Kakeya and Restriction Problems in Harmonic Analysis

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A Thesis Submitted in Partial Fulfilment
of the Requirements for the Degree of
Master of Philosophy
in
Mathematics

The Chinese University of Hong Kong
July 2017

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Abstract

In this master thesis we study restriction and Kakeya conjectures. We present some positive results obtained by mathematicians throughout the last few decades and some known implications between these conjectures. We will also explain the main harmonic analysis techniques used in the proofs, starting from some basic real, complex and functional analytic tools covered in a typical first year graduate curriculum.

摘要

這一篇碩士論文主要研究掛谷猜想以及限制猜想。我們主要講述近幾十年來數學家在這些問題上取得的成果，以及這些猜想之間的緊密聯繫。從研究生基礎課程的數學背景開始，我們會闡釋證明中用到的調和分析的基本技巧，這些技巧大多來自實分析，複分析以及泛函分析。

ACKNOWLEDGMENTS

I wish to express my sincere gratitude to my supervisor, Professor Po Lam Yung, for his constant support and guidance over the past two years. He participated in my presentations almost every week, providing me with considerable insights into harmonic analysis. He is also incredibly patient, considerate and kind, which greatly influenced my style of studying mathematics. I would also like to thank my classmates, notably Shu Shing Lai, Daniel Chun Ho Lau, Franky Jianhui Li and Kevin Chung Hang Kwan. We attended weekly seminars and discussed various topics in harmonic analysis, and generated interesting ideas through discussions. I would like to thank Yufei Zhang, with whom I shared lots of experience on research and conducting teaching assistant jobs. Lastly, I thank Professor Ka Sing Lau, Professor Kai Seng Chou, Professor De-jun Feng, Professor Renjun Duan and Professor Yong Yu, who instructed me in various graduate level courses. I feel so grateful for all of them in my MPhil studies, without whom I would not have completed this thesis.

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Chapter 1

Preliminaries and Notations

1.1 Introduction to the Restriction Conjecture

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the usual topology and Lebesgue measure. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function. If f is in $L^1(\mathbb{R}^n)$, we define its Fourier transform by:

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx$$

We know that this integral converges absolutely and that $\hat{f}$ is uniformly continuous. Thus it can be restricted to any subset $S \subseteq \mathbb{R}^n$.

For $f \in L^p(\mathbb{R}^n)$, $1 < p \leqslant 2$, the classical way to define $\hat{f}$ is to use the bounded linear extension theorem (2) and the Hausdorff Young inequality. For more general $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with slow growth at infinity, say $f(x) = O(|x|^N)$ for large $|x|$, another way to define its Fourier transform is via distribution theory. Since f is locally integrable and grows slowly at infinity, we may view it as a tempered distribution, $g \mapsto \int fg$ for any $g \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions. We define $\hat{f}$ to be the Fourier transform of this tempered distribution, which is another tempered distribution. If $1 \leqslant p \leqslant 2$, then such $\hat{f}$ becomes a function. Note that if $p > 1$, then $\hat{f}$ is only defined almost everywhere

in $\mathbb{R}^n$, and it is not meaningful to directly restrict $\hat{f}$ to S.

The Fourier restriction problem is to deal with the restriction of $\hat{f}$ to a subset $S \subseteq \mathbb{R}^n$, in particular a hypersurface (a smooth $n-1$ dimensional manifold). Such S can be shown to have zero n-dimensional Lebesgue measure. It carries a positive induced surface measure which we denote by $d\sigma$.

For $f \in L^1(\mathbb{R}^n)$ it is trivially done. For $f \in L^p(\mathbb{R}^n)$, $1 < p \leqslant 2$, in order to define such restriction, we may hope to prove an inequality of the form:

$$\|\hat{f}\|_{L^q(d\sigma)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}, \text{ where } C(S,n,p,q) \text{ is some constant,} \tag{1.1}$$

valid for all $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Hence by approximation by L^1 functions, we can meaningfully restrict $\hat{f}$ to S, uniquely up to a set $N \subseteq S$ with $\sigma(N) = 0$, when $f \in L^p(\mathbb{R}^n)$. Unfortunately it is clear that (1.1) cannot hold for any exponent q when $p = 2$. Indeed, the Plancherel formula shows that the Fourier transform is an isometry on $L^2(\mathbb{R}^n)$, thus one can never make sense of the restriction of an L^2-function to a set of measure zero. Thus $\hat{f}$ cannot be well-defined on S when $f \in L^2(\mathbb{R}^n)$.

However, an interesting story began to unfold with the observation by Elias M. Stein that if p is close to 1, and if S is a compact (and hence bounded in $\mathbb{R}^n$) hypersurface with non-vanishing Gaussian curvature, then (1.1) holds, for some exponent q. By saying non-vanishing Gaussian curvature we mean the following:

Definition 1. *Let $S \subseteq \mathbb{R}^n$ be an $n-1$ dimensional smooth manifold, which means that for each $P \in S$, there exists a neighbourhood of P on S such that it is locally represented as a graph (after relabeling the coordinates) of a smooth function $\phi : U \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$:*

$$Q := (\xi_1, \ldots, \xi_{n-1}, \phi(\xi_1, \ldots, \xi_{n-1})), \text{ near } P$$

We will denote $\xi := (\xi_1, \ldots, \xi_{n-1})$. Thus we have $Q = (\xi, \phi(\xi))$ near P. In this case, we say S has non-vanishing Gaussian curvature if for each such $P \in S$, ϕ has nonzero

Hessian determinant in U.

We finally state the restriction conjecture:

Conjecture 1 (Restriction Conjecture)**.** (1.1) *holds if S is a compact hypersurface with non-vanishing Gaussian curvature, with $1 \leqslant p < \frac{2n}{n+1}$ and $1 \leqslant q \leqslant \frac{n-1}{n+1}p'$.*

The case $n = 2$ has been completely verified in the 1970s. Zygmund [16] established (1.1) when $n = 2$, $1 \leqslant p < \frac{4}{3}$ and $1 \leqslant q \leqslant \frac{1}{3}p'$ in 1974. The same result when $1 \leqslant q < \frac{1}{3}p'$ is due to Fefferman and Stein [4] in 1970.

In higher dimensions $n \geqslant 3$, Stein and Tomas (See [9] and [13]) proved the following partial result:

Theorem 1. (1.1) *holds if S is a compact hypersurface with non-vanishing Gaussian curvature, $1 \leqslant p \leqslant \frac{2(n+1)}{n+3}$ and $1 \leqslant q \leqslant 2$.*

Notice that $\frac{2(n+1)}{n+3} < \frac{2n}{n+1}$ if $n \geqslant 2$.

The origin of these endpoints on exponents will be clarified in Chapter 4.

A typical case of compact hypersurface with non-vanishing Gaussian curvature is a compact piece of the paraboloid: $x_n := |x'|^2, |x_i| \leqslant 1, 1 \leqslant i \leqslant n-1$. Another example is the unit sphere $\mathbb{S}^{n-1}$. For simplicity sometimes we will consider only specific cases. For our purposes the specific choice of S is usually irrelevant as long as S has non-vanishing Gaussian curvature.

1.2 Introduction to the Kakeya Conjecture

The Kakeya Conjecture was first posed by the Japanese mathematician Sōichi Kakeya in 1917. At first sight this seems to be a totally unrelated question from the restriction problem, but we will see some deep connection later on. Consider a needle in $\mathbb{R}^2$ with length 1, and we would like to translate and rotate the needle (with respect to any centre in the plane) so that its direction will be reversed. In this process the trajectory of the needle forms a set in the plane; this is called a Kakeya needle set. Formally we have a definition:

Definition 2. *(Kakeya needle set) Let $S \subseteq \mathbb{R}^2$ be a set. We say S is a Kakeya needle set if there exists a unit line segment $l \subseteq S$ that can be rotated continuously by 180 degrees so that any part of it never leaves the set S.*

A trivial example is the unit closed ball in $\mathbb{R}^2$. Mathematicians are concerned with Kakeya needle sets with minimum area, and some positive results were obtained. In 1928, Besicovitch showed that for any $\varepsilon > 0$, there exists a Kakeya needle set in $\mathbb{R}^2$ that has Lebesgue measure less than ε. This result was rather striking. On the other hand, it was shown that such set cannot be too small, either: any Kakeya needle set must have positive Lebesgue measure. This solved the Kakeya needle problem to some extent.

Later mathematicians thought that the condition "continuously rotated" was too strong; they removed this condition and hoped to find a better answer for this. This is the so called Kakeya set, which has a natural generalisation to n-dimensions:

Definition 3. *(Kakeya Set) Let $S \subseteq \mathbb{R}^n$ be a compact set. We say S is a Kakeya set if it contains a unit line segment in every direction.*

In 1919, even before his work on the Kakeya needle sets, Besicovitch constructed a compact set with zero Lebesgue measure, using the sprouting method. This is quite astonishing compared with the Kakeya needle problem, and in contrast, mathematicians later thought a Kakeya set must be large in some sense. Indeed, a Borel set with zero Lebesgue measure

could have positive and even full Hausdorff dimension. This motivates the following well-known conjecture:

Conjecture 2. *(Kakeya Conjecture) Let $S \subseteq \mathbb{R}^n$ be a Kakeya set. Then S has Hausdorff dimension n.*

The restriction conjecture and the Kakeya conjecture seems not related at all at the beginning; but only after we go deeply into the details their relevance would become apparent.

1.3 Remarks and Conventions

1. We will use the conventional notation $|A(x)| \lesssim_N B(x)$, $A(x) = O_N(B(x))$ to mean that there is a constant $0 < C < \infty$ dependent on N such that for each x in the domain we are concerned (say, for large $x \to \infty$ or x close to some point x_0), we have $|A(x)| \leqslant CB(x)$. We will often drop the dependence on N if it is not important or it is clear from the context.

2. A typical feature of our analysis is the "loss of epsilon" in the local estimates. More precisely, in many cases we can only prove slightly weaker results in the following form:

$$\|Ef\|_{L^q(B_R)} \lesssim_\varepsilon R^\varepsilon \|f\|_{L^p(S)}$$

   Here E is the extension operator which is the adjoint operator to the Fourier restriction operator, and S is the hypersurface specified as above. The equation means that for arbitrarily small $\varepsilon > 0$, there exists a constant $C(\varepsilon, p, q) > 0$ such that $\|Ef\|_{L^q(B_R)} \leqslant C(\varepsilon, p, q) R^\varepsilon \|f\|_{L^p(S)}$ for any $R > 0$ large, and any $f \in L^p(S)$, where B_R is a ball in $\mathbb{R}^n$ whose centre often does not matter. If ε can be taken to be 0, then we obtain our original stronger estimate; otherwise we will have $\lim_{\varepsilon \to 0^+} C(\varepsilon, p, q) \to \infty$. Such ε may change from line to line, but they must be

arbitrarily close to 0 simultaneously. Note that sometimes we get slightly stronger estimates with R^ε for any $\varepsilon > 0$ replaced by $\log(R)$.

3. Below is a simple fact which is used repeatedly. Suppose some quantity $Q(x), x \in \mathbb{R}^n$ has rapid decay, in the sense that

$$|Q(x)| \leqslant C_N |x|^{-N}, \text{ for any } N = 0, 1, 2, \ldots, \tag{1.2}$$

Then it also satisfies the following decay estimate:

$$|Q(x)| \leqslant C'_N (1 + |x|)^{-N}, \text{ for any } N = 0, 1, 2, \ldots,$$

The elementary proof is omitted.

Chapter 2

Rudiments of Harmonic analysis

In this chapter we state without proof some of the basic theorems in real, complex, functional and harmonic analysis from which many estimates and results in this thesis are obtained. This theorems are standard, and many can be found in [10].

2.1 Results in Functional Analysis

Theorem 2. *(Bounded Linear Extension) Let E, F be Banach spaces and $D \subseteq E$ be a dense linear subspace. Let $T : D \to F$ be linear. Suppose T is bounded on D, that is, there exists $C > 0$ such that for any $x \in D$,*

$$\|T(x)\|_F \leqslant C\|x\|_E$$

Then there exists a unique extension $\tilde{T} : E \to F$ such that $\tilde{T}$ is linear, $\tilde{T}|_D = T$, and that

$$\|\tilde{T}(x)\|_F \leqslant C\|x\|_E$$

Theorem 3. *(TT^* Theorem for L^p-Spaces) Let $(X, \mu), (Y, \nu)$ be sigma-finite measure spaces and let T^* be a linear operator mapping a dense class of test functions $f : X \to \mathbb{C}$ to measurable functions $T^*f : Y \to \mathbb{C}$. Let $1 \leqslant p \leqslant \infty$, $0 < A < \infty$. Then the followings*

are equivalent:

1. $\|T^*f\|_{L^2(X)} \leqslant A\|f\|_{L^p(Y)}$, *for any* $f \in L^p(Y)$.

2. $\|Tg\|_{L^{p'}(Y)} \leqslant A\|g\|_{L^2(X)}$, *for any* $g \in L^2(X)$.

3. $\|TT^*f\|_{L^{p'}(Y)} \leqslant A^2\|f\|_{L^p(Y)}$, *for any* $f \in L^p(Y)$.

2.2 Interpolation Theorems

Theorem 4. *(Riesz-Thorin Interpolation Theorem) Let* $(X, \mu), (Y, \nu)$ *be sigma-finite measure spaces and let* T *be a linear operator mapping the family of simple functions with finite measure support* $f : X \to \mathbb{C}$ *to measurable functions* $Tf : Y \to \mathbb{C}$, *such that the integral*

$$\int_Y (Tf)g \, d\nu$$

is absolutely convergent for any simple functions f, g *with finite measure support.*

Suppose $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ *and for* $i = 0, 1$, *we have:*

$$\|Tf\|_{q_i} \leqslant A_i\|f\|_{p_i},$$

for some $A_0, A_1 > 0$, *for any simple function* f *with finite measure support.*

Then $\|Tf\|_{q_\theta} \leqslant A_\theta\|f\|_{p_\theta}$, *for any simple function* f *with finite measure support, where:*

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad A_\theta := A_0^{1-\theta}A_1^\theta, \quad 0 \leqslant \theta \leqslant 1.$$

We remark that if $p_\theta < \infty$, then the bounded linear extension theorem shows that we can extend T to be defined on all of L^{p_θ} with the same bound.

Next is a remarkable discovery by Elias M. Stein, a generalisation to the above interpolation theorem:

Theorem 5. *(Stein Interpolation Theorem) Let $(X, \mu), (Y, \nu)$ be sigma-finite measure spaces and Let $\{T_z\}_z$ be a family of linear operators mapping the family of simple functions with finite measure support $f : X \to \mathbb{C}$ to measurable functions $T_z f : Y \to \mathbb{C}$, such that whenever f, g are simple functions with finite measure support,*

$$z \mapsto \int_Y (T_z f) g \, d\nu$$

is absolutely convergent, continuous on the strip $z \in \{0 \leqslant \mathrm{Re}(z) \leqslant 1\}$ and analytic in its interior, with order of growth $\leqslant 1$.

Suppose $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and for $\mathrm{Re}(z) = i$, $i = 0, 1$,we have:

$$\|T_z f\|_{q_i} \leqslant A_i \|f\|_{p_i},$$

for some $A_0, A_1 > 0$, for any simple function f with finite measure support.

Then $\|T_\theta f\|_{q_\theta} \leqslant A_\theta \|f\|_{p_\theta}$, for any simple function f with finite measure support, where:

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad A_\theta := A_0^{1-\theta} A_1^\theta, \quad 0 \leqslant \theta \leqslant 1.$$

The Schur's test is also very useful.

Theorem 6 (Schur's test)**.** *Let $Tf(y) := \int_X K(x, y) f(x) dx$ be an integral operator with kernel $K : X \times Y \to \mathbb{C}$. Suppose we have the following two estimates:*

$$\sup_{x \in X} \int_Y |K(x, y)| dy \leqslant A$$

and

$$\sup_{y \in Y} \int_X |K(x, y)| dx \leqslant B$$

Then T is bounded from $L^p(X)$ to $L^p(Y)$, with norm bounded by $A^{\frac{1}{p}} B^{\frac{1}{p'}}$, $1 \leqslant p \leqslant \infty$. In

particular, T is bounded from $L^2(X)$ to $L^2(Y)$ with norm bounded by $\sqrt{AB}$.

Lastly, we state some standard results in Lorentz spaces. We only list what we will use. The interested reader should investigate the whole theory of Lorentz spaces.

Definition 4. *Let (X, μ) be a sigma-finite measure space. For $1 \leqslant p, q \leqslant \infty$, the (p, q)-Lorentz quasi-norms are defined to be*

$$\|f\|_{L^{p,q}(X,d\mu)} := \begin{cases} p^{\frac{1}{q}} \left\| t\mu\{x \in X : |f(x)| > t\}^{\frac{1}{p}} \right\|_{L^q(\mathbb{R}^+, \frac{dt}{t})}, & \text{if } p, q < \infty \\ \left\| t\mu\{x \in X : |f(x)| > t\}^{\frac{1}{p}} \right\|_{L^\infty(\mathbb{R}^+, \frac{dt}{t})}, & \text{if } p < \infty, q = \infty \\ \|f\|_{L^\infty(X,d\mu)}, & \text{if } p = q = \infty. \end{cases}$$

In particular, if $(X, \mu) = (\mathbb{Z}^n, c)$ where c is the counting measure, we denote

$$\|b\|_{l^{p,q}} = \|b\|_{L^{p,q}(\mathbb{Z}^n, dc)}.$$

We have the following fact.

Proposition 1 (Dyadic Decomposition)**.** *If $1 \leqslant p < \infty, q = 1$, then*

$$\|f\|_{L^{p,1}} \sim_p \sum_{l \in \mathbb{Z}} 2^l \mu \left\{x \in X : |f(x)| > 2^l\right\}^{\frac{1}{p}}.$$

The following theorem for dual space will be used.

Proposition 2. *Let $1 < p < \infty, 1 \leqslant q \leqslant \infty$. Then the dual space of $L^{p,q}$ is $L^{p',q'}$, in the sense that a linear operator $T : E \to L^{p',q'}$ is bounded if and only if $T^* : L^{p,q} \to E^*$ is bounded, where E is any normed space.*

We can state the following special case of the real interpolation theorem.

Theorem 7. *(Marcinkiewicz Interpolation Theorem) Let $(X, \mu), (Y, \nu)$ be sigma-finite measure spaces and let T be a sublinear operator mapping the family of simple functions with finite measure support $f : X \to \mathbb{C}$ to measurable functions $Tf : Y \to \mathbb{C}$.*

Suppose $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$, $p_0 < p_1, q_0 > q_1$ *and for* $i = 0, 1$, *we have:*

$$\|Tf\|_{q_i,\infty} \leqslant A_i \|f\|_{p_i},$$

for some $A_0, A_1 > 0$, *for any simple function* f *with finite measure support.*

Note that $\|Tf\|_{q_i,\infty} \lesssim \|Tf\|_{q_i}$, *hence the about weak type bound is indeed weaker than strong type* ($L^p \to L^q$) *bounds.*

Then $\|Tf\|_{q_\theta} \leqslant C A_\theta \|f\|_{p_\theta}$, *for any simple function* f *with finite measure support, where* C *depends on* p_0, q_0, p_1, q_1 *only, and*

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad A_\theta := A_0^{1-\theta} A_1^{\theta}, \quad 0 < \theta < 1.$$

under an additional assumption that $p_\theta \leqslant q_\theta$.

2.3 Theory of Stationary Phase

This is a list of results in the theory of stationary phase, which may be found in Chapter VIII of [9] or Chapter 6 of [14].

Theorem 8. *(Stationary Phase) Consider the following oscillatory integral:*

$$I(\lambda) := \int_{\mathbb{R}^n} e^{i\lambda\Phi(x)} \psi(x) dx,$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$ *is smooth,* $\psi \in C_c^\infty(\mathbb{R}^n)$, $\lambda > 0$.

