$, define (with the convention $\infty/\infty = 1$)

$$g_z(n,y) = \frac{g(n,y)}{|g(n,y)|} \frac{|g(n,y)|^{\frac{s}{s_z}}}{\|g(\cdot,y)\|_{L^s(N)}^{\frac{s}{s_z}}} \frac{\|g(\cdot,y)\|_{L^s(N)}^{\frac{q}{q_z}}}{\|g\|_{L^q(Y,L^s(N))}^{\frac{q}{q_z}}} \|g\|_{L^q(Y,L^s(N))},$$

so that

$$\|g_z\|_{L^{q_i}(Y,L^{r_i}(N))} = \|g\|_{L^q(Y,L^r(N))} \tag{3}$$

whenever $\mathrm{Re}(z) = i$, $i = 0, 1$, and $g_\theta = g$.

Since g is simple, we may similarly write

$$g_z = \sum_{k=1}^{K} b_{k,z} 1_{B_k},$$

where B_k is a measurable set in $N \times Y$ and $b_{k,z} \in \mathbb{C}$ is analytic and has order of growth 1 in $z \in S$.

Step 3. Applying maximum modulus principle. With the notation above, for each $\varepsilon > 0$, we can define the following nonnegative function which is analytic in S, continuous up to the boundary, and decays to 0 as $|\mathrm{Im}(z)| \to \infty$:

$$F_\varepsilon(z) = e^{\varepsilon z^2} A_0^{z-1} A_1^{-z} \iint T f_z(n,y) g_z(n,y) dndy.$$

Here, we have used the order of growth of the constants A_0^{z-1}, A_1^{-1}, $a_{j,z}$, and $b_{k,z}$.

Take $K = K(\varepsilon)$ such that for all $|\mathrm{Im}(z)| \geq K$, we have $|F_\varepsilon(z)| < \|f\|_{L^p(X,L^r(M))}$. By the maximum modulus principle for analytic functions and our choice of K, it suffices to show $F_\varepsilon(z)$ is bounded by $(1 + O(\varepsilon))\|f\|_{L^p(X,L^r(M))}$ for all Re(z) $= i$, $i = 0, 1$.

We have not used the operator norms of T yet, and now is the time to use it. Using Hölder's inequality, for $\mathrm{Re}(z) = 0$, we have

$$\begin{aligned}
|F_\varepsilon(z)| &\leq A_0^{-1} \|T f_z\|_{L^{q_0}(Y,L^{s_0}(N))} \|g_z\|_{L^{q_0'}(Y,L^{s_0'}(N))} \\
&= A_0^{-1} \|T f_z\|_{L^{q_0}(Y,L^{s_0}(N))} \|g_z\|_{L^{q_0}(Y,L^{s_0}(N))} \\
&\leq \|f_z\|_{L^{p_0}(X,L^{r_0}(M))} \|g_z\|_{L^{q_0}(Y,L^{s_0}(N))} \\
&= \|f\|_{L^p(X,L^r(M))} \|g\|_{L^q(Y,L^s(N))} \\
&= \|f\|_{L^p(X,L^r(M))},
\end{aligned}$$

where in the second to last line we have used (2) and (3).

Similarly, for $\mathrm{Re}(z) = 1$, we have

$$|F_\varepsilon(z)| \leq e^{\varepsilon} \|f\|_{L^p(X,L^r(M))}.$$

Thus, letting $\varepsilon \to 0$, we have for any $z \in \overline{S}$ that

$$|A_0^{z-1} A_1^{-z}| \left| \iint T f_z(n,y) g_z(n,y) dndy \right| \leq \|f\|_{L^p(X,L^r(M))},$$

which is (1) if we let $z = \theta$.

The case of infinite exponents. We finally consider the special case $p = r = \infty$. In particular, this means that $p_0 = p_1 = p$ and $r_0 = r_1 = r$. In this case, we use log-convexity and Hölder's inequality:

$$\begin{aligned}
\|Tf\|_{L^q(Y,L^s(N))} &\leq \left\| \|Tf(\cdot,y)\|_{L^{s_0}(N)}^{1-\theta} \|Tf(\cdot,y)\|_{L^{s_1}(N)}^{\theta} \right\|_{L^q(Y)} \\
&\leq \|Tf\|_{L^{q_0}(Y,L^{s_0}(N))}^{1-\theta} \|Tf\|_{L^{q_1}(Y,L^{s_1}(N))}^{\theta} \\
&\leq A_0^{1-\theta} \|f\|_{L^p(X,L^r(M))}^{1-\theta} A_1^{\theta} \|f\|_{L^p(X,L^r(M))}^{\theta} \\
&= A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p(X,L^r(M))}.
\end{aligned}$$

where in the second line we have used the relation $1/q = (1-\theta)/q_0 + \theta/q_1$.

