Lecture 2: Introduction to Fractals

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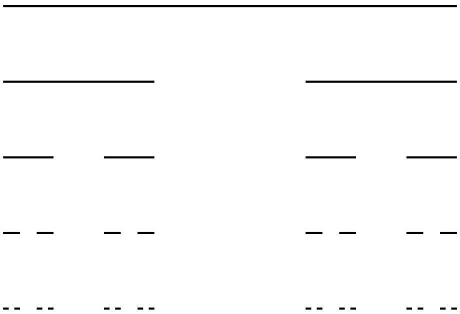
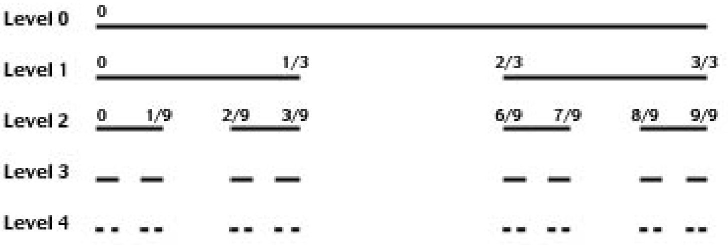
Cantor Set

The most classical nontrivial example of a fractal is the Cantor set.



Cantor Set: Construction



Base 3 Expansions

Restrict ourselves to $[0, 1]$. Observation:

$$\cdot \text{All numbers between 0 and } 1/3$$
$$\approx \text{All numbers such that the first digit is 0 in base 3 expansion}$$
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Question: Why "$\approx$"?

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Question: Why "$\approx$"?
Ans: some numbers have 2 different digit expansions. For example, $0.99999\cdots = 1$!

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Using long division from last time, we have

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To make the process more "symmetric", we would like to remove both $1/3$ and $2/3$ from the first step.
Convention: In general, if a number has 2 different digit expansions in base 3, such that either one has no "1" among the digits, include that number in the Cantor set.

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Exercise: Express C_1, C_2, C_3 using interval notations and logic symbols in set theory.

Cantor Set

Theorem

The Cantor set consists of exactly (with the convention above) all numbers in $[0, 1]$ *such that the digit "1" never appears in their base 3 expansions.*

Exercise

Are the following numbers in the Cantor set?

- $1/2$
- $4/5$
- $2/9$
- $1/4$
- $\pi/10 \approx 0.314159265\cdots$

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Fun fact: if you pick 2 numbers from the Cantor set and consider their sum, and if you put all such sums into a collection of numbers, the collection is exactly the interval $[0, 2]$!

How Large is the Cantor Set?

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The answer is:

$$1 \times \frac{1}{3} + 2 \times \frac{1}{9} + 4 \times \frac{1}{3^3} + 2^3 \times \frac{1}{3^4} + \cdots = 1.$$

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- Suppose you choose a number from $0, 1, 2$ randomly with equal probability. What is the probability that you do NOT get 1?
- Probability of not getting 1 in each step is $2/3$. After infinitely many times, the probability of getting a number in the Cantor set equals

$$\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \cdots = 0.$$

Dimension of a Fractal: Intuition

Consider $\mathbb{R}^3$, our 3D world.

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From 3D perspective, both a rectangle and a line segment have 0 volume, but rectangle $>$ line segment significantly.

Fractal Dimension: Intuition

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- What is the dimension of a sphere in 3D?
- What is the dimension of a coil (helical) spring in 3D (ignoring thickness)?
- What is the dimension of a parabola in 2D?

Dimension of a Fractal

Various ways; the easiest is the Minkowski dimension. In this lecture, dimension=Minkowski dimension.

Figure: German Mathematician: Hermann Minkowski

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Definition (Covering Number)

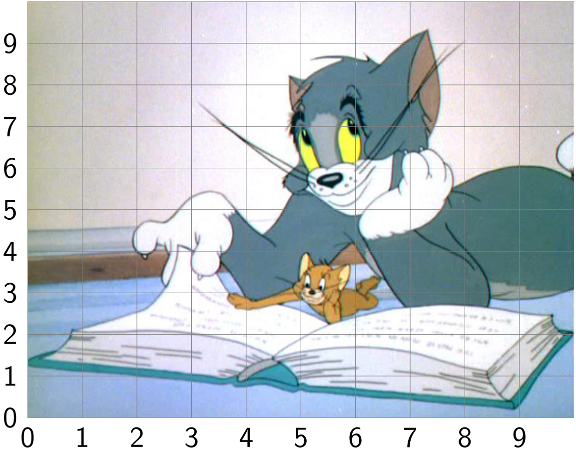
Let $\delta > 0$ be a small positive number (like 1, 0.1, 0.01, etc.) For $d = 1, 2, 3$, we cover the whole real line, plane, space, respectively, by d-dimensional grids of side length δ.