Then:

1. *Assume that* $|\nabla\Phi| \geqslant c > 0$ *on the support of* ψ. *Then we have: for each* $N > 0$,

   $$|I(\lambda)| \lesssim_N (c\lambda)^{-N}, \text{ for large } \lambda.$$

2. *Assume that* $\nabla\Phi$ *vanishes at some point on the support of* ψ *but*

$$\det_{1\leqslant i,j\leqslant n}\left[\frac{\partial^2\Phi}{\partial x_i\,\partial x_j}\right](x)\geqslant c^n,\qquad near\qquad \xi=0$$

Then

$$|I(\lambda)|\lesssim(c\lambda)^{-\frac{n}{2}},\ \textit{for large}\ \lambda.$$

3. *In the case* $n=1$*, we have a more general result. Assume that* $\Phi',\Phi'',\ldots,\Phi^{(k)}$ *all vanishes at some point* x_0 *but* $|\Phi^{(k+1)}(x_0)|\geqslant c$*, where* $k\geqslant 1$*. Also assume that the function* ψ *vanishes to some order* $l\geqslant 0$ *at* x_0*. Then*

$$|I(\lambda)|\lesssim(c\lambda)^{-\frac{1+l}{k+1}}$$

Using the stationary phase we can obtain the following well known decay estimate of the Fourier transform of a surface measure.

Theorem 9. *Let* $S\subseteq\mathbb{R}^n$ *be a smooth compact manifold of dimension* $n-1$ *with nonzero Gaussian curvature. Then there is* $c\neq 0$ *such that for large* $|x|$*,*

$$\widehat{d\sigma}(x):=\int_S e^{2\pi ix\cdot\eta}d\sigma(\eta)=c|x|^{\frac{1-n}{2}}+O(|x|^{-\frac{n}{2}}).$$

Chapter 3

The Tomas-Stein Theorem

As discussed in the introduction, the Tomas-Stein Theorem is a partial result for the restriction conjecture. We present here the version of Hörmander, which is applicable to a more general family of oscillatory integrals and has the advantage that it proves the endpoint case $p = \frac{2(n+1)}{n+3}$ without any ε-loss. It also has a disadvantage, however. Bourgain [1] showed that if we only assume the below Hörmander's condition, we cannot go beyond the exponent $p = \frac{2(n+1)}{n+3}$. The first section is devoted to the formulation of the theorem.

3.1 Introduction to the Key Estimate

Consider the following family of oscillatory integral operators:

$$T_\lambda^* f(\xi) := \int_{\mathbb{R}^n} e^{-i\lambda\Phi(x,\xi)}\overline{\psi(x,\xi)}f(x)dx, \tag{3.1}$$

where $x \in \mathbb{R}^n$, $\lambda > 0, \Phi : \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{R}$ is smooth, and $\psi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$.

Assume without loss of generality that ψ is supported in a neighbourhood of $(0, 0)$. Con-

sider the following mixed Hessian non-square matrix

$$M := \left[\frac{\partial^2 \Phi}{\partial x_i \, \partial \xi_j}\right]_{n\times(n-1)}.$$

We require that $\operatorname{rank}(M) = n-1$. Then there exists a unique nonzero (up to a constant multiple) vector $u \in \mathbb{R}^n$ satisfying $uM = 0$. Define $\zeta(\xi) := u \cdot \nabla_x \Phi(x,\xi)\big|_{x=0}$, and consider the following Hessian. We also require that:

$$\det_{1\leqslant i,j\leqslant n-1}\left[\frac{\partial^2 \zeta}{\partial \xi_i \, \partial \xi_j}\right](\xi) \neq 0, \qquad \text{near} \qquad \xi = 0$$

Note that the above was actually a condition on a third-order derivative. Also, by continuity we only need to assume the above Hessian is nonzero at $\xi = 0$.

The previous two assumptions on the phase function Φ are referred to as Hörmander's conditions.

With the above, we can state Hörmander's restriction estimate:

Theorem 10 (Hörmander's Restriction Estimate)**.** *Consider the operator T^*_λ defined as in (3.1). Suppose Φ satisfies Hörmander's conditions near $(0,0)$. Then we have the following estimate:*

$$\|T^*_\lambda f\|_{L^q(\mathbb{R}^n)} \lesssim \lambda^{-\frac{n}{p'}} \|f\|_{L^p(\mathbb{R}^n)}, \tag{3.2}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and all large $\lambda > 0$, where $1 \leqslant p \leqslant \frac{2(n+1)}{n+3}$, $1 \leqslant q \leqslant \frac{n-1}{n+1}p'$ and the implicit constant does not depend on f, λ.

This leads to the following restriction estimate:

$$\|\hat{f}\|_{L^q(d\sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \tag{3.3}$$

for any $f \in \mathcal{S}(\mathbb{R}^n)$, where $1 \leqslant p \leqslant \frac{2(n+1)}{n+3}$ and $1 \leqslant q \leqslant \frac{n-1}{n+1}p'$.

Indeed, in the following we are going to show that (3.2) indeed implies (3.3).

Write $\xi := (\xi_1, \ldots, \xi_{n-1})$ as before. Since the surface is compact, there are finitely many points on the surface each having a neighbourhood, whose union covers the surface.

By a partition of unity we only need to consider a single neighbourhood, with a mapping $\phi : \xi \in U \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$ and we assume $0 \in U$. By translation and rotation we may assume that $\phi(0) = 0, D\phi(0) = 0$, that is, the neighbourhood is a graph of a smooth function ϕ whose normal at the origin is e_n. Since S has non-vanishing Gaussian curvature, we have:

$$\det_{1 \leqslant i,j \leqslant n-1} \left[\frac{\partial^2 \phi}{\partial \xi_i \, \partial \xi_j} \right] (\xi) \neq 0, \qquad \text{near} \qquad \xi = 0$$

Hence the restriction estimate is equivalent to the following:

$$\left\| \int_{\mathbb{R}^n} e^{-2\pi i (x' \cdot \xi + x_n \cdot \phi(\xi))} b(\xi) f(x) dx \right\|_{L^q(\mathbb{R}^{n-1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \tag{3.4}$$

where $b \in C_c^\infty(\mathbb{R}^{n-1})$ is a bump function that equals to $\sqrt{1 + |D\phi(\xi)|^2}$ on U.

The reason why we put the adjoint and the complex conjugation sign is that we would like to define the extension operator T_λ by:

$$T_\lambda g(x) := \int_{\mathbb{R}^{n-1}} e^{i\lambda \Phi(x,\xi)} \psi(x, \xi) g(\xi) d\xi, \tag{3.5}$$

thus it agrees with the standard notation TT^*-method.

Write $\Phi(x, \xi) := 2\pi(x' \cdot \xi + x_n \cdot \phi(\xi))$, and set $\psi(x, \xi) := a(x)b(\xi)$, where b is the bump function specified as above, and a is any bump function that equals to 1 at 0. We may assume they are real-valued.

Apply the key estimate (3.2) to the scaled function $f_\lambda(x) := f(\lambda x)$, and obtain:

$$\|T_\lambda^* f_\lambda\|_{L^q(\mathbb{R}^{n-1})} \lesssim \lambda^{-\frac{n}{p'}} \|f_\lambda\|_{L^p(\mathbb{R}^n)},$$

By scaling, we have

$$\|T_\lambda^* f_\lambda\|_{L^q(\mathbb{R}^{n-1})} = \lambda^{-n} \Big\| \int_{\mathbb{R}^n} e^{-i\Phi(x,\xi)} a(\lambda^{-1}x) b(\xi) f(x) dx \Big\|_{L^q(\mathbb{R}^{n-1})}$$

On the other hand, we have:

$$\lambda^{-\frac{n}{p'}} \|f_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{-\frac{n}{p'}} \lambda^{-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)} = \lambda^{-n} \|f\|_{L^p(\mathbb{R}^n)}$$

Combining these equations we get

$$\Big\| \int_{\mathbb{R}^n} e^{-i\Phi(x,\xi)} a(\lambda^{-1}x) b(\xi) f(x) dx \Big\|_{L^q(\mathbb{R}^{n-1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Lastly, we let $\lambda \to \infty$, and by dominated convergence theorem, we have:

$$\Big\| \int_{\mathbb{R}^n} e^{-i\Phi(x,\xi)} b(\xi) f(x) dx \Big\|_{L^q(\mathbb{R}^{n-1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

since we chose $a(0) = 1$ at the beginning. Thus we have established (3.4).

3.2 The TT^* method

We see above that the main ingredient was the key estimate (3.2), which we will state and prove here.

Before we come to the proof of this estimate, we first note that if we take $\Phi(x, \xi) := 2\pi(x' \cdot \xi + x_n \cdot \phi(\xi))$, a simple calculation shows that

$$M := 2\pi \left[\begin{array}{cccc} & I_{n-1} & & \\ \hline \frac{\partial \phi}{\partial \xi_1} & \frac{\partial \phi}{\partial \xi_2} & \cdots & \frac{\partial \phi}{\partial \xi_{n-1}} \end{array} \right]$$

Thus we may take $u := (\frac{\partial \phi}{\partial \xi_1}, \frac{\partial \phi}{\partial \xi_2}, \ldots, \frac{\partial \phi}{\partial \xi_{n-1}}, -1)$. In this case, $\zeta(\xi) = 2\pi(\xi_1 \frac{\partial \phi}{\partial \xi_1} + \cdots + \xi_{n-1} \frac{\partial \phi}{\partial \xi_{n-1}} - \phi(\xi))$. A straightforward calculation then shows that the Hessian of ζ is

exactly 2π times the Hessian of ϕ at $\xi = 0$. Since the latter is nonzero, we see that Φ indeed satisfies the Hörmander's condition.

To prove the key estimate, we are going to use the TT^* method, which we formulated in Theorem 3 in Chapter 2.

First we notice that by a simple Lemma 1 (See first part of Chapter 4) it suffices to prove the case $p = \frac{2(n+1)}{n+3}$ and $q = 2$. So it suffices to show

$$\|T_\lambda T_\lambda^* f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)} \lesssim \lambda^{-\frac{n(n-1)}{(n+1)}} \|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n)}.$$

Written this into an integral operator, we have

$$T_\lambda T_\lambda^* f(x) = \int_{\mathbb{R}^n} K_\lambda(x,y) f(y)\, dy,$$

where the kernel is given by:

$$K_\lambda(x,y) := \int_{\mathbb{R}^{n-1}} e^{i\lambda[\Phi(x,\xi)-\Phi(y,\xi)]} \psi(x,\xi)\overline{\psi(y,\xi)}\, d\xi$$

Write $U := T_\lambda T_\lambda^*$.

We remark that the change of order of integration was valid since we can assume a priori that f is a smooth function with compact support and hence Fubini's theorem may be applied; we omit further these kind of arguments.

3.3 Analytic Family of Operators

In this subsection we will appreciate the powerful Stein's interpolation theorem (Theorem 5), which deals with the interpolation of an analytic family of linear operators. More

precisely, we will construct a complex analytic family of linear operators $\{U^s\}$ for $\frac{1-n}{2} \leqslant \mathrm{Re}(s) \leqslant 1$, so that

$$\begin{cases} \|U^s f\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-n} \|f\|_{L^2(\mathbb{R}^n)}, & \text{if} \quad \mathrm{Re}(s) = 1 \\ \|U^s f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}, & \text{if} \quad \mathrm{Re}(s) = \frac{1-n}{2} \\ U^0 = U \end{cases}$$

The choice of endpoints may seem rather surprising at first, but it will be clear in the end that both endpoints are chosen so that the operator norms are easy to bound, indeed $\frac{1-n}{2}$ is related to the decay of the Fourier Transform of the surface measure $d\sigma$.

Suppose such family could be constructed, and assume it has limited growth in the parameter z as stated in Theorem 5. Then by simple translation and scaling mapping the endpoints $\frac{1-n}{2}$ and 1 to be 0 and 1, respectively, we have:

$$U = U^0 : L^{p_\theta} \to L^{p'_\theta},$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{1} + \frac{\theta}{2}$ and $\theta = \left(\frac{n-1}{2}\right) \Big/ \left(\frac{n+1}{2}\right) = \frac{n-1}{n+1} \in (0,1)$. Solving this gives

$$\|U^0 f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)} \lesssim \lambda^{-\frac{n(n-1)}{n+1}} \|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n)}$$

By Theorem 3, the above is equivalent to the restriction estimate when $q = 2$.

Next we start our construction of the analytic family. The advantage is then we can use the previously known theory of oscillatory integrals, in particular, Theorem 11 to be stated in the next section.

- We extend the phase function $\tilde{\Phi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by setting

$$\tilde{\Phi}(x, \tilde{\xi}) := \Phi(x, \xi) + \xi_n \Phi_0(x),$$

where $\tilde{\xi} := (\xi, \xi_n)$ and Φ_0 is chosen so that the $n \times n$ Hessian of $\tilde{\Phi}$ is nonvanishing; however, by the Hörmander's condition stated in Section 1, it suffices to choose any Φ_0 so that

$$(u \cdot \nabla_x)\Phi_0(x) \neq 0, \qquad \text{near} \qquad x = 0.$$

Note that $\Phi_0(x) := \exp(u_1 x_1 + \cdots + u_n x_n)$ would suffice.

- We would like to embed the one dimensional δ-function to an analytic family of distributions with compact support. To do this we fix any $z \in C_c^\infty$ with $z(y) = 1$ for $|y| \leqslant 1$, $z(y) = 0$ for $|y| \geqslant 2$ and consider the family $\{\alpha_s\}$ of distributions on $\mathbb{R}$ that arises by analytic continuation to all $s \in \mathbb{C}$ of the family $\{\alpha_s\}$ of functions, initially given when $\mathrm{Re}(s) > 0$ by:

$$\alpha_s(y) := \begin{cases} \frac{e^{s^2}}{\Gamma(s)} y^{s-1} z(y), & \text{if} \quad y > 0 \\ 0, & \text{if} \quad y \leqslant 0 \end{cases}$$

The bump function is introduced to make the distribution compactly supported. The gamma function is introduced from integration by parts. Indeed, let f be any smooth function. We will do a typical step of this analytic continuation:

$$\begin{aligned} \int_{\mathbb{R}} y^{s-1} z(y) f(y)\, dy &= \int_0^\infty \frac{1}{s} z(y) f(y)\, d(y^s) \\ &= -\frac{1}{s} \int_0^\infty y^s (zf)'(y)\, dy, \qquad \text{if} \quad \mathrm{Re}(s) > 0 \end{aligned}$$

The good news is that the integral on the last line is now well defined for $\mathrm{Re}(s) > -1$. In particular, multiplying the factor $\frac{e^{s^2}}{\Gamma(s)}$ in the above expression, letting $s = 0$ and noting that $\lim_{s \to 0} s\Gamma(s) = 0$, the integral in the last displayed equation is equal to $z(0)f(0) = f(0)$, which is the δ-function acting on f.

More generally, for each $N > 0$, we can define $\{\alpha_s\}$ for $-N < \mathrm{Re}(s) \leqslant -N + 1$ by

iterated integration by parts:

$$\begin{aligned}\alpha_s(f) &:= (-1)^N \frac{e^{s^2}}{\Gamma(s)s(s+1)\cdots(s+N-1)} \int_0^\infty y^{s+N-1}(zf)^{(N)}(y)\,dy \\ &= \frac{e^{s^2}}{\Gamma(s+N)} \int_0^\infty y^{s+N-1}(zf)^{(N)}(y)\,dy.\end{aligned}$$

With $\{\alpha_s\}$ legitimately defined, we can finally state our definition of the analytic family of operators:

$$U^s f(x) := \int_{\mathbb{R}^n} K^s(x,y) f(y)\,dy,$$

where

$$K^s(x,y) := \alpha_s\Big(\int_{\mathbb{R}^{n-1}} e^{i\lambda[\tilde{\Phi}(x,\tilde{\xi})-\tilde{\Phi}(y,\tilde{\xi})]} \psi(x,\xi)\overline{\psi(y,\xi)} d\xi \Big) \tag{3.6}$$

The integral on the right hand side is a function of ξ_n, and the expression on the right hand side is α_s acting on this function of ξ_n. If $\mathrm{Re}(s) > 0$, then

$$\begin{aligned}K^s(x,y) &= \int_{\mathbb{R}^n} e^{i\lambda[\tilde{\Phi}(x,\tilde{\xi})-\tilde{\Phi}(y,\tilde{\xi})]} \psi(x,\xi)\overline{\psi(y,\xi)} \alpha_s(\xi_n) d\tilde{\xi} \\ &= \int_{\mathbb{R}} K_\lambda(x,y) e^{i\lambda y[\Phi_0(x)-\Phi_0(y)]} \alpha_s(\xi_n) d\xi_n\end{aligned}$$

3.4 The Interpolation Argument

A first observation is that for any smooth $u : \mathbb{R} \to \mathbb{C}$, $\{\alpha_s(u)\}$ is bounded on any strip $\{a \leqslant \mathrm{Re}(s) \leqslant b\}$, mainly thanks to the factor e^{s^2} which has a rapid decay as $|\mathrm{Im}(s)| \to \infty$. This shows that the analytic family satisfies the order of growth condition in Stein's interpolation theorem.

The nontrivial part are the two endpoint bounds.

3.4.1 Bound at $\mathrm{Re}(s) = 1$

In this case the initial definition of α_s is applicable. Let $\tilde{z}$ be a bump function which equals 1 on the support of z, so that $\overline{\psi(y,\xi)}\tilde{z}(\xi_n) \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

Write $U^s = S_2 \circ S_1$, where

$$\begin{aligned} S_1 f(\tilde{\xi}) &:= \int_{\mathbb{R}^n} e^{-i\lambda\tilde{\Phi}(y,\tilde{\xi})}\overline{\psi(y,\xi)}\tilde{z}(\xi_n) f(y) dy \\ S_2 g(x) &:= \int_{\mathbb{R}^n} e^{i\lambda\tilde{\Phi}(x,\tilde{\xi})}\psi(x,\xi) z(\xi_n) \xi_n^{s-1} \frac{e^{s^2}}{\Gamma(s)} g(\tilde{\xi}) d\tilde{\xi} \\ &= \frac{e^{s^2}}{\Gamma(s)} \int_{\mathbb{R}^n} e^{i\lambda\tilde{\Phi}(x,\tilde{\xi})} [\psi(x,\xi) z(\xi_n)][\xi_n^{s-1} g(\tilde{\xi})] d\tilde{\xi} \end{aligned}$$

Here we must invoke another theorem of Hörmander, which is similar but easier than the Hörmander's restriction estimate:

Theorem 11. *(Hörmander's Oscillatory integral Estimate) Let $\tilde{\Phi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be smooth near a neighbourhood of (0,0), and suppose its mixed Hessian is nonvanishing:*

$$\det_{1 \leqslant i,j \leqslant n} \left[\frac{\partial^2 \tilde{\Phi}}{\partial x_i \, \partial \tilde{\xi}_j} \right] (0,0) \neq 0.$$

Consider a family of operators defined by:

$$S_\lambda f(\tilde{\xi}) := \int_{\mathbb{R}^n} e^{i\lambda\tilde{\Phi}(x,\tilde{\xi})} \psi'(x,\tilde{\xi}) f(x) dx, \ \textit{where } \psi' \in C_c^\infty(x,\tilde{\xi}).$$

Then for large $\lambda > 0$, we have:

$$\|S_\lambda f\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)}, \tag{3.7}$$

We are going to apply this theorem first. For S_1, apply $\psi' = \overline{\psi} \cdot \tilde{z}$ and by the boundedness

of $\tilde{z}$, we have

$$\|S_1 f\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)}.$$

For S_2, apply $\psi' = \psi \cdot z$, $\tilde{g}(\tilde{\xi}) := g(\tilde{\xi})\xi_n^{s-1}$ and by the boundedness of $\frac{e^{s^2}}{\Gamma(s)}$ in the strip, we have

$$\|S_2 g\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-\frac{n}{2}} \|\tilde{g}\|_{L^2(\mathbb{R}^n)} \leqslant \lambda^{-\frac{n}{2}} \|g\|_{L^2(\mathbb{R}^n)}.$$

The last inequality is why we consider the endpoint $\mathrm{Re}(s) = 1$.

Combining the above two estimates, we obtain the bound at $\mathrm{Re}(s) = 1$.