□

2.2 Application in decoupling theory

Now we can use Theorem 6 proved above to prove Theorem 5. For technical reasons, we consider another version of decoupling, which can be shown to be equivalent to the decoupling constant defined in Definition 1 (see [4], Section 2.)

Definition 8. *Let $\delta \in \mathbb{N}^{-2}$. For $r > 0$ and $1 \leq j \leq \delta^{-1/2}$, we define $\mathcal{N}_j(r)$ to be the r-neighbourhood of the graph of (s, s^2) over $s \in [(j-1)\delta^{1/2}, j\delta^{1/2}]$.*

For $0 < p \leq \infty$, let $D_p'(\delta)$ be the best constant such that for any sequence of functions $f_j \in L^p(\mathbb{R}^2)$, $1 \leq j \leq \delta^{-1}$ each with Fourier support in $\mathcal{N}_j(\delta)$, we have

$$\left\| \sum_{j=1}^{\delta^{-1/2}} f_j \right\|_{L^p(\mathbb{R}^2)} \leq D'_{p,2}(\delta) \left(\sum_{j=1}^{\delta^{-1/2}} \|f_j\|_{L^p(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}. \tag{4}$$

Proposition 9. *For $2 \leq p \leq \infty$, we have $D'_{p,2}(\delta) \sim_p D_{p,2}(\delta)$.*

Proof of Theorem 5. Using the proposition above, it suffices to prove $D'_{p,2}(\delta) \lessapprox \delta^{-\frac{1}{4}+\frac{3}{2p}}$, that is,

$$\left\| \sum_{j=1}^{\delta^{-1/2}} f_j \right\|_{L^p(\mathbb{R}^2)} \lessapprox \delta^{-\frac{1}{4}+\frac{3}{2p}} \left(\sum_{j=1}^{\delta^{-1/2}} \|f_j\|_{L^p(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}. \tag{5}$$

By Theorem 2 we have $D_{p,2}(\delta) \lessapprox 1$ for $2 \leq p \leq 6$. We also know that $D_{\infty,2}(\delta) \leq \delta^{-\frac{1}{4}}$ by the trivial triangle inequality and Cauchy-Schwarz. All we have to do is interpolate.

The interpolation theorem cannot be applied directly since we have to define a linear operator first. To do this, we first deal with the Fourier restriction condition. For each j, define the multiplier operator T_j such that $\widehat{T_j f} = \psi_j \hat{f}$, where $\psi_j \in C_c^\infty(\mathcal{N}_j(2\delta))$ and equals 1 in $\mathcal{N}_j(\delta)$.

Then (5) is true (up to a constant depending on p only) if and only if for arbitrary $g_j \in L^p(\mathbb{R}^2)$ we have

$$\left\| \sum_{j=1}^{\delta^{-1/2}} T_j g_j \right\|_{L^p(\mathbb{R}^2)} \lessapprox \delta^{-\frac{1}{4}+\frac{3}{2p}} \left(\sum_{j=1}^{\delta^{-1/2}} \|g_j\|_{L^p(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}. \tag{6}$$

Indeed, for the "if" side, given f_j, we may just take $g_j = f_j$ and note that $T_j f_j = f_j$ by the Fourier support condition of f_j. For the "only if" side, given arbitrary g_j, we take $f_j = T_j g_j$ and note that $\widehat{f_j}$ is supported on $\mathcal{N}_j(2\delta)$, which is slightly larger than $\mathcal{N}_j(\delta)$, but we still have (5) up to a constant using a simple tiling argument. Lastly we bound the right hand side $\|T_j g_j\|_{L^p(\mathbb{R}^2)} \lesssim \|g_j\|_{L^p(\mathbb{R}^2)}$ by Young's inequality.

Hence, it suffices to prove 6 instead. Define the linear operator T that acts on sequences $\{g_j\}_{j=1}^{\delta^{-1/2}}$ by the relation $T\{g_j\} = \sum_j T_j g_j$.

With this, the assumptions in Theorem 6 are satisfied, with the following choices. (To avoid ambiguity, we change the exponent p of the decoupling inequality to t.) $X = \{1, 2, \ldots, \delta^{-1/2}\}$, $M = N = \mathbb{R}^2$ and no Y involved. The objective exponents are $p = 2$, $s = r = t < \infty$ (so we are not in the unsolved special case) and no q involved. The boundary exponents are given by $p_0 = p_1 = 2$, $s_0 = r_0 = 6$, $s_1 = r_1 = \infty$. Thus $\theta = 1 - 6/t$ and we have the desired interpolation theorem. □

3 Sharpness of decoupling

In this section, we shall prove Theorem 3. A proof of this theorem in the case $p > 6$ can be found in Theorem 12.22 of [3], but our proof also works for $2 \le p \le 6$ and does not rely directly on an application of the restriction estimate.

