Lecture 3: Introduction to Fractals
Diversity in Mathematics, 2019

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Question: How can you generalise the definition of the Cantor set?

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Notation: Middle-third Cantor set

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Using a Different Base

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- Number of intervals removed.
- The positions of intervals being removed.

Question: How does the dimension of the resulting Cantor set change?

Theorem

If the base is A and we choose B intervals at each time (regardless of their positions), then the resulting Cantor set has dimension

$$\frac{\log B}{\log A}.$$

Create Your Own Cantor Set

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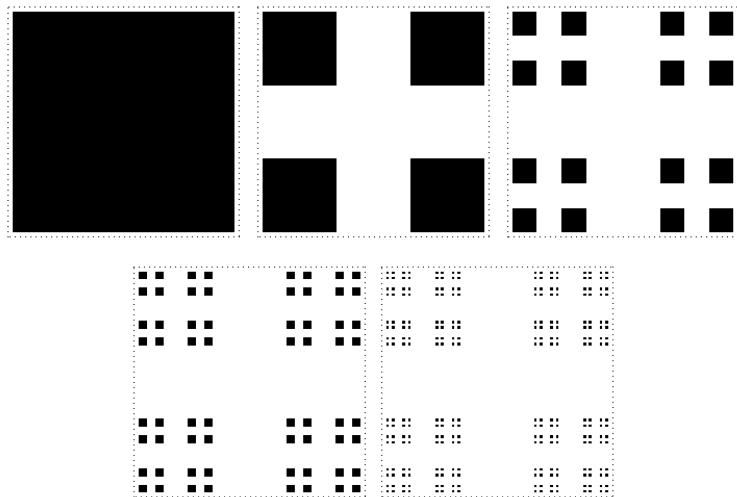 - Only once
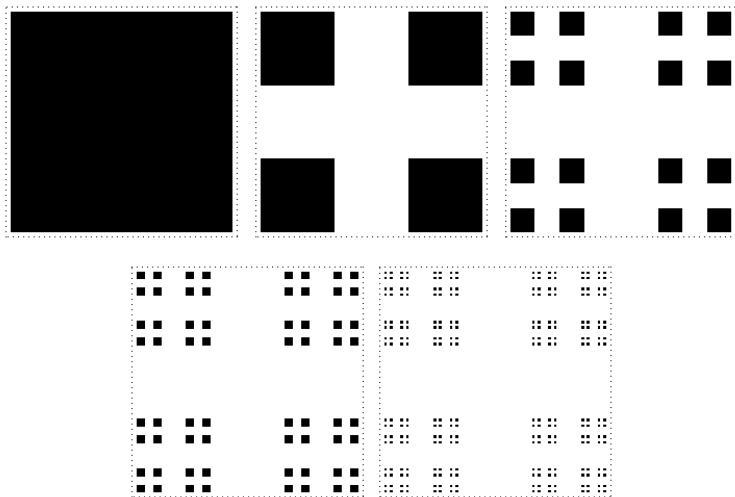 - Roll a dice at each step.

Figure: 2D middle-third Cantor Dust
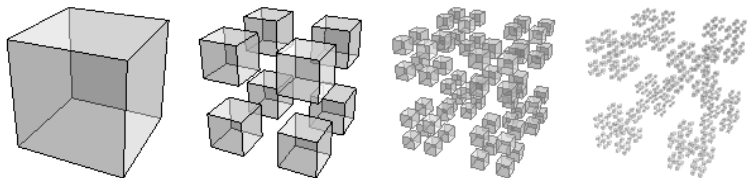
Figure: 2D middle-third Cantor Dust

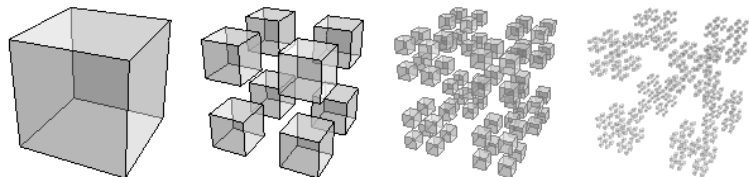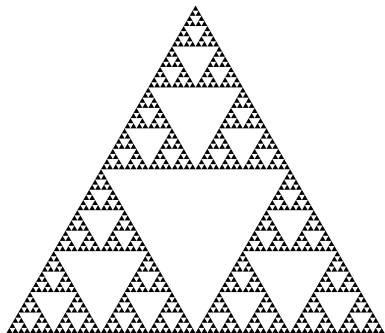Figure: 3D Cantor Dust
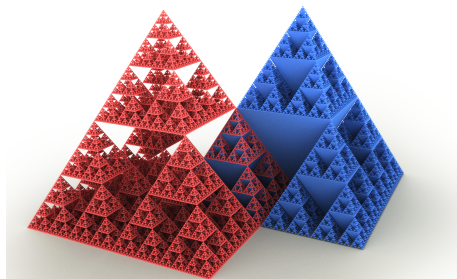
Higher Dimensions

Figure: 3D Cantor Dust

Question: Find the dimension.

(a) Sierpinski Triangle

(b) Sierpinski Pyramid

What is the dimension of Sierpiński triangle?

- The Smith-Volterra-Cantor set ("fat" Cantor set).
- Collatz conjecture ($3n + 1$ conjecture) and Collatz fractal.

- At step n, remove the middle intervals with length 4^{-n} from each interval remaining.
- What is the length remaining after infinitely many steps?
- Is there any "solid" interval left?

The Collatz Fractal

Define a function

$$f(n) = \begin{cases} 3n+1, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$
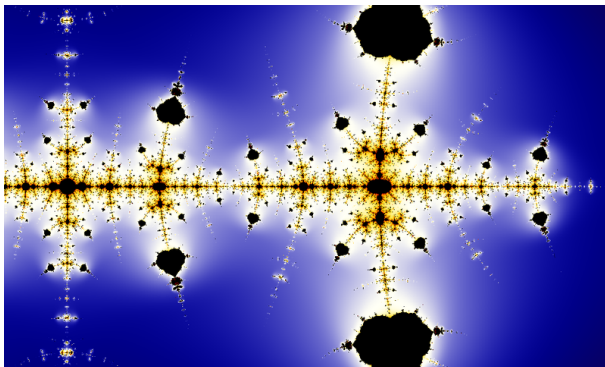
Conjecture:

Figure: The Collatz Fractal

The End