3.4.2 Bound at $\mathrm{Re}(s) = (1-n)/2$

By Minkowski's inequality, it suffices to show that

$$|K^s(x,y)| \lesssim 1.$$

Referring to 3.6, we rewrite the above as:

$$K^s(x,y) = K_\lambda(x,y)\hat{\alpha}_s(\lambda[\Phi_0(x) - \Phi_0(y)]),$$

where $\hat{\alpha}_s$ is the Fourier transform of α_s.

- Estimate of K_λ :

  Recall that

$$K_\lambda(x,y) := \int_{\mathbb{R}^{n-1}} e^{i\lambda[\Phi(x,\xi)-\Phi(y,\xi)]}\psi(x,\xi)\overline{\psi(y,\xi)}\, d\xi.$$

Write $\Psi(x,y,\xi) := \Phi(x,\xi) - \Phi(y,\xi)$. By Taylor expansion, for each multi-index β,

$$\left(\frac{\partial}{\partial \xi}\right)^{\beta} \left[\Psi(x,y,\xi) - \nabla_x \Phi(x,\xi)(x-y)\right] = O_\beta(|x-y|^2), \text{ as } y \to x.$$

By a standard partition of unity argument we may assume that ψ has small support so that the Taylor expansion is valid. Then we have two cases:

Case 1. The unit vector in the direction $x-y$ or $y-x$ is close to the critical direction u.

In this case, we will need the non-vanishing third order derivative of the Hörmander's condition. With β running through all second derivatives in ξ, we have:

$$\left| \det_{1 \leqslant i,j \leqslant n-1} \left[\frac{\partial^2 \Psi}{\partial \xi_i \, \partial \xi_j}\right] \right| (0) \gtrsim |x-y|^{n-1} \qquad \text{near} \qquad \xi = 0.$$

By the stationary phase estimate in Theorem 8, we have

$$|K_\lambda(x,y)| \lesssim \lambda^{\frac{1-n}{2}} |x-y|^{\frac{1-n}{2}}.$$

Here is where the endpoint $\frac{1-n}{2}$ emerged.

Case 2. The unit vector in the direction $x-y$ or $y-x$ is away from the critical direction u.

In this case we use again the stationary phase estimate and get

$$|K_\lambda(x,y)| \lesssim_N \lambda^{-N} |x-y|^{-N}, \text{ for any real } N > 0.$$

Taking $N = \frac{n-1}{2}$, we obtain the same estimate as in the other case.

Finally, we will need the Fourier decay estimate of α_s:

$$|\hat{\alpha}_s(u)| \leqslant A_\sigma(1+|u|^{-\sigma}), s := \sigma + it, \sigma \leqslant 1.$$

The exponents could be canceled exactly if we choose $\mathrm{Re}(s) = \sigma := \frac{1-n}{2} \leqslant 1$:

$$|\hat{\alpha}_s(\lambda[\Phi_0(x) - \Phi_0(y)])| \leqslant C\Big\{1 + |\lambda[\Phi_0(x) - \Phi_0(y)]|\Big\}^{\frac{n-1}{2}}.$$

Since our choice of Φ_0 is locally Lipschitz, combining with the previous estimate, we finally obtain:

$$|K^s(x,y)| \lesssim 1.$$

3.5 Remaining Estimates

In this subsection we are going to complete our verification of the previous assertions.

3.5.1 Proof of Hörmander's Oscillatory Integral Estimate

We now give the proof of Theorem 11. The key idea is to use Taylor's expansion and integration by parts. The whole proof is adapted from Page 378, 379 of [9].

For simplicity we drop all tildes and primes in the notations, but we keep in mind that they are different from those in the settings of our final theorem. Using again TT^*-method, it suffices to show that

$$\|S_\lambda S_\lambda^* f\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-n}\|f\|_{L^2(\mathbb{R}^n)},$$

The kernel of $S_\lambda S_\lambda^*$ is given by

$$K_\lambda(x,y) := \int_{\mathbb{R}^n} e^{i\lambda[\Phi(x,\xi)-\Phi(y,\xi)]}\psi(x,\xi)\overline{\psi(y,\xi)}\, d\xi.$$

Let $M(x,\xi)$ be the mixed Hessian matrix and for any $a \in \mathbb{R}^n$, we use ∇_ξ^a denote differentiation in the direction a. Fix (x,y) first, and write

$$\Delta(x,y,\xi) := \nabla_\xi^{a(\xi)}[\Phi(x,\xi) - \Phi(y,\xi)].$$

By Taylor expansion, we have $\Delta = (x-y)^T M a(\xi) + O(|x-y|^2)$. Since M is invertible by assumption, we may choose

$$a := M^{-1}\left(\frac{x-y}{|x-y|}\right),$$

which gives $(x-y)^T M a(\xi) = |x-y|$. Again by partition of unity we may take suppψ to be sufficiently small so that near the support of K_λ, we have

$$|\Delta(x,y,\xi)| \geqslant c|x-y|$$

Note that Δ, a, M are all smooth.

We set $D_\xi := [i\lambda\Delta]^{-1}\nabla_\xi^{a(\xi)}$ to be the modified differential operator so that

$$D_\xi^N(e^{i\lambda[\Phi(x,\xi)-\Phi(y,\xi)]}) = e^{i\lambda[\Phi(x,\xi)-\Phi(y,\xi)]}$$

Using definition of inner products repeatedly gives

$$K_\lambda(x,y) := \int_{\mathbb{R}^n} e^{i\lambda[\Phi(x,\xi)-\Phi(y,\xi)]}(D_\xi^N)^t\left(\psi(x,\xi)\overline{\psi(y,\xi)}\right)\, d\xi,$$

where $(D_\xi^N)^t$ denotes the Hilbert space adjoint of D_ξ^N.

But notice that $(D_\xi^N)^t$ is essentially a differential operator which satisfies

$$\left|(D_\xi^N)^t\left(\psi(x,\xi)\overline{\psi(y,\xi)}\right)\,d\xi\right| \lesssim (\lambda|x-y|)^{-N}, N=0,1,2,\ldots.$$

But recall the fact (1.2) and using the condition that ψ has compact support in both variables, we have

$$|K_\lambda(x,y)| \lesssim_N (1+\lambda|x-y|)^{-N}, N=0,1,2,\ldots.$$

If we take, say, $N=n+1$, then

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K_\lambda(x,y)|dy \lesssim_N \lambda^{-n},$$

The same is true with x and y interchanged.

Lastly, recall that

$$S_\lambda S_\lambda^* f = \int_{\mathbb{R}^n} K_\lambda(x,y)f(y)dy,$$

we have, by Schur's test (6), that $S_\lambda S_\lambda^*$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with operator norm bounded by $C\sqrt{\lambda^{-2n}} = C\lambda^{-n}$.

This finishes the proof of (3.7).

3.5.2 Proof of the Fourier Decay Estimate

We are going to prove:

$$|I(u)| := |\hat{\alpha}_s(u)| \leqslant A_\sigma(1+|u|^{-\sigma}), s := \sigma+it, \sigma \leqslant 1. \tag{3.8}$$

This is similar to the standard Paley-Wiener theorem. However, it cannot be applied directly as it offers no information on the dependence on the parameter s.

We first prove the case $0 < \sigma \leqslant 1$. In this case α_s is the original function, which is in $L^1(\mathbb{R})$. Hence the bound (3.8) is trivial for small $|u|$.

For large $|u|$, consider $I = I_1 + I_2$, where

$$I_1(u) := \int_0^{|u|^{-1}} \alpha_s(y) e^{iyu} dy$$

$$I_2(u) := \int_{|u|^{-1}}^{\infty} \alpha_s(y) e^{iyu} dy$$

For I_1, since $\frac{e^{s^2}}{\Gamma(s)}$ is bounded on $0 < \sigma \leqslant 1$, we have, after changing variables:

$$|I_1(u)| \lesssim_\sigma \int_0^1 |u|^{-\sigma} |y|^{\sigma-1} dy \sim |u|^{-\sigma}$$

For I_2, integration by parts shows that

$$\begin{aligned}
|I_2(u)| &\lesssim_\sigma |u|^{-\sigma} \left| \int_1^\infty y^{s-1} z(|u|^{-1}y) e^{iy} dy \right| \\
&= |u|^{-\sigma} \left| \int_1^\infty y^{s-1} z(|u|^{-1}y) d(e^{iy}) \right| \\
&= |u|^{-\sigma} \left| \int_1^\infty \left[(s-1) y^{s-2} z(|u|^{-1}y) + y^{s-1} z'(|u|^{-1}y) |u|^{-1} \right] e^{iy} dy \right| \\
&\lesssim_\sigma |u|^{-\sigma} \left[\int_1^\infty y^{\sigma-2} dy + \int_1^{2|u|} y^{\sigma-1} |u|^{-1} dy \right], \text{ recall } z \text{ vanishes if } |u|^{-1}|y| \geqslant 2 \\
&\lesssim_\sigma |u|^{-\sigma} [1 + |u|^{\sigma-1}] \\
&\lesssim |u|^{-\sigma}, \text{ for large } u, \text{ since } \sigma \leqslant 1.
\end{aligned}$$

This finishes the proof for the base case. For $-N < \sigma \leqslant -N+1$, $N \geqslant 0$, we have:

$$\begin{aligned}
\alpha_s(e^{iuy}) &:= (-1)^N \frac{e^{s^2}}{\Gamma(s) s(s+1) \cdots (s+N-1)} \int_0^\infty y^{s+N-1} (z e^{iuy})^{(N)}(y)\, dy \\
&= \frac{e^{s^2}}{\Gamma(s+N)} \int_0^\infty y^{s+N-1} (z e^{iuy})^{(N)}(y)\, dy,
\end{aligned}$$

The factor $\frac{e^{s^2}}{\Gamma(s+N)}$ is bounded on any vertical strip. For u small, again the inequality

holds trivially. For u large, splitting the integral into two parts as above, using repeated integration by parts and the Leibniz formula, we have $|I(u)| \lesssim_\sigma |u|^{-\sigma}$.

Chapter 4

The Fourier Restriction Conjecture

Recall the statement of the Fourier restriction conjecture from Chapter 1.1:

Conjecture 3. *Let $S \subseteq \mathbb{R}^n$ be a compact $n-1$-dimensional hypersurface with non-vanishing Gaussian curvature, and $1 \leqslant p < \frac{2n}{n+1}$ and $1 \leqslant q \leqslant \frac{n-1}{n+1}p'$. Then for any $f \in C_c^\infty(\mathbb{R}^n)$, we have:*

$$\|\hat{f}\|_{L^q(S)} \leqslant C(p,q,n,S)\|f\|_{L^p(\mathbb{R}^n)}.$$

In this chapter, we begin by discussing an example due to Knapp, showing the above range of exponents is best possible. We then discuss a local restriction estimate, and prove an ε-removal lemma.

4.1 Necessary Conditions for the Restriction Conjecture

In this section we discuss the necessary conditions for the restriction conjecture.

We first show that it suffices to push down the exponents p, q. We have:

Lemma 1. *Suppose the restriction estimate holds for some $p_0, q_0 \geqslant 1$, and let $1 \leqslant p \leqslant$*

$p_0, 1 \leqslant q \leqslant q_0$. *Then it holds for* p, q *also.*

Proof. We fix q first. By the trivial $L^1 \to L^\infty$ bound and $L^{p_0} \to L^{q_0}$ bound and the Riesz-Thorin interpolation theorem 4, we get the $L^p \to L^{\tilde{q}}$ bound where $\tilde{q} \geqslant q_0$. Next we fix p and note that the hypersurface is compact and hence having finite surface measure. By Hölder's inequality, we get the $L^p \to L^q$ bound, since $q \leqslant q_0 \leqslant \tilde{q}$. □

4.1.1 Necessity of Nonvanishing Gaussian Curvature

Let $p > 1$, otherwise the restriction conjecture holds trivially. Then the condition of non-vanishing Gaussian curvature is necessary, as the following example (see [11]) shows. Let $\psi(x_2, \ldots, x_n)$ be a nonzero bump function and define

$$f_k(x_1, x_2, \ldots, x_n) := \psi(x_2, \ldots, x_n) u_k(x_1)$$

where $u_k(x_1) := \frac{1}{1+|x_1|} \chi_{\{|x_1| \leqslant k\}}$ and $u(x_1) := \frac{1}{1+|x_1|}$. Consider the restriction of $\hat{f}_k$ onto the hypersurface $\{\xi_1 = 0\}$. By definition we have

$$\hat{f}_k(0, \xi_2, \ldots, \xi_n) = \hat{\psi}(\xi_2, \ldots, \xi_n) \|u_k\|_{L^1(\mathbb{R})}.$$

Then for $p > 1$,

$$\frac{\|\hat{f}_k(0, \cdot)\|_{L^q(\mathbb{R}^{n-1})}}{\|f_k\|_{L^p(\mathbb{R}^n)}} = \frac{\|\hat{\psi}\|_{L^q(\mathbb{R}^{n-1})}}{\|\psi\|_{L^p(\mathbb{R}^{n-1})}} \cdot \frac{\|u_k\|_{L^1(\mathbb{R})}}{\|u_k\|_{L^p(\mathbb{R})}} \to \infty, \quad \text{as } k \to \infty,$$

since $\|u_k\|_{L^1(\mathbb{R})} \to \infty$ but $\|u_k\|_{L^p(\mathbb{R})} \to \|u\|_{L^p(\mathbb{R})} < \infty$.

4.1.2 Necessity of the Upper Bound for p:

We will consider the extension operator:

$$\mathcal{E}g(x) := \int_S g(\xi) e^{2\pi i x \cdot \eta} \, d\sigma(\eta)$$

Then the restriction estimate is equivalent to:

$$\|\mathcal{E}g\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|g\|_{L^{q'}(d\sigma)} \tag{4.1}$$

Now we let $g := 1$. Then we see right hand side is a constant. By the decay of Fourier transform (9), we see that the left hand side is finite only if $p' \cdot \frac{n-1}{2} > n$. Solving this gives $p < \frac{2n}{n+1}$.

4.1.3 Necessity of the Upper Bound of q:

The following construction is called Knapp's example. The key idea it utilizes is the so called uncertainty principle.

Consider a tiny cap on S, and without loss of generality, assume that it is given by the graph of the function $\phi : [-\delta, \delta]^{n-1} \to \mathbb{R}$ by $\phi(\xi) := |\xi|^2$. Let χ_K denote the characteristic function of that small cap K. Notice that the whole cap is contained in some rectangle T with dimensions $\delta \times \cdots \times \delta \times \delta^2$ with its shortest side normal to the plane $\xi_n = 0$. We claim that

$$|(\chi_K d\sigma)^\vee(x)| = \left| \int_{[-\delta,\delta]^{n-1}} e^{2\pi i x\cdot(\xi,|\xi|^2)} \sqrt{1+4|\xi|^2}\, d\xi \right| \gtrsim \delta^{n-1},$$

for any x in the "dual rectangle T^*" given by $|x_1|, |x_2|, \ldots, |x_{n-1}| \leqslant c\delta^{-1}$, $|x_n| \leqslant c\delta^{-2}$ with the same orientation as T for some unimportant constant c dependent on n only.

Proof. If $x = 0$, then $(\chi_K d\sigma)^\vee(x) = \sigma(K) \sim \delta^{n-1}$. We know that if $x \in T^*$, then $|x \cdot (\xi, |\xi|^2)| < nc$. By continuity, if $c > 0$ were chosen sufficiently small, then if $|x \cdot (\xi, |\xi|^2)| < nc$, we have $|\cos(2\pi x \cdot (\xi, |\xi|^2))| + |\sin(2\pi x \cdot (\xi, |\xi|^2))| > \frac{1}{2}$, thus

$$\left| \int_{[-\delta,\delta]^{n-1}} e^{2\pi i x\cdot(\xi,|\xi|^2)} \sqrt{1+4|\xi|^2}\, d\xi \right| \geqslant \frac{1}{4}\sigma(K),$$

whence $|(\chi_K d\sigma)^\vee(x)| \gtrsim \delta^{n-1}\chi_{T^*}$. □

With this result, the extension estimate (4.1) is true only if

$$\delta^{n-1}|T^*|^{\frac{1}{p'}} \lesssim \sigma(K)^{\frac{1}{q'}}, \text{ i.e. } \delta^{n-1}\delta^{-\frac{n+1}{p'}} \lesssim \delta^{\frac{n-1}{q'}}.$$

For this to hold as $\delta \to 0^+$, we need $q \leqslant \frac{n-1}{n+1}p'$.

4.2 The Local Restriction Conjecture

In this section we deal with the local restriction estimate and an equivalent form with thickening of the surface. We first state the local restriction conjecute:

Conjecture 4 (Localised Extension)**.**

$$\|(gd\sigma)^\vee\|_{L^{p'}(B_R)} \lesssim_\varepsilon R^\varepsilon \|g\|_{L^{q'}(d\sigma)}, \tag{4.2}$$

where $1 \leqslant p \leqslant \frac{2n}{n+1}$ *and* $1 \leqslant q \leqslant \frac{n-1}{n+1}p'$*, and* g *is a smooth function supported on* S*.*

It can be shown that Conjecture 4 is equivalent to Conjecture 1.

Conjecture 4 has an equivalent formulation, which we present here:

Let N_R denote the CR^{-1} neighbourhood of S (a thickening of the surface); $N_R := \{\eta \in \mathbb{R}^n : d(\eta, S) \leqslant CR^{-1}\}$. We also denote $\tilde{N}_R$ as the CR^{-1} neighbourhood of S in the normal direction: $\tilde{N}_R := \{(\xi + tR^{-1}) : \xi \in S, |t| \leqslant C\}$. Note then $\tilde{N}_R \subseteq N_R$.

We endow N_R with the usual n-dimensional Lebesgue measure, which is often easier to deal with than surface measures.

This conjecture is formulated as follows:

Conjecture 5 (Localised Extension with Thickening)**.**

$$\|G^\vee\|_{L^{p'}(B_R)} \lesssim_\varepsilon R^\varepsilon R^{-\frac{1}{q}} \|G\|_{L^{q'}(N_R)}, \tag{4.3}$$

where $1 \leqslant p \leqslant \frac{2n}{n+1}$ *and* $1 \leqslant q \leqslant \frac{n-1}{n+1}p'$*,* B_R *is centred at* 0 *and* G *is a smooth function*

supported on N_R.

Notice that we allow $p = \frac{2n}{n+1}$ here; the loss is at the exponent R^{ε} for arbitrarily small $\varepsilon > 0$. Also, the blurring gives us with a power $R^{-\frac{1}{q}}$ on the right hand side, which is compatible with the uncertainty principle.

Actually we have the following:

Lemma 2 (The Thickening Lemma)**.** *Conjectures 4 and 5 are equivalent.*

Proof. The proof is mainly from [6].

- (4.2) $\Longrightarrow$ (4.3) This part is easier to handle. Let $R > 1$ be large and by partition of unity, assume that G is supported near 0, so that G is supported in $\tilde{N}_R$ also. By Fubini's theorem and change of variables,

$$\begin{aligned}\check{G}(x) &= \int_{|t|<CR^{-1}} \int_{\xi\in[-1,1]^{n-1}} G(\xi, |\xi|^2 + t) e^{2\pi i(x'\cdot\xi + x_n(|\xi|^2+t))} d\xi dt \\ &= \int_{|t|<CR^{-1}} (G|_{S_t} d\sigma_t)^{\vee}(x) dt,\end{aligned}$$

  where σ_t is the natural surface measure on $S_t := S + (0, 0, \ldots, t)$.

