Lecture 1: Introduction to Fractals

Diversity in Mathematics, 2019

Tongou Yang

The University of British Columbia, Vancouver

toyang@math.ubc.ca

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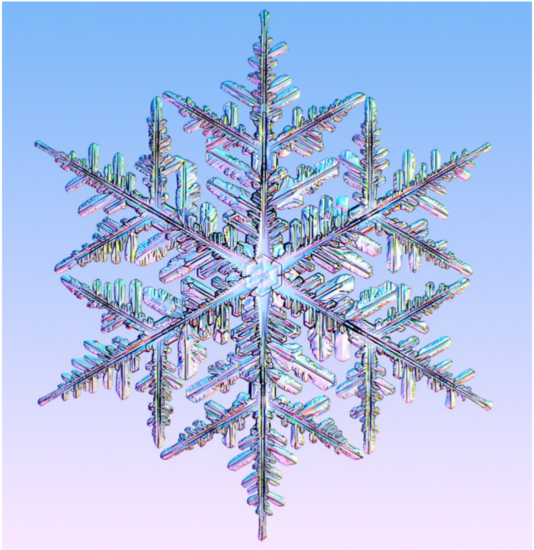
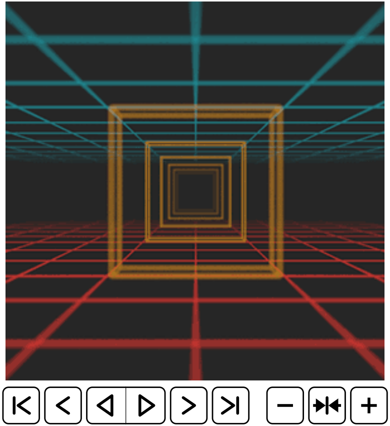
Does it Stop?

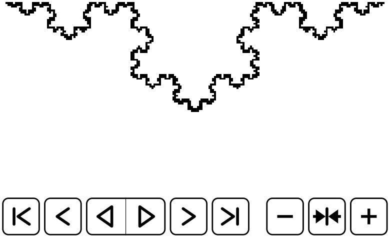
More Examples

More Examples



More Regular Shapes





Length of Coastlines

Trivia: Which country has the longest coastline in the world? Which is the runner-up?

List of countries by length of coastline

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(a) Map of Canada

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List of countries by length of coastline

(a) Map of Canada

(b) Map of Nunavut

Length of Coastlines

Figure: Map of Russia

Norwegian: Fjord

Figure: Map of Norway (South Part)

Crystal Structure

(a) Crystal of Sodium Chloride

Crystal Structure

(a) Crystal of Sodium Chloride

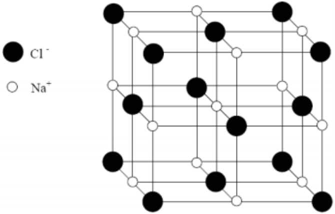


Figure 7.16: Crystal structure of NaCl

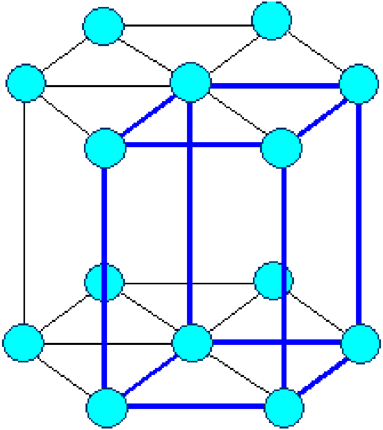
(b) Lattice Structure of Sodium Chloride

Crystal Structure

(a) Crystal of Emerald

Crystal Structure

(a) Crystal of Emerald



(b) Lattice Structure of Emerald

Fractal Geometry

Fractal Geometry is the study of geometric objects possessing self-similarity or approximate self-similarity.

Self-similarity

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Self-similarity

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- A and B are said to be *similar* if one could be turned into the other using the following transformations: translations, rotations, mirror reflections, and dilations.
- If A is a geometric figure, we say it is *self-similar* if some part of A is similar to the whole of A.

Are These Self-similar?