3.1 A lemma on exponential sums

Lemma 10. *For $\delta \in \mathbb{N}^{-2}$ we define the function on $\mathbb{T}^2$ as*

$$f(x,y) = \sum_{j=1}^{\delta^{-1/2}} e\left(jx + j^2 y\right).$$

Then for all $2 \le p \le \infty$, we have

$$\|f\|_{L^p(\mathbb{T}^2)} \approx \begin{cases} \delta^{-\frac{1}{4}} & \text{if } 2 \le p \le 6 \\ \delta^{-\frac{1}{2}+\frac{3}{2p}} & \text{if } 6 \le p \le \infty \end{cases}.$$

Proof. We first prove the upper bound by testing

$$g(s) = \sum_{j=1}^{\delta^{-1/2}} \Delta_{j\delta^{1/2}}(s),$$

where Δ_a is the delta-mass at a. (A more rigorous argument is to take an approximation to the identity at each of the delta masses.) Then we have

$$Eg(x,y) = \sum_{j=1}^{\delta^{-1/2}} e\left(j\delta^{1/2}x + j^2\delta y\right) = f(\delta^{1/2}x, \delta y).$$

Also, for each j we have (we may choose the aforesaid approximation to the identity slightly to the left at each delta mass)

$$Eg_j(x,y) = e\left(j\delta^{1/2}x + j^2\delta y\right).$$

Thus $\|Eg_j\|_{L^p(w_B)} \sim \delta^{-\frac{2}{p}}$, and so

$$\left\| \|Eg_j\|_{L^p(w_B)} \right\|_{l^2(j)} \sim \delta^{-\frac{1}{4}}\delta^{-\frac{2}{p}}.$$

But by periodicity, we have

$$\|Eg\|_{L^p(B)} = \delta^{-\frac{2}{p}} \|f\|_{L^p(\mathbb{T}^2)}.$$

Combining with (2) and (5), we get the desired upper bound.

Now we come to the lower bound. The case $p = \infty$ is trivial, taking $x = y = 0$. The case $6 \le p < \infty$ follows by considering (x,y) near the origin; the detail can be found in Theorem 2.2 of [5] (with $s = p/2$, and the proof there is easily seen to work for all real numbers $s > 0$.) The case $p = 2$ and $p = 4$ follows from the first (trivial) bound of Theorem 2.3 of [5].

Thus, the only case remaining is the case when $2 < p < 6$ and $p \neq 4$. Assume $2 < p < 4$ first. Then $4 \in (p, 6)$ and using the log-convexity of L^p-norms, we have

$$\|f\|_{L^4(\mathbb{T}^2)} \le \|f\|_{L^p(\mathbb{T}^2)}^{1-\theta} \|f\|_{L^6(\mathbb{T}^2)}^{\theta},$$

where $\frac{1-\theta}{p} + \frac{\theta}{6} = \frac{1}{4}$. But since $\|f\|_{L^4(\mathbb{T}^2)} \sim \delta^{-1/4}$ and $\|f\|_{L^6(\mathbb{T}^2)} \sim \delta^{-1/4}$, we also have $\|f\|_{L^p(\mathbb{T}^2)} \gtrsim \delta^{-1/4}$. The case $4 < p < 6$ is similar. □

3.2 Proof of Theorem 3

Now we are ready to prove Theorem 3.

Proof. The upper bound is an easy consequence of Hölder's inequality and Theorems 2 and 5. So it suffices to prove the lower bound.

We use the same test function g in the proof of Lemma 10 above:

$$g = \sum_{j=1}^{\delta^{-1/2}} \Delta_{j\delta^{1/2}}.$$

We also have

$$\left\| \|Eg_j\|_{L^p(w_B)} \right\|_{l^q(j)} \sim \delta^{-\frac{1}{2q}}\delta^{-\frac{2}{p}}.$$

On the left hand side, we have again

$$Eg(x,y) = f(\delta^{1/2}x, \delta y),$$

so by periodicity and Lemma 10 we have

$$\|Eg\|_{L^p(B)} \sim \begin{cases} \delta^{-\frac{2}{p}-\frac{1}{4}} & \text{if } 2 \le p \le 6 \\ \delta^{-\frac{2}{p}-\frac{1}{2}+\frac{3}{2p}} & \text{if } 6 \le p < \infty \end{cases}.$$

Hence by comparing with the right hand side, we have

$$D_{p,q}(\delta) \gtrsim \begin{cases} \delta^{-\frac{2}{p}-\frac{1}{4}+\frac{1}{2q}+\frac{2}{p}} = \delta^{-\frac{1}{4}+\frac{1}{2q}} & \text{if } 2 \le p \le 6 \\ \delta^{-\frac{2}{p}-\frac{1}{2}+\frac{3}{2p}+\frac{1}{2q}+\frac{2}{p}} = \delta^{-\frac{1}{2}+\frac{3}{2p}+\frac{1}{2q}} & \text{if } 6 \le p \le \infty \end{cases}.$$

□

Acknowledgement. I thank Jianhui Li for useful discussions on the case of non-integral p's in Lemma 10 .

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