  By (4.2) applied to each slice of the translates of S (this estimate is obviously translation invariant), we obtain the following:

$$\|(G|_{S_t} d\sigma_t)^{\vee}\|_{L^{p'}(B_R)} \lesssim_{\varepsilon} R^{\varepsilon} \|G|_{S_t}\|_{L^{q'}(S_t)},$$

for all $|t| < CR^{-1}$. Since $p \geqslant 1$, we compute:

$$\begin{aligned}
&\|\check{G}\|_{L^{p'}(B_R)} \\
\text{(By Minkowski)} &\leqslant \int_{|t|<CR^{-1}} \|G|_{S_t} d\sigma_t\check{}\|_{L^{p'}(B_R)}\, dt \\
&\lesssim_\varepsilon R^\varepsilon \int_{|t|<CR^{-1}} \|G|_{S_t}\|_{L^{q'}(S_t)}\, dt \\
\text{(By Hölder)} &\leqslant R^{\varepsilon-\frac{1}{q}} \left(\int_{|t|<CR^{-1}} \|G|_{S_t}\|_{L^{q'}(S_t)}^{q'}\, dt \right)^{\frac{1}{q'}} \\
&= R^{\varepsilon-\frac{1}{q}} \|G\|_{L^{q'}(N_R)}.
\end{aligned}$$

- (4.3) $\implies$ (4.2) Let $R > 1$ be large. Fix $\phi \in C_c^\infty(\mathbb{R})$ with support in $B(0,1)$ such that $|\check{\psi}(x)| \gtrsim 1$ for all $|x| \leqslant 1$. Define $G := \psi_R * (gd\sigma)$ where $\psi_R(\xi) := R^n\psi(R\xi)$. Then

$$\|(gd\sigma)\check{}\|_{L^{p'}(B_R)} \lesssim \|(gd\sigma)\check{}\check{\psi}_R\|_{L^{p'}(B_R)} = \|\check{G}\|_{L^{p'}(B_R)}$$

Since G is supported in N_R, we may apply (4.3) to deduce

$$\|(gd\sigma)\check{}\|_{L^{p'}(B_R)} \lesssim_\varepsilon R^{\varepsilon-\frac{1}{q}} \|\psi_R * (gd\sigma)\|_{L^{q'}(N_R)}$$

Thus it remains to show

$$\|\psi_R * (gd\sigma)\|_{L^{q'}(\mathbb{R}^n)} \lesssim R^{\frac{1}{q}} \|g\|_{L^{q'}(d\sigma)}. \tag{4.4}$$

This estimate is trivial in the case $q' = 1$ by the Fubini's theorem. It suffices to show that (4.4) holds in the case $q' = \infty$. This is in turn reduced to showing

$$\sup_{\xi\in\mathbb{R}^n} \int_S |\psi_R(\xi-\eta)| d\sigma(\eta) \lesssim R. \tag{4.5}$$

Heuristically, the above is true because the support of ψ_R intersects S on at most

an $\sim (R^{-1})^{n-1}$ cap and the height of ψ_R is bounded above by $\sim R^n$, leading to the bound $R^{-(n-1)} \times R^n = R$. To prove it more rigourously, we can pose this problem into the following lemma:

Lemma 3. *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$, $S \subseteq \mathbb{R}^n$ be a compact hypersurface (without any curvature conditions). Then we have:*

$$\sup_{\xi \in \mathbb{R}^n} \int_S |\psi_R(\xi - \eta)| d\sigma(\eta) \lesssim R.$$

If this lemma is true, then we have proved that (4.4) holds.

Proof of the Lemma. This lemma is purely technical. By rapid decay of ψ, we may bound it by the integral

$$I(\xi) := R^n \int_S \frac{1}{(1 + R|\xi - \eta|)^n} d\sigma(\eta).$$

Form a dyadic decomposition based on the size of $R|\xi - \eta|$: more precisely, fix $\xi \in \mathbb{R}^n$ and denote

$$A_{-1}(\xi) := \{\eta \in \mathbb{R}^n : R|\xi - \eta| \leqslant 1\}$$

and for $k \geqslant 0$,

$$A_k(\xi) := \{\eta \in \mathbb{R}^n : 2^k < R|\xi - \eta| \leqslant 2^{k+1}\}$$

Let $S_k(\xi) := A_k(\xi) \cap S$. Then we can bound

$$\begin{aligned} I(\xi) &\lesssim R^n \sum_{k=-1}^{\infty} \int_{S_k(\xi)} \frac{1}{(1+R|\xi-\eta|)^n} d\sigma(\eta) \\ &\lesssim R^n \sum_{k=-1}^{\infty} \sigma(S_k(\xi)) \frac{1}{2^{kn}} \\ &\leqslant R^n \sum_{k=-1}^{\infty} \sigma(B(\xi, \frac{2^{k+1}}{R}) \cap S) 2^{-kn} \\ &\lesssim R^n \sum_{k=-1}^{\infty} \left(\frac{2^{k+1}}{R}\right)^{n-1} 2^{-kn} \\ &\sim R. \end{aligned}$$

The second last line holds by a simple geometric observation: for small R the surface measure is bounded essentially by 1; for large R, the surface is locally flat, hence $\sigma(B(\xi, \frac{2^{k+1}}{R}) \cap S)$ is roughly a cap with radius at most $\frac{2^{k+1}}{R}$ on $\mathbb{S}^{n-1}$. This gives the estimate. □

□

The thickening lemma 2 has an immediate corollary:

Corollary 1. *The following localised extension estimate holds:*

$$\|(gd\sigma)^{\check{}}\|_{L^2(B_R)} \lesssim R^{\frac{1}{2}} \|g\|_{L^2(d\sigma)}, \tag{4.6}$$

for every $g \in L^2(d\sigma)$.

Thus we see that although the extension estimate is trivially false at $p = 2$, the localised version is true with a loss of factor of $R^{\frac{1}{2}}$.

Proof. By the thickening lemma 2, (4.6) is equivalent to the following:

$$\|G^{\check{}}\|_{L^2(B_R)} \lesssim R^{\frac{1}{2}} R^{-\frac{1}{2}} \|G\|_{L^2(N_R)} = \|G\|_{L^2(N_R)},$$

for any smooth G supported in N_R. By the Plancherel identity, the about trivially holds:

$$\|G^\vee\|_{L^2(B_R)} \leqslant \|G^\vee\|_{L^2(\mathbb{R}^n)} = \|G\|_{L^2(N_R)}.$$

Hence we have (4.6). □

4.3 The Epsilon Removal Theorem

In the last section we prove the ε-removal theorem, which is a useful tool that allows us to recover the global restriction estimates from local ones. This is first proved by Tao [12].

Notation: for $1 \leqslant p \leqslant 2$, $\varepsilon > 0$ be small, we denote $R(p, \varepsilon)$ to be the localised restriction estimate:

$$\|\hat{f}\|_{L^p(d\sigma)} \lesssim_\varepsilon R^\varepsilon \|f\|_{L^p(B(0,R))}, \tag{4.7}$$

for any f with support in $B(0, R)$ and all $R > 0$. Here σ denotes the surface measure on $S := \mathbb{S}^{n-1}$.

The theorem is precisely stated as follows:

Theorem 12 (ε-removal theorem)**.** *There exists a large $A > 0$ and small $0 < \varepsilon_0 < 1$, such that $R(p, \varepsilon)$ implies $R(q, 0)$ whenever $0 < \varepsilon < \varepsilon_0$ and*

$$\frac{1}{q} > \frac{1}{p} + \frac{A}{\log \frac{1}{\varepsilon}}.$$

The constants A and ε_0 depend on the dimension only. Note that q is slightly smaller than p. In this give and take, our gain is a global restriction estimate without any epsilon loss, but our loss is in the exponent p (q is slightly smaller than p is ε is slightly positive).

This theorem has the following consequence:

Corollary 2. *The followings are equivalent:*

1. *We have the restriction estimate near the endpoint: for any* $1 \leqslant p < \frac{2n}{n+1}$,

$$\|\hat{f}\|_{L^p(d\sigma)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}, \tag{4.8}$$

2. *We have the following localised restriction estimate at the conjectured endpoint:*

$$\|\hat{f}\|_{L^{\frac{2n}{n+1}}(d\sigma)} \lesssim_\varepsilon R^\varepsilon \|f\|_{L^{\frac{2n}{n+1}}(B_R)}, \tag{4.9}$$

for any smooth function f *supported on* B_R, *all* $R > 0$ *and all* $\varepsilon > 0$.

We show the forward direction first. By duality, (4.8) is equivalent to the following extension estimate:

$$\|(gd\sigma)^\vee\|_{L^{p'}(\mathbb{R}^n)} \leqslant C\|g\|_{L^{p'}(d\sigma)}, \tag{4.10}$$

for all $g \in L^{p'}(d\sigma)$ and all $\frac{2n}{n-1} < p' \leqslant \infty$. Let $\varepsilon > 0$ be given, and let p' so that $\frac{1}{p'} = \frac{n-1}{2n} - \varepsilon$. By interpolation between the two endpoints (p', p') and $(1, \infty)$ due to the trivial estimate $\|(gd\sigma)^\vee\|_{L^\infty(\mathbb{R}^n)} \leqslant \|g\|_{L^1(d\sigma)}$, we obtain the estimate

$$\|(gd\sigma)^\vee\|_{L^{p'+\varepsilon'}(\mathbb{R}^n)} \leqslant C_\varepsilon \|g\|_{L^{\frac{2n}{n-1}}(d\sigma)},$$

for all $g \in L^{\frac{2n}{n-1}}(d\sigma)$, where $\varepsilon' = O(\varepsilon)$.

Lastly, by Hölder's inequality, we have the following:

$$\|(gd\sigma)^\vee\|_{L^{\frac{2n}{n-1}}(B_R)} \leqslant CR^{\varepsilon''}\|(gd\sigma)^\vee\|_{L^{p'+\varepsilon'}(B_R)} \leqslant C_\varepsilon R^{\varepsilon''}\|g\|_{L^{\frac{2n}{n-1}}(d\sigma)},$$

where $\varepsilon'' = O(\varepsilon)$. This is equivalent to (4.9).

For the other direction, let $1 \leqslant p < \frac{2n}{n+1}$ be arbitrary, and $\varepsilon < \varepsilon_0$ be such that $\frac{1}{p} > \frac{1}{\frac{2n}{n+1}} + \frac{A}{\log \frac{1}{\varepsilon}}$. Then Theorem 12 shows that we have $R(p, 0)$.

We remark here that the equivalence theorem is slightly imperfect: if the local estimate

(4.7) is true, by interpolating the above L^p-L^p estimate (4.8) with the trivial L^1-L^∞ estimate, we can only prove the restriction estimates when $1 \leqslant p < \frac{2n}{n+1}$ and $1 \leqslant q < \frac{n-1}{n+1}p'$. This is best seen from the interpolation diagram.

The following materials are devoted to the proof of Theorem 12.

4.3.1 The Sparse Support Lemma

We start with a lemma which bootstraps local estimates to global estimates.

Definition 5. *Let $R > 1$. A collection $\{B(x_i, R)\}_{i=1}^N$ of R-balls in $\mathbb{R}^n$ is said to be C-sparse if there exists a large $C > 1$ so that for any $i \neq j$, we have $|x_i - x_j| \geqslant R^C N^C$.*

Lemma 4. *Suppose $R(p, \varepsilon)$ holds. Then for any f supported on $\bigcup_i B(x_i, R)$ which is 100-sparse, we have:*

$$\|\hat{f}\|_{L^p(d\sigma)} \lesssim_\varepsilon R^\varepsilon \|f\|_{L^p(\mathbb{R}^n)}. \tag{4.11}$$

We will denote $C := 100$ from the remaining part of this section.

Proof. By the thickening lemma 2 we see that $R(p, \varepsilon)$ is equivalent to

$$\|\hat{f}\|_{L^p(N_R)} \lesssim R^{-\frac{1}{p}+\varepsilon} \|f\|_{L^p(\mathbb{R}^n)}, \tag{4.12}$$

for all f supported on $B(0, R)$. By translational invariance we see that the above remains true with the same constant if we replace $B(0, R)$ by $B(x_i, R)$.

Fix ϕ to be a Schwartz function such that $\phi \geqslant 1$ on the unit ball and its Fourier transform is supported on the unit ball. Write $\phi_i := \phi(\frac{x-x_i}{R})$, so that $\hat{\phi}_i(\xi) = R^n e^{ix_i\xi}\hat{\phi}(R\xi)$ and hence it is supported on $B(0, R^{-1})$. Also, decompose $f = \sum_i f_i\phi_i$ where we set $f_i := \frac{f}{\phi_i}$ on each disjoint ball.

It suffices to show the following estimate:

$$\left\| \sum_i F_i * \hat{\phi}_i \right\|_{L^p(d\sigma)} \lesssim R^{\frac{1}{p}} \left(\sum_i \|F_i\|^p_{L^p(N_R)} \right)^{\frac{1}{p}}, \tag{4.13}$$

whenever F_i are bump functions supported on N_R. If this is true, denote $F_i := \hat{f}_i$ which is supported on N_R. With the observation that $\hat{f}\Big|_S = \sum_i \hat{f}_i * \hat{\phi}_i\Big|_S$, we have:

$$\begin{aligned}
\|\hat{f}\|_{L^p(d\sigma)} &= \left\| \sum_i \hat{f}_i * \hat{\phi}_i \right\|_{L^p(d\sigma)} \\
&= \left\| \sum_i F_i * \hat{\phi}_i \right\|_{L^p(d\sigma)} \\
&\lesssim R^{\frac{1}{p}} \left(\sum_i \|F_i\|^p_{L^p(N_R)} \right)^{\frac{1}{p}}, \text{by (4.13)} \\
&= R^{\frac{1}{p}} \left(\sum_i \|\hat{f}_i\|^p_{L^p(N_R)} \right)^{\frac{1}{p}} \\
&\lesssim R^{\frac{1}{p}} \left(\sum_i R^{p\varepsilon} R^{-p\cdot\frac{1}{p}} \|f_i\|^p_{L^p(B(x_i,R))} \right)^{\frac{1}{p}}, \text{by a translated version of (4.12)} \\
&\leqslant R^{\varepsilon} \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Proof of (4.13)*:* When $p = 1$, we have:

$$\begin{aligned}
\left\| \sum_i F_i * \hat{\phi}_i \right\|_{L^1(\mathbb{S}^{n-1})} &\leqslant \sum_i \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} |F_i(y)\hat{\phi}_i(x-y)| dy d\sigma(x) \\
&\leqslant \sum_i \int_{\mathbb{R}^n} |F_i(y)| \sup_{y\in\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |\hat{\phi}_i(x-y)| d\sigma(x) dy \\
&= \sum_i \|F_i\|_{L^1(\mathbb{R}^n)} \sup_{y\in\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |\hat{\phi}_i(x-y)| d\sigma(x).
\end{aligned}$$

Hence it suffices to prove $\sup_{y\in\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |\hat{\phi}_i(x-y)| d\sigma(x) \lesssim R$. But this is just Lemma 3.

By interpolation it suffices to show the case $p = 2$. By Plancherel's theorem, (4.13) is

equivalent to the following:

$$\left\| \widehat{\left(\sum_i g_i \phi_i \right)} \right\|_{L^2(d\sigma)} \lesssim R^{\frac{1}{2}} \left(\sum_i \|g_i\|^2_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{2}}, \tag{4.14}$$

whenever g_i are Schwartz functions whose Fourier transforms are supported on N_R. Then (4.13) follows by taking $g_i := F_i^{\check{}}$.

Now we will denote $\mathcal{R}$ to be the restriction operator on S. By squaring both sides it is equivalent to:

$$\langle \mathcal{R}(\sum_i g_i \phi_i), \mathcal{R}(\sum_j g_j \phi_j) \rangle \lesssim R \sum_i \|g_i\|^2_{L^2(\mathbb{R}^n)}.$$

For simplicity, in the following we will just denote $\|\cdot\|_{L^2(\mathbb{R}^n)}$ as $\|\cdot\|$.

Using Cauchy-Schwarz inequality twice:

$$\begin{aligned} \langle \mathcal{R}(\sum_i g_i \phi_i), \mathcal{R}(\sum_j g_j \phi_j) \rangle &= \sum_i \sum_j \langle g_i, \overline{\phi_i} \mathcal{R}^* \mathcal{R}(g_j \phi_j) \rangle \\ &\leqslant \sum_i \sum_j \|g_i\| \|\phi_i \mathcal{R}^* \mathcal{R}(g_j \phi_j)\| \\ &\leqslant \left(\sum_i \|g_i\|^2 \right)^{\frac{1}{2}} \left[\sum_i \left(\sum_j \|\phi_i \mathcal{R}^* \mathcal{R}(g_j \phi_j)\| \right)^2 \right]^{\frac{1}{2}} \end{aligned}$$

Hence it suffice to show:

$$\left[\sum_i \left(\sum_j \|\phi_i \mathcal{R}^* \mathcal{R}(g_j \phi_j)\| \right)^2 \right]^{\frac{1}{2}} \lesssim R \left(\sum_i \|g_i\|^2 \right)^{\frac{1}{2}}.$$

It further suffices to show: for each i, j,

$$\|\phi_i \mathcal{R}^* \mathcal{R} \phi_j\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \lesssim R, \text{ if } i = j$$

$$\|\phi_i \mathcal{R}^* \mathcal{R} \phi_j\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \lesssim R^{-1} N^{-1}, \text{ if } i \neq j$$

Thus we have, by Cauchy-Schwarz inequality,

$$\left[\sum_i \left(\sum_j \|\phi_i \mathcal{R}^* \mathcal{R}(g_j \phi_j)\|\right)^2\right]^{\frac{1}{2}} \lesssim \left[\sum_i R^2 \|g_i\|^2 + R^{-2} N^{2-2} \sum_j \|g_j\|^2\right]^{\frac{1}{2}}$$
$$\lesssim R \left(\sum_i \|g_i\|^2\right)^{\frac{1}{2}},$$

since $1 \leqslant i, j \leqslant N$.

Hence it remains to show the above two estimates. The first inequality follows by applying the following inequality twice:

$$\|(\mathcal{R}\phi_i f)\|_{L^2(d\sigma)} \lesssim R^{\frac{1}{2}} \|\phi_i f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)},$$

which in turn follows from the thickening lemma 2, with almost the same proof as that of (4.6).

To prove the second estimate, we will recall the Schur's test (Theorem 6). Fix $i \neq j$. Write $T := \phi_i \mathcal{R}^* \mathcal{R} \phi_j$, $Tf(x) := \int_{\mathbb{R}^n} K(x, y) dy$. Then we can calculate

$$K(x, y) = \phi_i(x) (d\sigma)^\vee (x - y) \phi_j(y),$$

By Schur's test and symmetry in x and y it suffices to show that

$$\sup_x \int_{\mathbb{R}^n} |K(x, y)| dy \lesssim R^{-1} N^{-1}.$$

Indeed, for $|x - x_i| \leqslant R^2 N$, if $|y - x_j| \leqslant R^2 N$, we have $|y - x| \geqslant \frac{1}{2} R^C N^C$, by the sparse assumption. By the decay of the Fourier transform (Theorem 9), we have

$$\int_{|y - x_j| < R^2 N} |K(x, y)| dy \lesssim (R^2 N)^n (RN)^{C\frac{1-n}{2}} \leqslant (RN)^{-1},$$

where we require that $2n + C\frac{1-n}{2} < -1$. This is always true since $C = 100$ and $n \geqslant 2$.

On the other hand, by the rapid decay of ϕ_j, and the fact that $\phi_i(x) = O(1)$, $(d\sigma)^\vee(x-y) = O(1)$, we have

$$\int_{|y-x_j|>R^2N} |K(x,y)|dy \lesssim (RN)^{-n-1}R^n \leqslant R^{-1}N^{-1}.$$

Hence for $|x - x_i| \leqslant R^2N$, we have

$$\int_{\mathbb{R}^n} |K(x,y)|dy \lesssim R^{-1}N^{-1},$$

If $|x - x_i| > R^2N$, then we use the rapid decay of ϕ_i. We have:

$$\int_{\mathbb{R}^n} |K(x,y)|dy \lesssim (RN)^{-n-1}\|(d\sigma)^\vee\|_{L^\infty(\mathbb{R}^n)}\|\phi_j\|_{L^1(\mathbb{R}^n)} \lesssim (RN)^{-n-1}R^n \leqslant R^{-1}N^{-1}.$$

To conclude, we have shown that

$$\sup_x \int_{\mathbb{R}^n} |K(x,y)|dy \lesssim R^{-1}N^{-1}.$$

□

This finishes the proof of the sparse support lemma. □

4.3.2 Decomposition of Cubes

This lemma gives rise to the sparse collections of cubes.