- A solid disk on the plane

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- A solid irregular triangle

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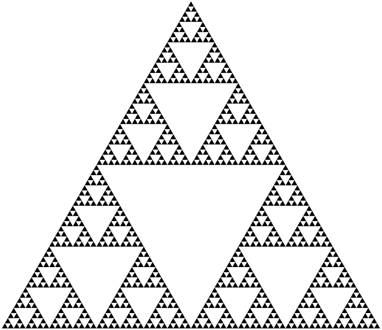
Every (bounded) shape which contains “a solid part” is self-similar; some shape without “a solid part” is not self-similar.

Sierpiński Triangle: A Non-Solid Example

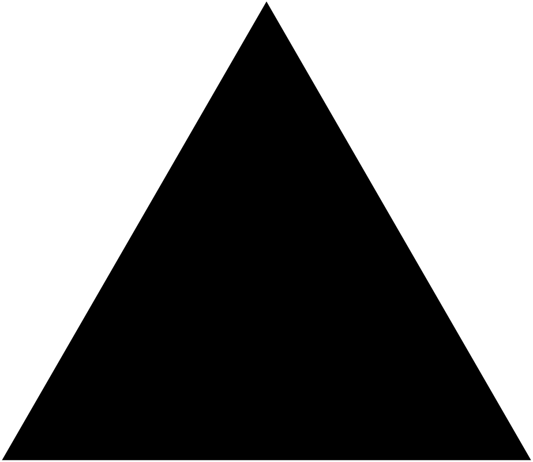


Wacław Sierpiński: Polish Mathematician

va-tswaf share-pin-ski (English approximation)

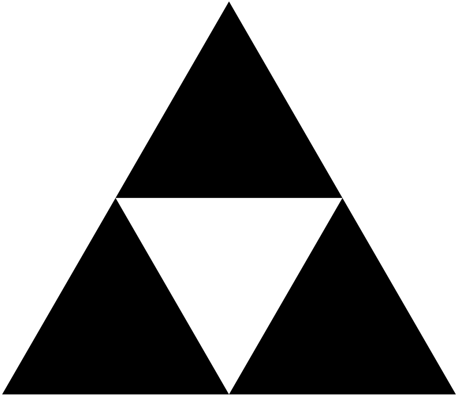
Sierpiński Triangle: Construction

Step 0: Start with a solid triangle (with its boundary).



Sierpiński Triangle: Construction

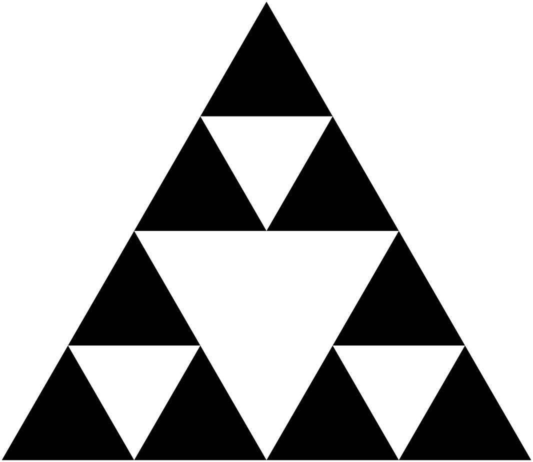
Step 1: Find the midpoint of each side. Remove the inner triangle formed (keeping all boundaries). Get 3 solid triangles.



Question: What does the boundary of the shape look like? Is it self-similar?

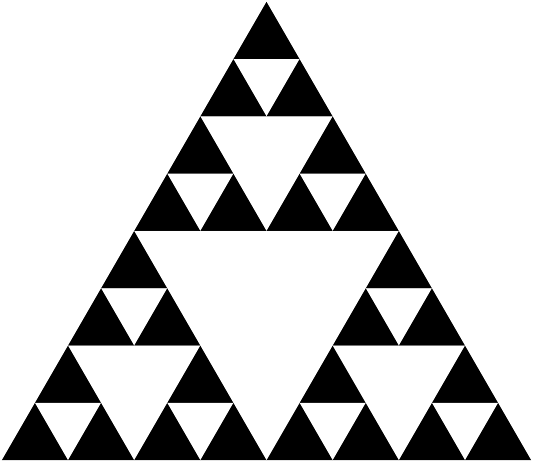
Sierpiński Triangle: Construction

Step 2: Do the same for the 3 solid sub-triangles formed by the last step. Get $9 = 3^2$ triangles.