If S is any geometric figure, we define

$$N_\delta(S) = \textit{The number of grids which contain some part of } S.$$

We call $N_\delta(S)$ the covering number of S by grids of side length δ.

Covering by grids



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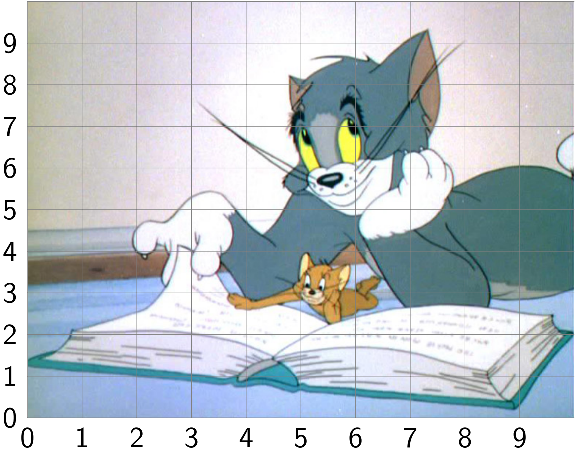

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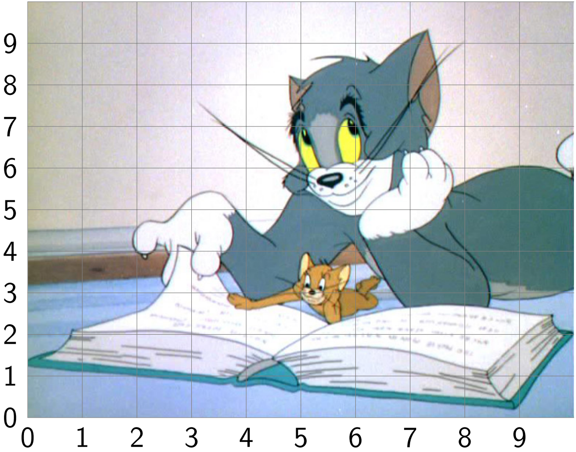

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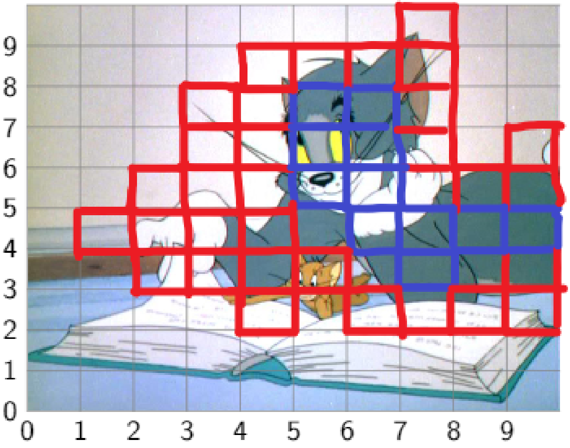
Question 2: what if $S =$ boundary of Tom?

Covering by grids: 2D Example



Question 1: if $S =$ Tom and $\delta = 1$, what is $N_\delta(S)$? 46

Question 2: what if $S =$ boundary of Tom? $46 - 11 = 35$

Dimension of a Fractal

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$$\dim(S) = \lim_{\delta \to 0} \frac{\log N_\delta(S)}{\log(\delta^{-1})}.$$

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  Roughly speaking, to compute the dimension, we need to pick a suitable integer k, let $\delta = k^{-n}$, and let n (the number of steps in the construction) go to infinity.
- This definition applies to $d = 1, 2, 3$ (and even larger dimensions!)

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- Identify the underlying dimension: $d = 1$.
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- Since we trisect the intervals, let us take $\delta = 3^{-n}$.
- Then $N_\delta(S) = 2^n$.
- The answer is

$$\dim(C) = \frac{\log(2^n)}{\log(3^n)} = \frac{\log 2}{\log 3} \approx 0.63.$$

- 3=Total number of intervals in each step;
  2=Number of intervals remaining in each step.

Dimension of Regular Shapes

Let $d = 3$. Find the dimensions of the following figures:

- A (solid) cube.
- A sphere.
- An arbitrary smooth curve (like a helix).
- A collection of 100 scattered points.
- The surface of the earth, along with all aircraft (regarded as points) in flight at some moment.

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If there are finitely many objects, the dimension of the union is equal to the largest among the individual dimensions.

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- A (solid) cube. $\mathrm{dim} = 3$
- A sphere. $\mathrm{dim} = 2$
- An arbitrary smooth curve (like a helix). $\mathrm{dim} = 1$
- A collection of 100 scattered points. $\mathrm{dim} = 0$
- The surface of the earth, along with all aircraft (regarded as points) in flight. $\mathrm{dim} = 2$

Generalisation of Cantor Set

Question: How can you generalise the definition of the Cantor set?

The End