Lemma 5. *Let E be the union of finitely many unit cubes of the form $[k_1, k_1 + 1] \times \cdots \times [k_n, k_n + 1], k_j \in \mathbb{Z}$. For any $C > 1$ and any $N \geqslant 1$, there exist $O(N|E|^{\frac{1}{N}})$ C-sparse collections of balls that cover E, such that the balls in each collection have radius $O(|E|^{2C^N})$. The implicit constants here depend on n only.*

Proof. Define the radii R_k for $0 \leqslant k \leqslant N$ by $R_0 := 1$, $R_{k+1} := |E|^C R_k^C$. In this way $R_k \leqslant |E|^{C^1+C^2+\cdots+C^k} \leqslant |E|^{2C^k}$ for each k since trivially $|E| \geqslant 1$. Starting with $k = 1$, we

set E_k to be the set of all $x \in E \backslash \bigcup_{j=1}^{k-1} E_j$ such that

$$|E \cap B(x, R_k)| \leqslant |E|^{\frac{k}{N}}.$$

Thus for $1 \leqslant k \leqslant N$, $x \in E_k$, we have

$$|E \cap B(x, R_{k-1})| \geqslant c|E|^{\frac{k-1}{N}}.$$

(For $k = 1$, we have $|E \cap B(x, 1)| \geqslant \frac{|B_1|}{2^n}$ if $x \in E$.) Then for $x \in E_k$, the set $E \cap B(x, R_k)$ can be covered by $O(|E|^{\frac{1}{N}})$ R_{k-1}-balls.

Fix k temporarily. Cover E_k by R_k-balls B_j. We allow them to overlap finitely many times, that is, there is $C(n)$ so that for each $x \in E_k$, x can belong to at most $C(n)$ such balls B_j, but we want the B_j's to be "as disjoint as possible". Note that trivially the number of balls B_j is $O(|E|)$. Now for each j, cover $E_k \cap B_j$ by $O(|E|^{\frac{1}{N}})$ R_{k-1}-balls $\{B_{j,l}\}_l$, i.e. $\#l = O(|E|^{\frac{1}{N}})$. By a simple combinatorics argument, we obtain $O(|E|^{\frac{1}{N}})$ collections (essentially each collection is chosen to be $\{B_{j,l}\}_j$, but we should further split into, say, in one dimensional case, $\{B_{j,l}\}_{j \text{ odd}}$, $\{B_{j,l}\}_{j \text{ even}}$) such that each of them consists of R_k-separated R_{k-1}-balls, and that each collection has cardinality $O(|E|)$. Thus each collection is C-sparse by the relation $R_k := |E|^C R_{k-1}^C$.

Lastly, note that $\bigcup_k E_k = E$. For each k we obtain $O(|E|^{\frac{1}{N}})$ collections of C-sparse R_{k-1}-balls, such that each collection has cardinality $O(|E|)$. Taking unions in $1 \leqslant k \leqslant N$, we obtain $O(N|E|^{\frac{1}{N}})$ collections of C-sparse balls, such that the balls in each collection have radii $O(R_N) = O(|E|^{2C^N})$.

□

4.3.3 A Discretization Argument

In this theorem we prove another lemma which is used in the proof of the ε-removal lemma.

Theorem 13. *Let $p, q \in [1, \infty)$, and let $S := \mathbb{S}^{n-1}$ with surface measure $d\sigma$. Let $\mathcal{R}$ be the restriction operator:*

$$\mathcal{R}f(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx, \ \xi \in S. \tag{4.15}$$

Suppose we have the following estimate:

$$\left\| \sum_{j\in\mathbb{Z}^n} b_j e^{-2\pi i j\cdot\xi} \right\|_{L^p(S,d\sigma)} \lesssim \|b_j\|_{l^{q,1}(\mathbb{Z}^n)}. \tag{4.16}$$

Then the restriction estimate is true:

$$\|\mathcal{R}f\|_{L^p(S,d\sigma)} \lesssim \|f\|_{L^{q,1}(\mathbb{R}^n)}. \tag{4.17}$$

If this is true, then in order to show (4.17), it suffices to assume that f is constant on 1-cubes. For, let $f(x) := b_j$ when $\|x - j\|_\infty < \frac{1}{2}$, $j \in \mathbb{Z}^n$, then f is defined a.e. and constant on 1-cubes. Substitution gives (4.16):

$$\begin{aligned}
&\left\| \sum_{j\in\mathbb{Z}^n} b_j e^{-2\pi i j\cdot\xi} \right\|_{L^p(S,d\sigma)} \\
&= \left\| \mathcal{R}\left(\sum_{j\in\mathbb{Z}^n} b_j \delta_j \right) \right\|_{L^p(S,d\sigma)} \quad \text{where } \delta_j \text{ is the dirac delta function at } j \\
&\leqslant \left\| \mathcal{R}\left(\sum_{j\in\mathbb{Z}^n} b_j \delta_j \right) \hat{\chi_Q} \right\|_{L^p(S,d\sigma)}, \left(Q := \left[-\frac{1}{2}, \frac{1}{2}\right]^n\right) \\
&= \left\| \mathcal{R}\left(\sum_{j\in\mathbb{Z}^n} b_j \delta_j * \chi_Q \right) \right\|_{L^p(S,d\sigma)} \\
&= \|\mathcal{R}f\|_{L^p(S,d\sigma)}
\end{aligned}$$

Here actually we need Q to be small such that $\hat{\chi_Q} \geqslant 1$ on S. For simplicity we just assume $Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ suffices.

Thus (4.17) is true.

Our proof needs a lemma in measure theory:

Lemma 6. *Let $A \subseteq B$ be bounded open sets in $\mathbb{R}^n$ such that $\partial A \subseteq B$, from which it follows that $d(\partial A, \partial B) > 0$. Then for N large enough, we have the following:*

$$\# \left\{ k \in \mathbb{Z}^n : \frac{k}{N} \in A \right\} N^{-n} \leqslant |B|, \tag{4.18}$$

where $|\cdot|$ denotes the n-dimensional Lebesgue measure.

Proof of the Lemma. Take N so large that $\frac{1}{N} < \frac{d(\partial A, \partial B)}{100\sqrt{n}}$. This guarantees that for any dyadic cube in a grid with size less than $\frac{1}{N}$, if it intersects A, then it must be strictly contained in B. Now if $\frac{k}{N} \in A$, the cube in which $\frac{k}{N}$ lies is strictly contained in B. (More precisely, we can take such cubes to be of the form $[a_1, b_1) \times \cdots \times [a_n, b_n)$). Thus

$$\# \left\{ k \in \mathbb{Z}^n : \frac{k}{N} \in A \right\} N^{-n} \leqslant |B|, \tag{4.19}$$

since the n-dimensional Lebesgue measure for such a cube is N^{-n}. □

Proof of the theorem. We need a fact that $C_c(\mathbb{R}^n)$ is dense in $L^{q,1}(\mathbb{R}^n)$, which the reader can refer to texts about Lorentz spaces. Now let $f \in C_c(\mathbb{R}^n)$. Then the Riemann sum of the integral in (4.15) converges to the integral pointwisely for any $\xi \in S$:

$$\lim_{N\to\infty} \frac{1}{N^n} \sum_{k\in\mathbb{Z}^n} f\left(\frac{k}{N}\right) e^{-2\pi i \frac{k}{N}\cdot\xi} = \mathcal{R}f(\xi). \tag{4.20}$$

This is actually a finite sum for each N since f has compact support. By Fatou's Lemma, we have the following:

$$\|\mathcal{R}f\|_{L^p(S,d\sigma)} \leqslant \liminf_{N\to\infty} \frac{1}{N^n} \left\| \sum_{k\in\mathbb{Z}^n} f\left(\frac{k}{N}\right) e^{-2\pi i \frac{k}{N}\cdot\xi} \right\|_{L^p(S,d\sigma)}.$$

The trick is to utilize the translational invariance in the summation. More specifically,

we have:

$$\begin{aligned}
&\left\| \sum_{k\in\mathbb{Z}^n} f\left(\frac{k}{N}\right) e^{-2\pi i \frac{k}{N}\cdot\xi} \right\|_{L^p(S,d\sigma)} \\
&= \left\| \sum_{c\in\{0,1,2,\ldots,N-1\}^n} \sum_{j\in\mathbb{Z}^n} f\left(\frac{jN+c}{N}\right) e^{-2\pi i\left(\frac{jN+c}{N}\cdot\xi\right)} \right\|_{L^p(S,d\sigma)} \\
&\leqslant \sum_{c\in\{0,1,2,\ldots,N-1\}^n} \left\| \sum_{j\in\mathbb{Z}^n} f\left(\frac{jN+c}{N}\right) e^{-2\pi i\left(\frac{jN+c}{N}\cdot\xi\right)} \right\|_{L^p(S,d\sigma)} \\
&= \sum_{c\in\{0,1,2,\ldots,N-1\}^n} \left\| \sum_{j\in\mathbb{Z}^n} f\left(\frac{jN+c}{N}\right) e^{-2\pi ij\cdot\xi} \right\|_{L^p(S,d\sigma)}
\end{aligned}$$

Now by (4.16) and Proposition 1, with $(X,\mu) = (\mathbb{Z}^n, c)$ where c is the counting measure, we have

$$\left\| \sum_{j\in\mathbb{Z}^n} f\left(\frac{jN+c}{N}\right) e^{-2\pi ij\cdot\xi} \right\|_{L^p(S,d\sigma)} \leqslant \left\| f\left(j+\frac{c}{N}\right) \right\|_{l^{q,1}(\mathbb{Z}^n)} \sim \sum_{l\in\mathbb{Z}} 2^l \left(\#\left\{ j : \left| f\left(j+\frac{c}{N}\right)\right| > 2^l \right\}\right)^{\frac{1}{q}}.$$

By Hölder's inequality applied to summation in $c \in \{0,1,2,\ldots,N-1\}^n$, we have:

$$\begin{aligned}
&\sum_{c\in\{0,1,2,\ldots,N-1\}^n} \left\| \sum_{j\in\mathbb{Z}^n} f\left(\frac{jN+c}{N}\right) e^{-2\pi ij\cdot\xi} \right\|_{L^p(S,d\sigma)} \\
&\sum_{c\in\{0,1,2,\ldots,N-1\}^n} \sum_{l\in\mathbb{Z}} 2^l \#\left\{ j\in\mathbb{Z}^n : \left| f\left(j+\frac{c}{N}\right)\right| > 2^l \right\}^{\frac{1}{q}} \\
&\leqslant \sum_{l\in\mathbb{Z}} 2^l \left(\sum_{c\in\{0,1,2,\ldots,N-1\}^n} \#\left\{ j\in\mathbb{Z}^n : \left| f\left(j+\frac{c}{N}\right)\right| > 2^l \right\}\right)^{\frac{1}{q}} (N^n)^{\frac{1}{q'}} \\
&= \sum_{l\in\mathbb{Z}} 2^l \left(\#\left\{ k\in\mathbb{Z}^n : \left| f\left(\frac{k}{N}\right)\right| > 2^l \right\}\right)^{\frac{1}{q}} (N^n)^{\frac{1}{q'}}.
\end{aligned}$$

Now with $A := \{x : f(x) > 2^l\}$, $B := \{x : f(x) > 2^{l-1}\}$ noting that $f \in C_c(\mathbb{R}^n)$, we may use Lemma 6 to obtain:

$$\#\left\{ k\in\mathbb{Z}^n : \left| f\left(\frac{k}{N}\right)\right| > 2^l \right\} \leqslant \left|\{x\in\mathbb{R}^n : |f(x)| > 2^{l-1}\}\right| N^n.$$

Finally we put everything together to get:

$$\begin{aligned}\|\mathcal{R}f\|_{L^p(S,d\sigma)} &\leqslant \liminf_{N\to\infty} \frac{1}{N^n} \left\| \sum_{k\in\mathbb{Z}^n} f\left(\frac{k}{N}\right) e^{-2\pi i \frac{k}{N}\cdot\xi} \right\|_{L^p(S,d\sigma)} \\ &\leqslant \sum_{l\in\mathbb{Z}} 2^l \left|\{x\in\mathbb{R}^n : |f(x)| > 2^{l-1}\}\right|^{\frac{1}{q}} \\ &\sim \|f\|_{L^{q,1}(\mathbb{R}^n)}\,,\end{aligned}$$

by Proposition 1 again.

□

Remark: This theorem holds as long as $1 \leqslant p, q < \infty$; however it highly relies on the power 1 on the Lorentz exponent $L^{q,1}$. The following simple theorem provides a little insight into this specific Lorentz space:

Theorem 14. *Let $p, q \in [1,\infty)$, T be a sublinear operator. Then $\|Tf\|_p \lesssim \|f\|_{q,1}$ if and only if the same holds for characteristic functions.*

Proof. By dyadic decompostion, given a test function f, there exists a pointwise bound:

$$|f(x)| \leqslant \sum_{l\in\mathbb{Z}} 2^l \chi_{E_l}(x), \tag{4.21}$$

where $E_l := \{x : |f(x)| > 2^l\}$, such that $\|f\|_{q,1} \sim \sum_l 2^l \mu(E_l)^{\frac{1}{q}}$.

Then $|Tf(x)| \leqslant \sum_{l\in\mathbb{Z}} 2^l |T(\chi_{E_l})(x)|$, hence

$$\begin{aligned}\|Tf\|_p &\leqslant \sum_{l\in\mathbb{Z}} 2^l \|T(\chi_{E_l})\|_p \\ &\lesssim \sum_{l\in\mathbb{Z}} 2^l \|\chi_{E_l}\|_{q,1} \\ &= \sum_{l\in\mathbb{Z}} 2^l \mu(E_l)^{\frac{1}{q}} \\ &\sim \|f\|_{q,1}\,,\end{aligned}$$

where the second inequality holds because we assumed $\|T\chi_E\|_p \lesssim \|\chi_E\|_{q,1}$ for any measurable set E. □

4.3.4 Proof of the Epsilon Removal Theorem

We finally come to the proof of the epsilon removal theorem. Recall our goal is to show that there exists a large $A > 1$ and a small $0 < \varepsilon_0 < 1$ such that for any $0 < \varepsilon < \varepsilon_0$ and $1 < p < 2$, $R(p, \varepsilon)$ implies

$$\|\mathcal{R}f\|_{L^q(S,d\sigma)} \lesssim \|f\|_{L^q(\mathbb{R}^n)},$$

where

$$\frac{1}{q} > \frac{1}{p} + \frac{A}{\log \frac{1}{\varepsilon}}.$$

First we can reduce the problem to proving the Lorentz space estimate:

$$\|\mathcal{R}f\|_{L^p(S,d\sigma)} \lesssim \|f\|_{L^{q_0,1}(\mathbb{R}^n)}, \tag{4.22}$$

where $\frac{1}{q_0} = \frac{1}{p} + \frac{A}{\log(1/\varepsilon)}$.

Indeed, if (4.22) is true, writing it into an extension estimate and use Proposition 2, we have

$$\|(gd\sigma)^\vee\|_{L^{q_0',\infty}(\mathbb{R}^n)} \lesssim \|g\|_{L^{p'}(d\sigma)}$$

together with the trivial bound:

$$\|(gd\sigma)^\vee\|_{L^{\infty}(\mathbb{R}^n)} \leqslant \|g\|_{L^{1}(d\sigma)}$$

By the Marcinkeiwicz interpolation theorem (7), since $q_0' < q' < \infty$, $p' > 1$, we have the

bound

$$\|(gd\sigma)^\vee\|_{L^{q'}(\mathbb{R}^n)} \lesssim \|g\|_{L^{\tilde{p}'}(d\sigma)}, 1 < \tilde{p}' < p' < q_0' < q' < \infty.$$

Lastly, Hölder inequality gives the bound:

$$\|(gd\sigma)^\vee\|_{L^{q'}(\mathbb{R}^n)} \lesssim \|g\|_{L^{q'}(d\sigma)},$$

which is equivalent to the restriction estimate $R(q, 0)$.

Hence we are going to prove (4.22), assuming $R(p, \varepsilon)$.

Since we have $R(p, \varepsilon)$, by the sparse support lemma, our assumption can be strengthened to:

$$\|\mathcal{R}f\|_{L^p(d\sigma)} \lesssim_\varepsilon R^\varepsilon \|f\|_{L^p(\mathbb{R}^n)}, \tag{4.23}$$

for any f supported on a sparse collection of balls.

On the other hand, the discretization argument (See Theorem 13 and the remark after it) reduces the problem to the case that f is constant on 1-cubes. By the argument in the proof of Theorem 14, in proving (4.22), we can do a further reduction by assuming $f = \chi_E$ where E is a finite union of cubes of the form $[k_1, k_1+1] \times \cdots \times [k_n, k_n+1], k_j \in \mathbb{Z}$. By Lemma 5, there are $O(N|E|^{\frac{1}{N}})$ C-sparse collections of balls that cover E (to be more precise, we will use the collections that cover E_k separately since the radii of the balls corresponding to different k are different), such that the balls in each collection have radius $O(|E|^{2C^N})$. By triangle inequality twice and (4.23) we have

$$\|\mathcal{R}\chi_E\|_{L^p(d\sigma)} \leqslant \sum_{k=0}^{N} M_\varepsilon R_k^\varepsilon |E|^{\frac{1}{N}} |E|^{\frac{1}{p}} \leqslant M_\varepsilon (|E|^{2C^N})^\varepsilon N |E|^{\frac{1}{N}} |E|^{\frac{1}{p}}.$$

We would like to have $M_\varepsilon(|E|^{2C^N})^\varepsilon N|E|^{\frac{1}{N}}|E|^{\frac{1}{p}} \lesssim |E|^{\frac{1}{q_0}} = |E|^{\frac{1}{p}+\frac{A}{\log(\frac{1}{\varepsilon})}}$, where $C = 100$, $0 < \varepsilon < \varepsilon_0$ and ε_0, N, A are all to be determined.

For this purpose we may take $\varepsilon_0 > 0$ so that $\frac{\log \frac{1}{\varepsilon_0}}{4 \log C} > 1$. Then given any $0 < \varepsilon < \varepsilon_0$, we may take N such that $\frac{\log \frac{1}{\varepsilon}}{4 \log C} \leqslant N \leqslant \frac{\log \frac{1}{\varepsilon}}{2 \log C}$. Then we may compute

$$2C^N \varepsilon = 2e^{N \log C} \varepsilon \leqslant 2e^{\log(\frac{1}{\varepsilon}) \cdot \frac{\log C}{2 \log C}} \varepsilon = 2\varepsilon^{\frac{1}{2}} \leqslant \frac{B}{\log(\frac{1}{\varepsilon})},$$

for some universal constant B, since $0 < \varepsilon < \varepsilon_0 < 1$. Then we have:

$$\begin{aligned}
&M_\varepsilon (|E|^{2C^N})^\varepsilon N |E|^{\frac{1}{N}} |E|^{\frac{1}{p}} \\
&\leqslant \frac{\log(\frac{1}{\varepsilon})}{2 \log C} |E|^{\frac{4 \log C}{\log(\frac{1}{\varepsilon})}} |E|^{2C^N \varepsilon} |E|^{\frac{1}{p}} \\
&\leqslant M_\varepsilon \frac{\log(\frac{1}{\varepsilon})}{2 \log C} |E|^{\frac{4 \log C + B}{\log(\frac{1}{\varepsilon})}} |E|^{\frac{1}{p}} \\
&\leqslant C_\varepsilon |E|^{\frac{A}{\log(\frac{1}{\varepsilon})}} |E|^{\frac{1}{p}},
\end{aligned}$$

where we have set $C_\varepsilon := M_\varepsilon \frac{\log(\frac{1}{\varepsilon})}{2 \log C}$, $A := 4 \log C + B$. This shows that $R(p, \varepsilon)$ implies (4.22), and the completes the proof of the ε-removal theorem.

Chapter 5

The Kakeya Conjecture

In this chapter we discuss the Kakeya conjecture in more detail. Recall our ultimate goal is to show that each Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension n.