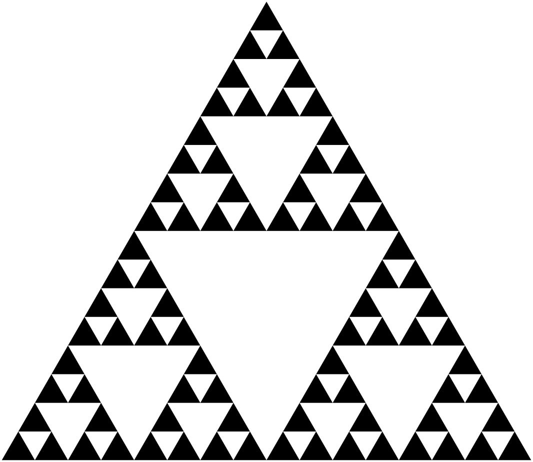
Sierpiński Triangle: Construction

Step 3: Do the same for the 3^2 solid sub-triangles formed by the last step. Get 3^3 triangles.



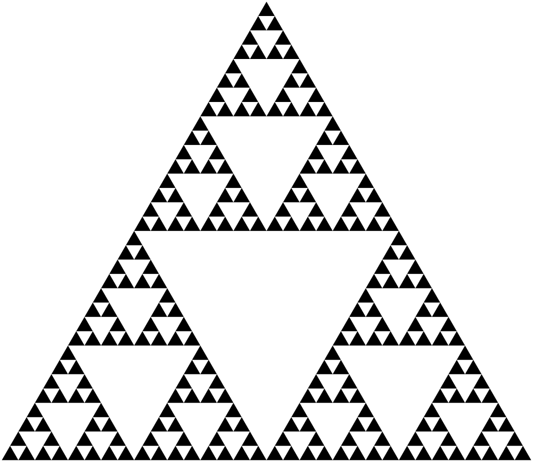
Sierpiński Triangle: Construction

Step 4: Do the same for the 3^3 solid sub-triangles formed by the last step. Get 3^4 triangles.



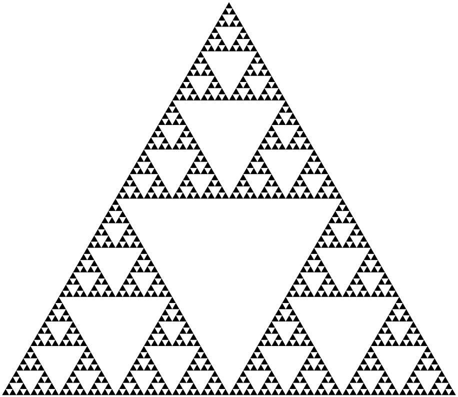
Sierpiński Triangle: Construction

Step 5: Do the same for the 3^4 solid sub-triangles formed by the last step. Get 3^5 triangles.



Sierpiński Triangle: Construction

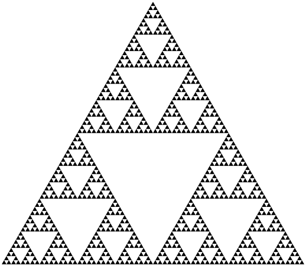
Step 6: Do the same for the 3^5 solid sub-triangles formed by the last step. Get 3^6 triangles.



Question: Is the boundary of the shape self-similar?

Sierpiński Triangle: Construction

Step ∞: Do this for infinitely many times.



- Is there any solid part remaining?
- What is its boundary? Is it self-similar?
- What is the "area" of the remaining figure?

Mystery Figure: Is There Something or Nothing?

We will need the following formula for an infinite sum of geometric sequence:

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}, \text{ if } 0 < r < 1.$$

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Assume the original triangle has area 1.

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$$3^{n-1} \times \left(\frac{1}{4}\right)^n = \frac{1}{4} \times \left(\frac{3}{4}\right)^{n-1}.$$

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  Nothing left? But at least the 3 sides of the original triangle are still there! (Recall we have kept all boundaries)

Preliminaries for the Lectures

1. Sum of geometric sequences
2. Intervals on the real line
3. Logarithmic functions
4. Basic set theory
5. Base *n* Digit expansions

1. Sum of Geometric Sequences: Derivation

We will first prove the partial sum formula

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

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As n goes to infinity, the term r^{n+1} in the partial sum becomes negligible since $0 < r < 1$.