It is more natural to handle an estimation for the analysts than to handle a purely geometric proposition. Here we introduce various versions of the Kakeya maximal conjectures, which will be shown to imply our desired Kakeya conjecture at the end of this chapter. One way to formulate the Kakeya maximal conjecture is the following:

Conjecture 6. *(Kakeya Maximal Conjecture, I) Suppose* $0 < \delta \ll 1$. *Let* $\mathbb{T}$ *be a family of tubes of size* $\delta^{n-1} \times 1$ *in* $\mathbb{R}^n$, *whose directions form a* δ*-separated set on* $\mathbb{S}^{n-1}$. *For each* $\frac{n}{n-1} < q \leqslant \infty$, *we have*

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^q(\mathbb{R}^n)} \leqslant C(q) \delta^{1-\frac{n}{q'}} \left(\sum_{T \in \mathbb{T}} \delta^{n-1} \right)^{\frac{1}{p}},$$

where p *is such that* $1 \leqslant p' \leqslant (n-1)q$, *and* $C(q)$ *is a constant independent of* δ *and* $\mathbb{T}$.

Note that the particular shape of the tubes is not important; it could be either cylindrical or rectangular, and its ends could be either rough or enclosed by a tiny cap, like a rod. Also, since $\sum_{T \in \mathbb{T}} \delta^{n-1} \lesssim 1$, if this is true at $p' = (n-1)q$, then it is also true for $p' \leqslant (n-1)q$. Another remark is that since the equation holds trivially at $q = \infty$, by interpolating with

the trivial bound $p = 1, q = \infty$, it suffices to prove the case $q = \frac{n}{n-1}$.

The following formulation is also used frequently.

Conjecture 7. *(Kakeya Maximal Conjecture, II) Suppose $0 < \delta \ll 1$. Let $\mathbb{T}$ be a family of tubes of size $\delta^{n-1} \times 1$ in $\mathbb{R}^n$, whose directions form a δ-separated set on $\mathbb{S}^{n-1}$. For each $\frac{n}{n-1} \leqslant q \leqslant \infty$, we have*

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^q(\mathbb{R}^n)} \leqslant C(q, \varepsilon) \delta^{1 - \frac{n}{q'} - \varepsilon} \left(\sum_{T \in \mathbb{T}} \delta^{n-1} \right)^{\frac{1}{p}},$$

where p is such that $p' \leqslant (n-1)q$, and $C(q, \varepsilon)$ is a constant independent of δ and $\mathbb{T}$.

Compared with Conjecture 6, we note the main difference here is that we allow $q = \frac{n}{n-1}$, but we lose an ε on the power of δ. The implication from Conjecture 6 to Conjecture 7 follows from Hölder's inequality in the same manner as in the proof of the forward direction of Corollary 2. We omit the proof. For the reverse direction, it is a consequence of the Pisier Factorisation theorem, which we cannot cover here, but the interested readers could refer to Bourgain [2], Pisier [8], or an exposition in Mattila [7] or Yung [15].

5.1 Necessary Conditions of Maximal Kakeya Conjecture

As before, we will show how the endpoint exponents emerged.

5.1.1 Necessity of Bounds for β, q

Let $\beta \in \mathbb{R}$, we are to find necessary conditions on β, q such that the following holds:

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^q(\mathbb{R}^n)} \lesssim_\varepsilon \delta^{-\beta},$$

for all family $\mathbb{T}$ of δ-separated $\delta^{n-1} \times 1$ tubes and all $\delta > 0$.

1. Necessarily $\beta \geqslant 0$, since the tubes $\mathbb{T}$ may be all disjoint, in which case $\|\sum_{T\in\mathbb{T}} \delta^{n-1}\|_{L^q(\mathbb{R}^n)}$ could be as large as 1.

2. Necessarily $\beta \geqslant -1 + \frac{n}{q'}$. To see this, let $\#T \sim \delta^{1-n}$, which is the maximum number of tubes due to the condition that they are δ-separated. Assume the tubes are identical, centred at the origin and their directions are uniformly separated. (Imagine the shape of a sea urchin). Then there is a small ball of radius δ centred at the origin that is contained in all δ^{1-n} tubes. This gives

$$\left\| \sum_{T\in\mathbb{T}} \delta^{n-1} \right\|_{L^q(\mathbb{R}^n)} \gtrsim \delta^{1-n} (\delta^n)^{\frac{1}{q}} = \delta^{1-\frac{n}{q'}}$$

3. If further $q > 1$, then necessarily $\beta > 0$. Indeed, by the construction of Besicovitch sets (see Chapter X of [9]), for any $\varepsilon > 0$, there exists $\delta > 0$ and a family $\mathbb{T}$ of δ-separated $\delta^{n-1} \times 1$ tubes such that $\mathbb{T}$ has $\sim \delta^{1-n}$ tubes, and $\bigcup_{T\in\mathbb{T}} T$ has Lebesgue measure $\leqslant \varepsilon$.

   Then

$$1 = \left\| \sum_{T\in\mathbb{T}} \chi_T \right\|_{L^1(\mathbb{R}^n)} \leqslant \left\| \sum_{T\in\mathbb{T}} \chi_T \right\|_{L^q(\mathbb{R}^n)} \left| \bigcup_{T\in\mathbb{T}} T \right|^{\frac{1}{q'}} \leqslant \varepsilon^{\frac{1}{q'}} \left\| \sum_{T\in\mathbb{T}} \chi_T \right\|_{L^q(\mathbb{R}^n)}.$$

   Since ε can be arbitrarily small, this is incompatible with $\beta = 0$.

In particular, this suggests us to study whether

$$\left\| \sum_{T\in\mathbb{T}} \chi_T \right\|_{L^q(\mathbb{R}^n)} \lesssim_\varepsilon \delta^{1-\frac{n}{q'}-\varepsilon},$$

for all family $\mathbb{T}$ of δ-separated $\delta^{n-1} \times 1$ tubes in $\mathbb{R}^n$, all $\delta > 0$ and all $\frac{n}{n-1} \leqslant q \leqslant \infty$.

5.1.2 Necessity of Bounds for p

More generally, given $1 \leqslant p \leqslant \infty$, we ask whether

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^q(\mathbb{R}^n)} \lesssim_\varepsilon \delta^{1-\frac{n}{q'}-\varepsilon} \left(\sum_{T \in \mathbb{T}} \delta^{n-1} \right)^{\frac{1}{p}},$$

for all $\mathbb{T}$ as above, all $\delta > 0$ and all $\frac{n}{n-1} \leqslant q \leqslant \infty$.

For this to hold we must have $p' \leqslant (n-1)q$. Indeed, if $\mathbb{T}$ has only one tube, then

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^q(\mathbb{R}^n)} = \delta^{\frac{n-1}{q}},$$

while $\delta^{1-\frac{n}{q'}} \left(\sum_{T \in \mathbb{T}} \delta^{n-1}\right)^{\frac{1}{p}} = \delta^{1-\frac{n}{q'}} \delta^{\frac{n-1}{p}}$.

So for the inequality to hold for all $\varepsilon > 0$, we need

$$\frac{n-1}{q} \geqslant 1 - \frac{n}{q'} + \frac{n-1}{p},$$

that is, $p' \leqslant (n-1)q$.

Therefore, we arrive at one form of the Kakeya conjectures as in Conjecture 7.

5.2 A Dual Formulation of the Maximal Kakeya Conjecture

In this section we introduce Bourgain's Kakeya maximal functions and the Kakeya maximal function conjecture, basically following Wolff's notes [14]. We then prove an equivalence theorem relating the Kakeya maximal conjecture and the Kakeya maximal function conjecture.

For any $a \in \mathbb{R}^n$, $e \in \mathbb{S}^{n-1}$, $\delta > 0$, let $T_e^\delta(a)$ be the $\delta^{n-1} \times 1$ tube centred at a. Define the

Kakeya maximal function $f_\delta^* : \mathbb{S}^{n-1} \to \mathbb{R}$ by:

$$f_\delta^*(e) := \sup_{a \in \mathbb{R}^n} \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f|,$$

This gives rise to a sublinear operator mapping functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ to functions defined on $\mathbb{S}^{n-1}$.

The Kakeya maximal function conjecture states the following:

Conjecture 8 (Kakeya Maximal Function Conjecture)**.**

$$\|f_\delta^*\|_{L^n(\mathbb{S}^{n-1})} \lesssim_\varepsilon \delta^{-\varepsilon} \|f\|_{L^n(\mathbb{R}^n)},$$

for any $f \in L^n(\mathbb{R}^n)$ and any $\delta > 0$, for any $\varepsilon > 0$. We will see below that this corresponds to exactly the case $p' = q' = n$.

Theorem 15. *(Equivalence Theorem) Fix $\beta \in \mathbb{R}, 1 \leqslant p, q \leqslant \infty$. Then the followings are equivalent.*

1. *Let $\delta > 0$. For any family of tubes $\mathbb{T}$ given as in Conjecture 7, we have*

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^q(\mathbb{R}^n)} \lesssim_\varepsilon \delta^{-\beta-\varepsilon} \left(\sum_{T \in \mathbb{T}} \delta^{n-1} \right)^{\frac{1}{p}}, \tag{5.1}$$

   for any $\varepsilon > 0$.

2. *Let $\delta > 0$. For any family of tubes $\mathbb{T}$ given as in Conjecture 7 and any non-negative sequence $\{y_T\}_{T \in \mathbb{T}}$ indexed by $\mathbb{T}$, we have the following weighted Kakeya maximal inequality:*

$$\left\| \sum_{T \in \mathbb{T}} y_T \chi_T \right\|_{L^q(\mathbb{R}^n)} \lesssim_\varepsilon \delta^{-\beta-\varepsilon} \left(\sum_{T \in \mathbb{T}} y_T^p \delta^{n-1} \right)^{\frac{1}{p}}, \tag{5.2}$$

   for any $\varepsilon > 0$.

3. We have the bound:

$$\|f_\delta^*\|_{L^{p'}(\mathbb{S}^{n-1})} \lesssim_\varepsilon \delta^{-\beta-\varepsilon}\|f\|_{L^{q'}(\mathbb{R}^n)}, \tag{5.3}$$

for any $f \in L^{q'}(\mathbb{R}^n)$ and $\delta > 0$, for all $\varepsilon > 0$.

Proof.

"(1) $\Longrightarrow$ (2)" Let $\mathbb{T}$ and $\{y_T\}_{T\in\mathbb{T}}$ be as given, and fix q, p and ε as in (5.2). Fix C_ε to be the implicit constant in (5.1), corresponding to $\frac{\varepsilon}{2}$.

First we may normalize $\{y_T\}_{T\in\mathbb{T}}$ so that

$$\sum_{T\in\mathbb{T}} y_T^p \delta^{n-1} = 1, \tag{5.4}$$

so $y_T \leqslant \delta^{-(n-1)/p}$ for all $T \in \mathbb{T}$.

Next we classify the tubes T in $\mathbb{T}$ according to the size of the coefficients y_T. More precisely, by the triangle inequality,

$$\begin{aligned}
&\left\|\sum_{T\in\mathbb{T}} y_T\chi_T\right\|_{L^q(\mathbb{R}^n)} \\
&\leqslant \left\|\sum_{T\in\mathbb{T}:\ y_T\leqslant\delta^{-\frac{\varepsilon}{2}}} y_T\chi_T\right\|_{L^q(\mathbb{R}^n)} + \sum_{k\in\mathbb{N}:\ \delta^{-\frac{\varepsilon}{2}}<2^k\leqslant\delta^{-\frac{n-1}{p}}} \left\|\sum_{T\in\mathbb{T}:\ y_T\sim 2^k} y_T\chi_T\right\|_{L^q(\mathbb{R}^n)} \\
&\leqslant \delta^{-\frac{\varepsilon}{2}}\left\|\sum_{T\in\mathbb{T}:\ y_T\leqslant\delta^{-\frac{\varepsilon}{2}}} \chi_T\right\|_{L^q(\mathbb{R}^n)} + \sum_{k\in\mathbb{N}:\ \delta^{-\frac{\varepsilon}{2}}<2^k\leqslant\delta^{-\frac{n-1}{p}}} 2^k\left\|\sum_{T\in\mathbb{T}:\ y_T\sim 2^k} \chi_T\right\|_{L^q(\mathbb{R}^n)}
\end{aligned}$$

Each of the terms can be estimated using (5.1): indeed

$$\begin{aligned}
\delta^{-\frac{\varepsilon}{2}}\left\|\sum_{T\in\mathbb{T}:\ y_T\leqslant\delta^{-\frac{\varepsilon}{2}}} \chi_T\right\|_{L^q(\mathbb{R}^n)} &\leqslant \delta^{-\frac{\varepsilon}{2}} C_\varepsilon \delta^{-\beta-\frac{\varepsilon}{2}}\left(\sum_{T\in\mathbb{T}}\delta^{n-1}\right)^{\frac{1}{p}} \\
&\leqslant C_\varepsilon\delta^{-\beta-\varepsilon}
\end{aligned}$$

since $\mathbb{T}$, being a δ-separated family of tubes, contain at most $C\delta^{-(n-1)}$ tubes. Also,

$$\begin{aligned} 2^k \left\| \sum_{T\in\mathbb{T}:\ y_T\sim 2^k} \chi_T \right\|_{L^q(\mathbb{R}^n)} &\leqslant C_\varepsilon 2^k \delta^{-\beta-\frac{\varepsilon}{2}} \left(\sum_{T\in\mathbb{T}:\ y_T\sim 2^k} \delta^{n-1} \right)^{\frac{1}{p}} \\ &\leqslant C_\varepsilon \delta^{-\beta-\frac{\varepsilon}{2}} \left(\sum_{T\in\mathbb{T}} y_T^p \delta^{n-1} \right)^{\frac{1}{p}} \\ &= C_\varepsilon \delta^{-\beta-\frac{\varepsilon}{2}}, \end{aligned}$$

the last equality following from our normalization of $\{y_T\}_{T\in\mathbb{T}}$ in (5.4). This shows that we can estimate each term in the sum over k uniformly by $C_\varepsilon\delta^{-\beta-\frac{\varepsilon}{2}}$, which is independent of k. Since there are $\sim C\log(\delta^{-1})$ terms in the sum over k, we get

$$\begin{aligned} \left\| \sum_{T\in\mathbb{T}} y_T\chi_T \right\|_{L^q(\mathbb{R}^n)} &\leqslant C_\varepsilon\delta^{-\beta-\varepsilon} + C_\varepsilon\delta^{\beta-\frac{\varepsilon}{2}}\log(\delta^{-1}) \\ &\leqslant 2C_\varepsilon\delta^{-\beta-\varepsilon}. \end{aligned}$$

which gives the desired bound (5.2) in view of our normalization (5.4).

"(2) $\Longrightarrow$ (3)" If (2) is true, we can in particular fix a maximal δ-separated subset of $\mathbb{S}^{n-1}$ indexed by k, in which case there is a constant $C(n)$ such that $\bigcup_k B_{C\delta}(\omega_k) \supseteq \mathbb{S}^{n-1}$, where $B_\delta(\omega_k)$ denotes the δ-neighbourhood of ω_k on the surface $\mathbb{S}^{n-1}$.

We claim that for $|e-e'|<\delta$, $f_\delta^*(e)\leqslant Cf_\delta^*(e')$. For, given any $T_e^\delta(a)$, there are at most $C(n)$ tubes $T_{e'}^\delta(a_j)$, $1\leqslant j\leqslant C(n)$ such that $\bigcup_j T_{e'}^\delta(a_j)\supseteq T_e^\delta(a)$. Therefore

$$\begin{aligned} \|f_\delta^*\|_{L^{p'}(\mathbb{S}^{n-1})}^{p'} &\leqslant \sum_k \int_{B_\delta(\omega_k)} f_\delta^*(\omega)^{p'}\,d\sigma(\omega) \\ &\lesssim \sum_k \sigma(B_\delta(\omega_k)) f_\delta^*(\omega_k)^{p'} \\ &\sim \delta^{n-1}\sum_k f_\delta^*(\omega_k)^{p'}. \end{aligned} \tag{5.5}$$

Now by the duality of l^p, $l^{p'}$, there is a sequence $y_k \geqslant 0$ with $\sum_k y_k^p = 1$ such that

$$\sum_k f_\delta^*(\omega_k)^{p'} = \left(\sum_k y_k f_\delta^*(\omega_k) \right)^{p'}.$$

For each k, by definition of the Kakeya maximal function, there is a tube T_k centred somewhere with orientation ω_k such that

$$f_\delta^*(\omega_k) \leqslant 2 \frac{1}{|T_k|} \int_{T_k} |f|$$

Hence we have:

$$\begin{aligned}
\|f_\delta^*\|_{L^{p'}(\mathbb{S}^{n-1})} &\lesssim \delta^{\frac{n-1}{p'}} \sum_k y_k \frac{1}{\delta^{n-1}} \int_{T_k} |f| \\
&= \delta^{-\frac{n-1}{p}} \sum_k y_k \int_{T_k} |f| \\
&= \delta^{-\frac{n-1}{p}} \int_{\mathbb{R}^n} |f| \left(\sum_k y_k \chi_{T_k} \right) \\
&\leqslant \delta^{-\frac{n-1}{p}} \|f\|_{L^{q'}(\mathbb{R}^n)} \left\| \sum_k y_k \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \\
&\lesssim_\varepsilon \delta^{-\frac{n-1}{p}} \|f\|_{L^{q'}(\mathbb{R}^n)} \ \delta^{-\beta-\varepsilon} \left(\sum_k y_k^p \delta^{n-1} \right)^{\frac{1}{p}}, \text{ by (5.2)} \\
&= \delta^{-\beta-\varepsilon} \|f\|_{L^{q'}(\mathbb{R}^n)}, \text{ by our choice of } y_k \ .
\end{aligned}$$

Therefore if (5.1) is true, (5.2) is true.

"(3) $\Longrightarrow$ (1)" Let $\mathbb{T}$ be given. By the duality of L^q and $L^{q'}$, showing (5.1) is equivalent to showing that for any $f \in L^{q'}(\mathbb{R}^n)$ with $\|f\|_{L^{q'}(\mathbb{R}^n)} = 1$, we have:

$$\left| \int_{\mathbb{R}^n} f \left(\sum_{T \in \mathbb{T}} \chi_T \right) dx \right| \lesssim_\varepsilon \delta^{-\beta-\varepsilon} \left(\sum_{T \in \mathbb{T}} \delta^{n-1} \right)^{\frac{1}{p}}.$$

For each T, $\int_{\mathbb{R}^n} |f\chi_T|\, dx \lesssim \delta^{n-1} f_\delta^*(\omega_T)$, where ω_T is the orientation of T. Thus:

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} f \left(\sum_{T\in\mathbb{T}} \chi_T \right) dx \right| \\
&\leqslant \sum_{T\in\mathbb{T}} \int_T |f\chi_T|\, dx \\
&\lesssim \delta^{n-1} \sum_{T\in\mathbb{T}} f_\delta^*(\omega_T) \\
&\sim \sum_{T\in\mathbb{T}} \int_{B_\delta(\omega)} f_\delta^*(\omega) d\sigma(\omega), \text{ by (5.5).} \\
&= \int_U f_\delta^*(\omega) d\sigma(\omega), \text{ where } U := \bigcup_{T\in\mathbb{T}} B_\delta(\omega_T) \\
&\leqslant \left(\int_{\mathbb{S}^{n-1}} |f_\delta^*(\omega)|^{p'} d\sigma(\omega) \right)^{\frac{1}{p'}} \left(\int_U 1\, d\sigma(\omega) \right)^{\frac{1}{p}} \\
&\lesssim_\varepsilon \delta^{-\beta-\varepsilon} \|f\|_{L^{q'}(\mathbb{R}^n)} \left(\sum_{T\in\mathbb{T}} \delta^{n-1} \right)^{\frac{1}{p}}, \text{ by (5.3) and } \delta\text{-separation} \\
&= \delta^{-\beta-\varepsilon} \left(\sum_{T\in\mathbb{T}} \delta^{n-1} \right)^{\frac{1}{p}}.
\end{aligned}$$

This proves the equivalence theorem.

□

5.3 Kakeya Maximal Inequalities and the Hausdorff Dimension

In this section we prove a theorem that relates Kakeya maximal inequalities to the Hausdorff dimensions of a Kakeya set. In particular, we show that the maximal Kakeya inequality in the full range implies the Kakeya conjecture.

For completeness, let us recall the definitions of Hausdorff dimensions:

Let $\delta > 0$, $s > 0$. For each $A \subseteq \mathbb{R}^n$, we denote

$$\mathcal{H}^s_\delta(A) := \inf\{\sum_j (\mathrm{diam}(D_j))^s : \bigcup_j D_j \supseteq A, \mathrm{diam}(D_j) < \delta\}.$$

Then we see that as $\delta \searrow 0$, $\mathcal{H}^s_\delta(A)$ is increasing, and we denote $\mathcal{H}^s(A) := \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(A)$. Without loss of generality, we can assume that all D_j are open balls in $\mathbb{R}^n$, and that $\mathrm{diam}(D_j)$ can be arbitrarily small. Note that as $s \searrow 0$, $\mathcal{H}^s(A)$ is increasing. Moreover, one can show that for any $A \subseteq \mathbb{R}^n$, there exists a unique $0 \leqslant d \leqslant n$ with the property that $\mathcal{H}^s(A) = \infty$ for all $s < d$ and $\mathcal{H}^s(A) = 0$ for all $s > d$, which suggests that we define the Hausdorff dimension of such set A to be $\inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}$. We will denote such critical d by $\dim_{\mathcal{H}}(A)$.

Now our goal is to show that $d := \dim_{\mathcal{H}}(E) \geqslant n$ for any Kakeya set E. In fact, we will prove a more general theorem, which gives worse lower bounds on the Hausdorff dimensions in case of weaker maximal Kakeya function estimates:

Theorem 16 (Kakeya Maximal Inequality and the Hausdorff Dimension)**.** *Suppose we have the following estimate:*

$$\sigma\left\{e \in \mathbb{S}^{n-1} : f^*_\delta(\chi_E)(e) > \lambda\right\} \lesssim_\varepsilon \delta^{p'(-\beta-\varepsilon)} \lambda^{-p'} |E|^{\frac{p'}{q'}}, \tag{5.6}$$

for some $\beta \in \mathbb{R}$, some pair of exponents $1 \leqslant p, q \leqslant \infty$ with $p \leqslant q$, $q' < \infty$ and for all $\varepsilon > 0$ small. Equivalently, this is to say that the restricted weak type version of (5.3) holds. Then the Hausdorff dimension of a Kakeya set must be at least $n - \beta q'$.

Proof. Since $p \leqslant q$, $p' \geqslant q'$. Since $L^{p',\infty}(\mathbb{S}^{n-1})$ embeds continuously into $L^{q',\infty}(\mathbb{S}^{n-1})$, our assumption implies

$$\sigma\left\{e \in \mathbb{S}^{n-1} : f^*_\delta(\chi_E)(e) > \lambda\right\} \lesssim_\varepsilon \delta^{q'(-\beta-\varepsilon)} \lambda^{-q'} |E|. \tag{5.7}$$

We consider the definition $d := \dim_{\mathcal{H}}(A) := \sup\{s : \mathcal{H}^s(A) = \infty\}$. Let $\varepsilon > 0$ be any

positive number, and $q' < \infty$ as given in (5.7). We claim that $\mathcal{H}^{n-\beta q'-2q'\varepsilon}(E) \gtrsim 1$. If this is true, then we see that $n - \beta q' - 2q'\varepsilon \leqslant d$ for any $\varepsilon > 0$. Thus $n - \beta q' \leqslant d$ if we let $\varepsilon \to 0$.

To do this we show that with $\delta = \frac{1}{100}$, $\mathcal{H}_\delta^{n-\beta q'-2q'\varepsilon}(E) \gtrsim 1$. Given a covering of E by open balls $D_j = B_{r_j}(x_j)$, $r_j \leqslant \frac{1}{100}$. Partition the balls according to their sizes:

$$J_k := \{j : 2^{-k} \leqslant r_j < 2^{1-k}\}$$

Now since E is a Kakeya set, for any $e \in \mathbb{S}^{n-1}$, E contains a unit line segment I_e parallel to e. Similarly, we will do another partition of the directions according to their lengths covered by balls in the family k:

$$S_k := \left\{ e \in \mathbb{S}^{n-1} : m\left(I_e \cap \bigcup_{j\in J_k} D_j\right) \geqslant \frac{1}{100k^2} \right\},$$

where m is the one-dimensional Lebesgue measure. Then $I_e = \bigcup_k \left(I_e \cap \bigcup_{j\in J_k} D_j\right)$, and thus $\sum_k m\left(I_e \cap \bigcup_{j\in J_k} D_j\right) \geqslant m(I_e) = 1$. On the other hand, since $\sum_k \frac{1}{100k^2} < 1$, it follows that $\bigcup_{k=1}^\infty S_k = \mathbb{S}^{n-1}$.

Let $f := \chi_{F_k}$, where $F_k := \bigcup_{j\in J_k} B_{10r_j}(x_j)$. Then for each $e \in S_k$, denoting a_e to be the midpoint of I_e, we have

$$\begin{aligned} f^*_{2^{-k}}(e) &\geqslant \frac{1}{|T_e^{2^{-k}}(a_e)|}\left|F_k \cap T_e^{2^{-k}}(a_e)\right| \\ &\sim \frac{1}{2^{-k(n-1)}}\frac{1}{100k^2}2^{-k(n-1)} \\ &\sim k^{-2}. \end{aligned}$$

Hence $\|f^*_{2^{-k}}\|_{L^{q',\infty}(\mathbb{S}^{n-1})} \geqslant k^{-2}\sigma(S_k)^{\frac{1}{q'}}$.

On the other hand, by (5.7),

$$\|f^*_{2^{-k}}\|_{L^{q',\infty}(\mathbb{S}^{n-1})} \lesssim 2^{k(\varepsilon+\beta)}\ \left((\#J_k)2^{-kn}\right)^{\frac{1}{q'}}$$

Combining them we have $\sigma(S_k) \lesssim_\varepsilon (\#J_k)(k^2\, 2^{k(\varepsilon q' + \beta q' - n)}) \lesssim_\varepsilon (\#J_k) 2^{-k(n-\beta q' - 2q'\varepsilon)}$.

Therefore

$$\sum_j r_j^{n-\beta q'-2q'\varepsilon} \gtrsim \sum_k 2^{-k(n-\beta q'-2q'\varepsilon)}(\#J_k) \gtrsim \sum_k \sigma(S_k) \gtrsim 1.$$

This shows that the Hausdorff dimension of a Kakeya set is $\geqslant n - \beta q'$. In particular, if the maximal Kakeya inequality in Conjecture 7 holds, then by the Equivalence Theorem 15, we have (5.6) holds with $p' = q' = n$ and $\beta = 0$. Thus a Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension n.

□

Chapter 6

Relation Between Restriction and Kakeya Conjectures

In this chapter we will explain why the restriction and Kakeya conjectures are related. We will see that surprisingly, restriction theorem in the full range will implies the Kakeya conjecture in the full range. We will also talk about how a partial range of restriction theorem implies a partial result of the Kakeya conjecture.

6.1 Restriction Conjecture Implies Maximal Kakeya Conjecture

For this section, we reformulate the Maximal Kakeya Conjecture again:

Conjecture 9. *(Kakeya Maximal Conjecture, III)*

Suppose $0 < \delta \ll 1$. *Let* $\mathbb{T}$ *be a family of tubes of size* $(\delta^{-1})^{n-1} \times \delta^{-2}$ *in* $\mathbb{R}^n$, *whose directions form a* δ*-separated set on* $\mathbb{S}^{n-1}$. *We have*

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^{\frac{p'}{2}}(\mathbb{R}^n)} \lesssim_\varepsilon \delta^{1-n-\frac{2n}{p'}-\varepsilon} \left(\sum_{T \in \mathbb{T}} \delta^{n-1} \right)^{\frac{2}{q'}},$$

where $1 \leqslant p \leqslant \frac{2n}{n+1}$ *and* $1 \leqslant q \leqslant \frac{2(n-1)}{n+1-\frac{2}{p}}$.

This is different form Conjecture 7 in the following ways. First, we enlarged the tubes by a factor of δ^{-2}, so that each of them is essentially the dual rectangle of some δ^{n-1}-Knapp cap on the surface of the sphere. Second, the exponents are changed so that the p, q's corresponds exactly to those in the restriction conjecture. In other words, if we denote p_R, q_R, p_K, q_K to be the exponents in the restriction and Kakeya conjectures, respectively, we have the correspondence $\frac{p_R'}{2} = q_K$, $\frac{2}{q_R'} = \frac{1}{p_K}$. The restrictions on the endpoints follows exactly from $1 \leqslant p_K' \leqslant (n-1)q_K$ and $\frac{n}{n-1} \leqslant q_K \leqslant \infty$. The range of exponents (p, q) in Conjecture 9 is strictly contained in the range of exponents (p, q) in Conjecture 3 (except at the endpoint $p = \frac{2n}{n+1}$), since $\frac{2(n-1)}{n+1-\frac{2}{p}} < \frac{n-1}{n+1}p'$ whenever $1 \leqslant p < \frac{2n}{n+1}$. As before, to prove Conjecture 9, it suffices to prove it in the case when $p = \frac{2n}{n+1}$ and $q = \frac{2(n-1)}{n+1-\frac{2}{p}} = \frac{2n}{n+1}$.

6.1.1 Proof of the Implication

Now we hope to prove Conjecture 9 assuming that the following holds.

$$\|(gd\sigma)^\vee\|_{L^{p'}(B_R)} \lesssim_\varepsilon R^\varepsilon \|g\|_{L^{q'}(d\sigma)}, \tag{6.1}$$

where $1 \leqslant p \leqslant \frac{2n}{n+1}$ and $1 \leqslant q \leqslant \frac{2n}{n+1}$.

Unfortunately, we need more work to prove the global estimate; instead we will prove a slightly weaker localised version of the Kakeya maximal inequality:

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^{\frac{p'}{2}}(B_R)} \lesssim_\varepsilon \delta^{1-n-\frac{2n}{p'}-\varepsilon} \left(\sum_{T \in \mathbb{T}} \delta^{n-1} \right)^{\frac{2}{q'}}, \tag{6.2}$$

where $p = \frac{2n}{n+1}$ and $q = \frac{2n}{n+1}$, and B_R denotes a ball in $\mathbb{R}^n$ centred at 0 with radius $R \sim \delta^{-2}$.

The technique in the proof is mainly by randomisation. Let $\mathbb{T}$ be the scaled family of $(\delta^{-1})^{n-1} \times \delta^{-2}$-tubes as given, and denote $\{x_T\}$ to be their centres.

Without loss of generality, assume that the surface in the restriction conjecture is a piece

of a paraboloid. We consider the collection of "Knapp caps" κ_T on the surface, which is essentially the intersection of the surface with some n-dimensional ball in $\mathbb{R}^n$. (See also 4.1.3) where the normal of each cap κ_T is parallel to the tube T. The caps can be taken to be disjoint, each having surface measure $\sim \delta^{n-1}$.

We consider a randomised sum as follows:

$$g(\eta, \omega) := \sum_{T \in \mathbb{T}} \mathcal{E}_T(\omega) e^{2\pi i x_T \cdot \eta} \chi_{\kappa_T}(\eta),$$

where $\{\mathcal{E}_T\}_{T \in \mathbb{T}}$ is a family of i.i.d. random variables with distributions $P(\mathcal{E}_T = \pm 1) = \frac{1}{2}$. Then taking inverse Fourier transform,

$$(gd\sigma)^\vee(x, \omega) = \sum_{T \in \mathbb{T}} \mathcal{E}_T(\omega)(\chi_{\kappa_T})^\vee(x - x_T).$$

Applying (6.1), we get

$$\Big\| \sum_{T \in \mathbb{T}} \mathcal{E}_T(\omega)(\chi_{\kappa_T} d\sigma)^\vee(x - x_T) \Big\|_{L^{p'}(B_R)} \lesssim_\varepsilon R^\varepsilon \|g\|_{L^{q'}(d\sigma)} \tag{6.3}$$

To estimate both sides, we invoke Khinchin's inequality, whose proof can be found in many textbooks of probability theory:

Theorem 17. *(Khinchin's inequality) Let $\{\mathcal{E}_k\}$ be i.i.d. random variables with distributions $P(\mathcal{E}_k = \pm 1) = \frac{1}{2}$. Let $\{a_k\} \subseteq \mathbb{C}$. Then for $0 < p < \infty$, $N \in \mathbb{N}$, we have:*

$$\mathbb{E}\left(\left| \sum_{k=1}^N \mathcal{E}_k a_k \right|^p \right) \sim_p \left(\sum_{k=1}^N |a_k|^2 \right)^{\frac{p}{2}} \tag{6.4}$$

Using this theorem, we estimate the p'-th power of the left hand side of (6.3) in the

following way (the case $p = 1$ is trivial, so assume $p > 1$, thus $p' < \infty$):

$$\begin{aligned}
&\mathbb{E}\left(\left\|\sum_{T\in\mathbb{T}}\mathcal{E}_T(\omega)(\chi_{\kappa_T}d\sigma)^\vee(x-x_T)\right\|^{p'}_{L^{p'}(B_R)}\right)\\
&=\mathbb{E}\left(\int_{B_R}\left|\sum_{T\in\mathbb{T}}\mathcal{E}_T(\omega)(\chi_{\kappa_T}d\sigma)^\vee(x-x_T)\right|^{p'}dx\right)\\
&=\int_{B_R}\mathbb{E}\left(\left|\sum_{T\in\mathbb{T}}\mathcal{E}_T(\omega)(\chi_{\kappa_T}d\sigma)^\vee(x-x_T)\right|^{p'}\right)dx\\
&\sim\int_{B_R}\left(\sum_{T\in\mathbb{T}}\left|(\chi_{\kappa_T}d\sigma)^\vee(x-x_T)\right|^2\right)^{\frac{p'}{2}}dx
\end{aligned}$$

Recall the Knapp's example introduced in 4.1.3, and we have:

$$|(\chi_{\kappa_T}d\sigma)^\vee(x-x_T)|\gtrsim\delta^{n-1}\chi_T(x),$$

where each T is essentially the tubes given in the assumptions.

Applying this observation to the above, we get

$$\begin{aligned}
&\int_{B_R}\left(\sum_{T\in\mathbb{T}}\left|(\chi_{\kappa_T}d\sigma)^\vee(x-x_T)\right|^2\right)^{\frac{p'}{2}}dx\\
&\gtrsim\int_{B_R}\left(\sum_{T\in\mathbb{T}}\left(\delta^{n-1}\chi_T(x)\right)^2\right)^{\frac{p'}{2}}dx\\
&=\delta^{(n-1)p'}\int_{B_R}\left(\sum_{T\in\mathbb{T}}\chi_T(x)\right)^{\frac{p'}{2}}dx\\
&=\delta^{(n-1)p'}\left\|\sum_{T\in\mathbb{T}}\chi_T\right\|^{\frac{p'}{2}}_{L^{\frac{p'}{2}}(B_R)}.
\end{aligned}$$

Next we estimate the p'-th power of the right hand side of (6.3). By disjointness, we can

compute easily that

$$\begin{aligned}\mathbb{E}\left(\left(R^{\varepsilon}\|g\|_{L^{q'}(d\sigma)}\right)^{p'}\right) &= R^{\varepsilon}p'\mathbb{E}\left(\left\|\sum_{T\in\mathbb{T}}\mathcal{E}_T(\omega)e^{2\pi i x_T\cdot\eta}\chi_{\kappa_T}(\eta)\right\|^{p'}_{L^{q'}(d\sigma)}\right)\\ &\sim R^{\varepsilon}p'\mathbb{E}\left(\left(\sum_{T\in\mathbb{T}}\delta^{n-1}\right)^{\frac{p'}{q'}}\right)\\ &\sim R^{\varepsilon}p'\left(\sum_{T\in\mathbb{T}}\delta^{n-1}\right)^{\frac{p'}{q'}}\end{aligned}$$

Combining both sides, and recalling that $R\sim\delta^{-2}$, we have

$$\left\|\sum_{T\in\mathbb{T}}\chi_T\right\|_{L^{\frac{p'}{2}}(B_R)}\lesssim_{\varepsilon}\delta^{2(1-n)-\varepsilon}\left(\sum_{T\in\mathbb{T}}\delta^{n-1}\right)^{\frac{2}{q'}},$$

In particular, suppose the restriction conjecture is true. Then with $p=\frac{2n}{n+1}$, $q=\frac{2n}{n+1}$, we have:

$$\left\|\sum_{T\in\mathbb{T}}\chi_T\right\|_{L^{\frac{n}{n-1}}(B_R)}\lesssim_{\varepsilon}\delta^{2(1-n)-\varepsilon}\left(\sum_{T\in\mathbb{T}}\delta^{n-1}\right)^{\frac{n-1}{n}},$$

which is exactly the localised maximal Kakeya inequality (6.2) in the case $p=\frac{2n}{n+1}$, $q=\frac{2n}{n+1}$.

6.1.2 Partial Results and Hausdorff Dimensions

In this section we are going to investigate the partial results we can get by known restriction estimates. More precisely, suppose (6.1) does not hold for all, but just for some exponents p',q' in the feasible region. What is the implication for a lower bound for the Hausdorff dimension of Kakeya sets in $\mathbb{R}^n$? We argue as follows.

If we rescale the sizes of the tubes back to our original case ($\delta^{n-1}\times 1$), we obtain a family

of weaker Kakeya maximal inequalities:

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^{\frac{p'}{2}}(\mathbb{R}^n)} \lesssim_\varepsilon \delta^{2(1-n+\frac{2n}{p'})-\varepsilon} \left(\sum_{T \in \mathbb{T}} \delta^{n-1} \right)^{\frac{2}{q'}}, \tag{6.5}$$

With $\frac{p'_R}{2} = q_K$, $\frac{2}{q'_R} = \frac{1}{p_K}$, (6.5) is equivalent to the following:

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^{q_K}(\mathbb{R}^n)} \lesssim_\varepsilon \delta^{2(1-\frac{n}{q'_K})-\varepsilon} \left(\sum_{T \in \mathbb{T}} \delta^{n-1} \right)^{\frac{1}{p_K}}.$$

In this case, $\beta = -2(1 - \frac{n}{q'_K})$, whence $n - \beta q'_K = 2q'_K - n = \dfrac{2p_R}{2 - p_R} - n$.

Recall that by Theorem 16, if (5.1) is true for some $p_K \leqslant q_K$, $q'_K < \infty$ and some $\beta \in \mathbb{R}$, then we have a lower bound $n - \beta q'_K$ for the Hausdorff dimensions for any Kakeya set in $\mathbb{R}^n$. Now in the case of exponents p_R, q_R coming from the restriction estimates, we have $p_R < 2$, hence $q'_K < \infty$. Also, $p_K \leqslant q_K$ is equivalent to the condition $p_R \leqslant q_R$.

The problem is that $p_R \leqslant q_R$ may not always hold. Thus one needs the results of the Pisier factorisation theorem again; more precisely, this theorem tells us that if we can prove an extension estimate of the form $E : L^\infty(\mathbb{S}^{n-1}) \to L^{p'_R}(\mathbb{R}^n)$, then it automatically holds that $E : L^s(\mathbb{S}^{n-1}) \to L^s(\mathbb{R}^n)$ for all $s > p'_R$. Replacing p'_R by s and applying Theorem 16, we have a lower bound $\dfrac{2s'}{2 - s'} - n$ for the Hausdorff dimensions of a Kakeya set in $\mathbb{R}^n$. Lastly, letting $s \to p'_R$, we obtain our desired conclusion. The interested reader may see Mattila [7] or Yung [15] for more details.

Therefore we have the following results:

1. Suppose the restriction conjecture is true. Then (5.1) holds for $p_R = \frac{2n}{n+1}$, and hence any Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension n.

2. Let $p_R := 1$ be the trivial endpoint. Then any Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension at least $2 - n$. This provides no information.

3. Let $p_R := \frac{2(n+1)}{n+3}$ be the Tomas-Stein exponent, and hence any Kakeya set in $\mathbb{R}^n$ has

Hausdorff dimension at least 1. This is also useless.