Exercise

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   A Good Picture

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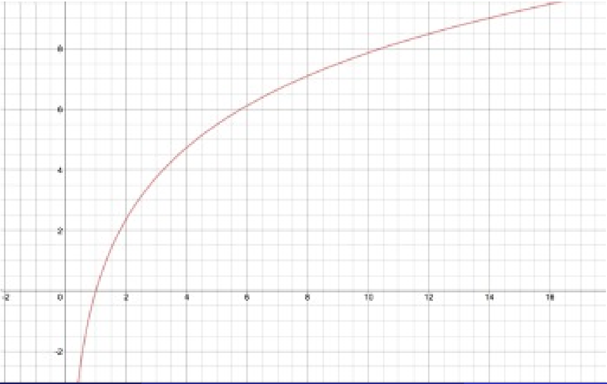
- We have the following formulas (where $x, y > 0$, n is an integer):

$$\ln(xy) = \ln x + \ln y, \quad \ln(x^n) = n \ln x, \quad \ln(x^{-1}) = -\ln x.$$

2. Logarithmic Functions: A Quick Review

We have:

$$\ln x = \begin{cases} \text{is not defined, if} x < 0 \\ < 0, \ \text{if } 0 < x < 1 \\ = 0, \ \text{if} x = 1 \\ > 0, \ \text{if} x > 1 \end{cases}.$$



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2. Simplify
$$\frac{\ln(2^n) + \ln(2 \times 3^n)}{\ln(6^n)},$$
where n is a natural number.

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- Similar definitions apply to $[a, b)$, $(a, b]$. We call them half-open-half-closed intervals.

The above are called bounded intervals.

3. Intervals on the Real Line: A Quick Review

- The interval (a, ∞) is equal to the set of all real numbers c satisfying $a < c$. This is an unbounded open interval.
- The interval $[a, \infty)$ is equal to the set of all real numbers c satisfying $a \leq c$. This is an unbounded closed interval.
- Similar definitions apply to $(-\infty, b)$, $(-\infty, b]$.

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3. $A \backslash B$ denotes the set of all elements that lie in A but not B.

Exercise

1. Find $[0,1) \cap (0.5,2)$.
2. Find $[0,1] \backslash (1/3,2/3)$.
3. Find

$$\Big([0,1/4] \cup [3/4,1]\Big) \backslash \Big((1/16,3/16) \cup (13/16,15/16)\Big).$$

5. Base n Digit Expansions

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Conventionally, if a number is in base 10, we omit the subscript $_{(10)}$.

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Theorem 0.1

Let $0 \leq x < 1$ be a real number and $n \geq 2$ be a natural number. Then there exists a (possibly infinite) sequence of numbers a_k $(k = 1, 2, 3, \dots)$, each one of them being one among $0, 1, 2, 3, \dots, n-1$, such that

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- The number 10 is nothing special at all! We are used to the base 10 systems mostly because we have 10 fingers!

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Therefore, $0.34375 = 0.112_{(4)}$.

Long Division

$$
\begin{array}{r l}
 & \color{red}{0.}112 \\
0.25 \,\big)\!\! & \overline{0.34375} \\
 & 0.25 \quad \color{blue}{= 0.25 \times 1} \\
 & \overline{0.09375} \\
 & 0.06250 \color{blue}{= 0.25^2 \times 1} \\
 & \overline{0.03125} \\
 & 0.03125 \color{blue}{= 0.25^3 \times 2} \\
 & \overline{0}
\end{array}
$$

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$$
\begin{array}{r|l l}
 & 0.1111\cdots & \\
1/3 & 1/2 & \\
 & 1/3 & = 1/3 \times 1 \\
\hline
 & 1/6 & \\
 & 1/9 & = (1/3)^2 \times 1 \\
\hline
 & 1/18 & \\
 & 1/27 & = (1/3)^3 \times 1 \\
\hline
 & 1/54 & \\
 & 1/81 & = (1/3)^4 \times 1 \\
\hline
 & 1/162 & \\
 & \cdots\cdots &
\end{array}
$$

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We guess that $1/2 = 0.11111111\cdots_{(3)}$.
Proof: we work backwards:

$$\begin{aligned} 0.1111\cdots_{(3)} &= 1 \times 3^{-1} + 1 \times 3^{-2} + 1 \times 3^{-3} + \cdots \\ &= (1/3) + (1/3)^2 + (1/3)^3 + \cdots \\ &= \frac{1}{1 - 1/3} = 1/2. \end{aligned}$$

More Examples!

Come up with more questions like this!

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Suggestion: work on base 2 and base 3 expansions first.

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Example: Write 0.314 into base 3 expansion.

The End