4. Let $p'_R := \frac{13}{4}$ be the exponent obtained by Guth [5], which is the best up to now in $n = 3$. Then any Kakeya set in $\mathbb{R}^3$ has Hausdorff dimension at least 2.2. Still, this estimate is worse than the easier bound given by Wolff's hairbrush (For details please refer to [7]), namely, $\frac{n+2}{2} = 2.5$.

Hence we see that this estimate is very rough, the main technical reason being the loss at the exponent by doubling the power of δ in Khinchin's inequality. In particular, the partial result given by Tomas-Stein estimate provides no information.

6.2 From Maximal Kakeya Conjecture to Restriction Conjecture

We saw that the restriction conjecture implies Maximal Kakeya Conjecture definitely. Naturally, people may ask whether the converse holds. The answer is not known. Nevertheless, if we assume the following additional square function estimate, then the implication holds.

6.2.1 The Square Function Estimate

We will introduce the useful wave packet decomposition technique and the square function estimate in this subsection. We formulate the problem as follows:

Let S be a hypersurface in $\mathbb{R}^n$ with non-vanishing Gaussian curvature. Let $R > 1$ be large, $\delta = R^{-\frac{1}{2}}$, and consider the R^{-1}-neighbourhood N_R of the hyper-surface S: $N_R := \{x \in \mathbb{R}^n : d(x, S) \lesssim R^{-1}\}$. This thickening of the surface was introduced in Chapter 4. Then we decompose N_R into δ-separated slabs $\Theta := \{S_\theta\}$ in the sense that the normal vectors at the centres of the slabs are δ-separated. To illustrate this, a typical example when S is the parabola on $[-1, 1]^{n-1}$ is as follows.

Cover $[-1,1]^{n-1}$ with cubes $\{Q_\theta\}$ satisfying:

- $\mathrm{diam}(Q_\theta) \sim \delta$.
- If a, b are centres of different cubes, then $d(a,b) \gtrsim \delta$.
- The cubes cover $[-1,1]^{n-1}$ and have bounded overlap: $1 \leqslant \sum_\theta \chi_{Q_\theta}(\xi) \leqslant C(n)$.

Then the slabs are defined by:

$$S_\theta := \{(\xi, \eta + |\xi|^2) : \xi \in Q_\theta, |\eta| \lesssim R^{-1}\}, \text{ where } c_\theta \text{ is the centre of } Q_\theta.$$

Notice that each slab S_θ is contained in some $(R^{-\frac{1}{2}})^{n-1} \times R^{-1}$ rectangle T_θ with the shortest side parallel to the normal direction of the slab. (The normal of each slab is defined by the normal to the hyper-surface at its centre.) In general, the partition of the coordinate plane may not consist of squares, but they should satisfy the properties listed above.

With the above settings, for each Schwartz function $f : \mathbb{R}^n \to \mathbb{C}$ whose Fourier transform is supported on N_R, we define $\hat{f}_\theta := \hat{f}\chi_\theta$, where $\chi_\theta := \chi_{S_\theta}$. In view of the finite overlapping, one expects that $\sum_\theta f_\theta$ would be similar to f. This suggests the following square function estimate:

Conjecture 10 (Square Function Estimate for Slabs)**.** *Let f be a Schwartz function with $\hat{f}$ supported on N_R. Then for $2 \leqslant p \leqslant \frac{2n}{n-1}$, we have:*

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim_\varepsilon R^\varepsilon \left\| \left(\sum_\theta |f_\theta|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \tag{6.6}$$

This conjecture is also called the Reverse Littlewood-Paley inequality for slabs. Actually, this conjecture is so strong that it can imply the Kakeya conjecture itself; See [3]. Thus it implies the restriction conjecture as well, by the proof we are going to present.

For $p = 2$ the conjecture is trivially true with no ε-loss, due to the Plancherel identity:

$$\begin{aligned}\left\|\sum_\theta f_\theta\right\|^2_{L^2(\mathbb{R}^n)} &= \left\|\sum_\theta \hat{f}_\theta\right\|^2_{L^2(\mathbb{R}^n)} \\ &= \int \sum_{\theta_1}\sum_{\theta_2} \hat{f}(\xi)\chi_{\theta_1}(\xi)\overline{\hat{f}(\xi)}\chi_{\theta_2}(\xi)d\xi \\ &= \int |\hat{f}(\xi)|^2 \sum_{\theta_1}\sum_{\theta_2} \chi_{\theta_1}(\xi)\chi_{\theta_2}(\xi)d\xi \\ &\sim \int |\hat{f}(\xi)|^2 d\xi = \|f\|^2_{L^2(\mathbb{R}^n)},\end{aligned}$$

by the finite overlapping assumption.

6.2.2 The Wave Packet Decomposition

The wave packet decomposition is an important technique. Recall that we have decomposed f with its frequency localised to each slab. In this section we are going to further decompose each slab into further sub-regions. We begin the technical part:

Fix $\phi \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transform is supported on $\left(-\frac{1}{2}, \frac{1}{2}\right)^n$ and equals to 1 on $\left[-\frac{1}{4}, \frac{1}{4}\right]^n$. For each rectangle T, denote $a_T : [-\frac{1}{4}, \frac{1}{4}]^n \to T$ be the natural invertible affine transformation. More precisely, write $a_T(x) = \rho(D(x)) + x_T$, where x_T is the centre of T, ρ, D are the rotations and (non-uniform) dilations, respectively. Next we define $\phi_T : T \to \mathbb{C}$ by $\phi_T = \phi \circ a_T^{-1}$, and note that $|T| \sim |D| := |\det(D)|$.

We will consider the $(R^{-\frac{1}{2}})^{n-1} \times R^{-1}$ rectangle T_θ discussed as above, and denote ω_θ as the normal to the slab S_θ. With such fixed T_θ, we wish to construct $\mathbb{T}(\theta)$ to be a collection of finitely overlapping rectangles of sizes $(R^{\frac{1}{2}})^{n-1} \times R$ with their longest sides parallel to ω_θ, and such that their union covers $\mathbb{R}^n$. More precisely, these rectangles are essentially the translates of the dual rectangle of T_θ, and by abuse of notations we write each dual rectangle in $\mathbb{T}(\theta)$ as T also. We define the wave packet adapted to $T \in \mathbb{T}(\theta)$ as:

$$\psi_T(x) := |T|^{-1} e^{2\pi i \xi_\theta \cdot x} \phi_T(x),$$

where ξ_θ denotes the centre of S_θ.

With all the backgrounds introduced, we state the following estimate:

Theorem 18. *Let f be a Schwartz function whose Fourier transform is supported on N_R, and fix a slab S_θ. Then there exists a collection $\mathbb{T}(\theta)$ as above, and a decomposition*

$$f_\theta(x) := \sum_{T \in \mathbb{T}_\theta} c_T \psi_T(x),$$

where c_T are constants satisfying

$$\left(\sum_{T \in \mathbb{T}(\theta)} |c_T|^2 \right)^{\frac{1}{2}} = |D|^{\frac{1}{2}} \|\hat{f}_\theta\|_{L^2(S_\theta)}.$$

The proof is based on multivariable Fourier series.

Proof. Let T_0 be the rectangle in $\mathbb{T}(\theta)$ centred at the origin, and let $a_{T_0} = \rho \circ D$ for some rotation ρ and some diagonal matrix D, whose entries are given by $D_{ii} = R^{\frac{1}{2}}$, $1 \leqslant i \leqslant n-1$, and $D_{nn} = R$. Consider

$$g_\theta(\xi) := \hat{f}_\theta(D^{-1}\rho(\xi) + \xi_\theta).$$

With suitable choice of constants, g_θ can be made to be supported on $(-\frac{1}{4}, \frac{1}{4})^n$. In such case, we may view g_θ as a smooth function defined on the torus $[-\frac{1}{2}, \frac{1}{2}]^n$. Thus it admits a Fourier series expansion:

$$g_\theta(\xi) := \sum_{k \in \mathbb{Z}^n} u_k e^{2\pi i k \cdot \xi},$$

where, by Parsevel identity, the u_k's satisfy:

$$\left(\sum_{k \in \mathbb{Z}^n} |u_k|^2 \right)^{\frac{1}{2}} = \|g_\theta\|_{L^2([-\frac{1}{2}, \frac{1}{2}]^n)}$$

However, by definition of g_θ, we also have:

$$\|g_\theta\|_{L^2([-\frac{1}{2},\frac{1}{2}]^n)} = |D|^{\frac{1}{2}}\|\hat{f}_\theta\|_{L^2(S_\theta)}$$

Thus it remains to observe that this decomposition of g_θ yields the desired decomposition of f_θ. Letting $\tilde{\xi} := \rho D^{-1}(\xi) + \xi_\theta$, we have:

$$\begin{aligned}\hat{f}_\theta(\tilde{\xi}) &= g_\theta(D\rho^{-1}(\tilde{\xi} - \xi_\theta)) \\ &= \sum_{k\in\mathbb{Z}^n} u_k e^{2\pi i k\cdot D\rho^{-1}(\tilde{\xi}-\xi_\theta)} \\ &= \sum_{k\in\mathbb{Z}^n} u_k e^{2\pi i k\cdot D\rho^{-1}(\tilde{\xi}-\xi_\theta)}\hat{\phi}(D\rho^{-1}(\tilde{\xi} - \xi_\theta)),\end{aligned}$$

the last equality following from the definition of ϕ and the support of $\hat{f}$.

Taking inverse Fourier transform,

$$f_\theta(x) = \sum_{k\in\mathbb{Z}^n} u_k|D|^{-1}e^{2\pi i x\cdot\xi_\theta}\phi(k + D^{-1}\rho^{-1}x) = \sum_{k\in\mathbb{Z}^n} u_k|D|^{-1}e^{2\pi i x\cdot\xi_\theta}\phi_{T_k}(x),$$

where T_k's are rectangles centred at $\rho(D(k))$ with equal dimensions $\sim (R^{\frac{1}{2}})^{n-1}\times R$, parallel to ω_θ, such that it covers $\mathbb{R}^n$. Now we can define our collection $\mathbb{T}(\theta) := \{T_k : k \in \mathbb{Z}^n\}$, and notice that in this special case we can make the rectangles uniformly distributed so that they touch but are non-overlapping (having disjoint interiors). Hence if for each $T \in \mathbb{T}(\theta)$, we set $c_T := u_k$, whenever $T = T_k$ for some $k \in \mathbb{Z}^n$, we have such decomposition, with

$$\left(\sum_{T\in\mathbb{T}(\theta)} |c_T|^2\right)^{\frac{1}{2}} = |D|^{\frac{1}{2}}\|\hat{f}_\theta\|_{L^2(S_\theta)}.$$

□

6.2.3 The Square Function Estimate Implies the Restriction Conjecture

Having assumed the square function estimate, we will show that the maximal Kakeya conjecture implies the restriction conjecture with the aid of the wave packet decomposition.

We will show the following extension estimate:

$$\|(gd\sigma)^{\vee}\|_{L^{\frac{2n}{n-1}}(B_R)} \lesssim_{\varepsilon} R^{\varepsilon}\|g\|_{L^{\frac{2n}{n-1}}(d\sigma)},$$

for all $g \in L^{\frac{2n}{n-1}}(d\sigma)$ and all $\varepsilon > 0$.

By the consequences of the Pisier factorisation theorem (See e.g. Mattila [7] or Yung [15]), it suffices to show the following:

$$\|(gd\sigma)^{\vee}\|_{L^{\frac{2n}{n-1}}(B_R)} \lesssim_{\varepsilon} R^{\varepsilon}\|g\|_{L^{\infty}(d\sigma)},$$

for all $g \in L^{\infty}(d\sigma)$ and all $\varepsilon > 0$.

This is in turn, by the thickening lemma 2, equivalent to the following:

$$\|f\|_{L^{\frac{2n}{n-1}}(B_R)} \lesssim_{\varepsilon} R^{\varepsilon-1}\|\hat{f}\|_{L^{\infty}(N_R)},$$

for all Schwartz function f whose Fourier transform is supported on N_R, where N_R denotes the R^{-1}-neighbourhood in $\mathbb{R}^n$ of the surface $(S, d\sigma)$. We can further normalise $\|\hat{f}\|_{\infty} = 1$.

We claim that

$$\left\| \left(\sum_{\theta} |f_{\theta}|^2 \right)^{\frac{1}{2}} \right\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^n)} \lesssim_{\varepsilon} R^{\varepsilon-1} \tag{6.7}$$

Then using the conjectured square function estimate (6.6), we are done.

To prove (6.7), the technical problem is that ϕ is not compactly supported. For each cap θ and each $T \in \mathbb{T}(\theta)$, where $\mathbb{T}(\theta)$ is defined as in the wave packet decomposition of Theorem 18, we decompose

$$\phi_T(x) = \sum_{l \in \mathbb{Z}^n} \phi_T(x) \cdot \chi_{T,l}(x),$$

where $\chi_{T,l}$ is the characteristic function of the rectangle $a_T([-\frac{1}{4}, \frac{1}{4}]^n + \frac{l}{2})$, so that $\{\chi_{T,l} : l \in \mathbb{Z}^n\}$ forms a partition of $\mathbb{R}^n$ (a.e.). By rapid decay of ϕ, we have, say, for $N = n + 1$,

$$|\phi_T(x)|\chi_{T,l}(x) \lesssim (1 + |l|)^{-N} \chi_{T,l}(x)$$

Hence

$$|\psi_T(x)| \lesssim |T|^{-1} \sum_{l \in \mathbb{Z}^n} (1 + |l|)^{-N} \chi_{T,l}(x)$$

Fix a slab S_θ. Decompose $f_\theta = \sum_{T \in \mathbb{T}(\theta)} c_T \psi_T$ as in the wave packet decomposition in Theorem 18, with

$$\left(\sum_{T \in \mathbb{T}(\theta)} |c_T|^2 \right)^{\frac{1}{2}} \sim |T|^{\frac{1}{2}} \|\hat{f}_\theta\|_{L^2(S_\theta)}.$$

Then

$$\begin{aligned} \left| \sum_{T \in \mathbb{T}(\theta)} c_T \psi_T \right|^2 &\lesssim \left(\sum_{T \in \mathbb{T}(\theta)} |c_T||T|^{-1} \sum_{l \in \mathbb{Z}^n} (1 + |l|)^{-N} \chi_{T,l}(x) \right)^2 \\ &= \left| \sum_{l \in \mathbb{Z}^n} (1 + |l|)^{-N} S_l(x) \right|^2, \end{aligned}$$

where $S_l(x) := \sum_{T \in \mathbb{T}(\theta)} |c_T||T|^{-1} \chi_{T,l}(x)$.

Thus

$$\begin{aligned}\left[\sum_{\theta}\left|\sum_{T\in\mathbb{T}(\theta)} c_T\psi_T\right|^2\right]^{\frac{1}{2}} &\lesssim \left[\sum_{\theta}\left|\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}S_l(x)\right|^2\right]^{\frac{1}{2}}\\ \text{(By Minkowski)} &\leqslant \sum_{l\in\mathbb{Z}^n}\left(\sum_{\theta}(1+|l|)^{-2N}|S_l(x)|^2\right)^{\frac{1}{2}}\\ &= \sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}\left(\sum_{\theta}|S_l(x)|^2\right)^{\frac{1}{2}}\end{aligned}$$

We consider the term $\sum_\theta |S_l(x)|^2$:

$$\sum_{\theta}|S_l(x)|^2 = \sum_{\theta}\left|\sum_{T\in\mathbb{T}(\theta)}|c_T||T|^{-1}\chi_{T,l}(x)\right|^2 = \sum_{\theta}\sum_{T\in\mathbb{T}(\theta)}|c_T|^2|T|^{-2}\chi_{T,l}(x),$$

by disjointness of the supports of $\chi_{T,l}$'s as l varies.

Therefore we have:

$$\begin{aligned}&\left\|\left[\sum_{\theta}\left|\sum_{T\in\mathbb{T}(\theta)} c_T\psi_T\right|^2\right]^{\frac{1}{2}}\right\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^n)}\\ &\lesssim \left\|\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}\left(\sum_{\theta}|S_l(x)|^2\right)^{\frac{1}{2}}\right\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^n)}\\ &\leqslant \sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}\left\|\left(\sum_{\theta}|S_l(x)|^2\right)^{\frac{1}{2}}\right\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^n)}\\ &= \sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}\left\|\sum_{\theta}\left|\sum_{T\in\mathbb{T}(\theta)}|c_T||T|^{-1}\chi_{T,l}(x)\right|^2\right\|^{\frac{1}{2}}_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}\\ &= \sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}\left\|\sum_{\theta}\sum_{T\in\mathbb{T}(\theta)}|c_T|^2|T|^{-2}\chi_{T,l}(x)\right\|^{\frac{1}{2}}_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}\\ &\sim R^{-\frac{n+1}{2}}\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}\left\|\sum_{\theta}\sum_{T\in\mathbb{T}(\theta)}|c_T|^2\chi_{T,l}(x)\right\|^{\frac{1}{2}}_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}.\end{aligned} \tag{6.8}$$

Our main technique is again randomisation. Note that $\sum_{T\in\mathbb{T}(\theta)}|c_T|^2 \leqslant |T|\|\hat{f}_\theta\|^2_{L^2(S_\theta)} \leqslant |T||S_\theta| \lesssim 1$. By scaling, there is a nonnegative sequence $\{d_T\}$ with $\sum_{T\in\mathbb{T}(\theta)} d_T = 1$ so that

$$\sum_{T\in\mathbb{T}(\theta)} |c_T|^2\chi_{T,l}(x) \lesssim \sum_{T\in\mathbb{T}(\theta)} d_T\chi_{T,l}(x).$$

We endow the space $\prod_\theta \mathbb{T}(\theta)$ with a probability measure P such that

$$P\Big((T_\theta)_\theta\Big) = \prod_\theta d_{T_\theta}.$$

Fix $x\in\mathbb{R}^n$. Consider a random variable $F_x\colon \prod_\theta \mathbb{T}(\theta)\to\mathbb{R}$, that sends a point $(T_\theta)_\theta$ to the number $\sum_\theta \chi_{T_\theta,l}(x)$. The expectation of F_x is $\sum_\theta\sum_{T\in\mathbb{T}(\theta)} d_T\chi_{T,l}(x)$, so

$$\sum_\theta\sum_{T\in\mathbb{T}(\theta)} |c_T|^2\chi_{T,l}(x) \leqslant \mathbb{E}\left(\sum_\theta \chi_{T_\theta,l}(x)\right).$$

By Minkowski's inequality,

$$\left\|\sum_\theta\sum_{T\in\mathbb{T}(\theta)} |c_T|^2\chi_{T,l}(x)\right\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leqslant \mathbb{E}\left\|\sum_\theta \chi_{T_\theta,l}(x)\right\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)},$$

and for each choice $(T_\theta)_\theta$, we have

$$\left\|\sum_\theta \chi_{T_\theta,l}(x)\right\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \lesssim R^\varepsilon R^{n-1}$$

by Conjecture 9.

Continuing the estimate in (6.8), we have:

$$\left\|\sum_\theta\sum_{T\in\mathbb{T}(\theta)} |c_T|^2\chi_{T,l}(x)\right\|^{\frac{1}{2}}_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \lesssim_\varepsilon R^\varepsilon R^{\frac{n-1}{2}}.$$

Thus to conclude

$$\begin{aligned}\left\|\left(\sum_{\theta}|f_\theta|^2\right)^{\frac{1}{2}}\right\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^n)} &\leqslant \left\|\left[\sum_{\theta}\left|\sum_{T\in\mathbb{T}(\theta)}c_T\psi_T\right|^2\right]^{\frac{1}{2}}\right\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^n)}\\ &\lesssim_\varepsilon R^\varepsilon R^{-\frac{n+1}{2}}\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}R^{\frac{n-1}{2}}\\ &\sim R^{-1+\varepsilon}.\end{aligned}$$

Hence we have proved that the maximal Kakeya conjecture, combined with the square function estimate, will imply the local version of the restriction conjecture with endpoints in the whole range.

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