Equivalence of decoupling constants

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The goal of this note is to give rigorous argument for the equivalence of various formulations of decoupling, which is a heuristics commonly used in decoupling theory. For simplicity of notation we will only consider the planar case. The technical treatment resembles that of [2] and [4], but our result generalises the counterparts in their articles.

1 Definition and notation

1.1 Convention and notation

In this note we use the following conventions and notations.

1. Throughout this article we say a Schwartz function $\eta : \mathbb{R}^2 \to \mathbb{C}$ is good if $1 \le |\eta(x)| \le 2$ on $[-C, C]^2$ and $\hat{\eta}$ is supported on $[-C', C']^2$, where C, C' are absolute constants. In the following, C will always represent an arbitrary absolute constant, and its value may vary from line to line.

2. Let $T \subseteq \mathbb{R}^2$ be an axis-parallel rectangle centred at (x_0, y_0) with base a and height b. We define a Schwartz function η_T adapted to T by
$$\eta_T(x, y) = \eta\left(\frac{x - x_0}{a}, \frac{y - y_0}{b}\right).$$
Also, for an integer $E \ge 100$ we define a weight function $w_{T,E}$ by
$$w_{T,E}(x, y) = \left(1 + \frac{|x - x_0|}{a} + \frac{|y - y_0|}{b}\right)^{-E}.$$
Note that $\eta_T \lesssim_E w_{T,E}$. The lower bound 100 is not important here, and in most applications we only care about large E's.

   In particular, this definition applies to an axis parallel square $B \subseteq \mathbb{R}^2$. Throughout the text we assume all squares B are axis-parallel, unless otherwise specified.

3. Throughout the text we let $\phi : [0, 1] \to \mathbb{R}$ be a C^2-function. A δ-neighbourhood of the graph of ϕ over some interval I, denoted by $\mathcal{N}_I^\phi(\delta)$, will always refer to the vertical neighbourhood, unless otherwise specified. In symbols,
$$\mathcal{N}_I^\phi(\delta) = \{(s, t) : s \in I, |t - \phi(s)| < \delta\}.$$
Also, for any function $f : \mathbb{R}^2 \to \mathbb{C}$ and any interval $I \subseteq \mathbb{R}$, we denote by f_I the Fourier restriction of f to the strip $I \times \mathbb{R}$:
$$\hat{f}_I(s, t) = \hat{f}(s, t)1_I(s).$$
Note that for $f \in L^p(\mathbb{R}^2)$ and $1 < p < \infty$, we have
$$\|f_I\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)},$$
by the boundedness of the Hilbert transform.

4. A partition of $[0, 1]$ in this note will be a finite collection of closed intervals with disjoint interiors.

1.2 Various formulations of decoupling

Definition 1 (Extension operator)**.** *Fix a C^2-function $\phi : [0,1] \to \mathbb{R}$. We define the extension operator $\mathcal{E}^\phi$ by*

$$\mathcal{E}^\phi g(x,y) = \int_0^1 g(s)e(xs + y\phi(s))ds,$$

for $g \in L^1([0,1])$. Here and throughout the text we write $e(z) := \exp(2\pi iz)$.

With this, we can formulate the first version of decoupling, which will be referred to as the extension operator formulation.

Definition 2. *Let $\mathcal{P}_\delta$ be a partition of $[0,1]$. For $0 < p, q \leq \infty$, we let $D^\phi_{l^q(L^p),E,}(\mathcal{P}_\delta)$ be the best constant such that for any $g \in L^1([0,1])$ and any square B with side length δ^{-1} we have*

$$\left\|\mathcal{E}^\phi g\right\|_{L^p(B)} \leq D^\phi_{l^q(L^p),E}(\mathcal{P}_\delta)\left\|\left\|\mathcal{E}^\phi g_I\right\|_{L^p(w_{B,E})}\right\|_{l^q(I\in\mathcal{P}_\delta)},$$

where $g_I = g1_I$.

For a good Schwartz function η defined in the beginning of this section, we also define the constant $D^\phi_{l^q(L^p),\eta}(\mathcal{P}_\delta)$ in the same way as above, except that the $w_{B,E}$ on the right hand side is replaced by η_B.

Remark. If $p \geq q$ and we have $w_{B,E}$ on the right hand side, then by Minkowski's inequality the left hand side can be replaced by $\left\|\mathcal{E}^\phi g\right\|_{L^p(w_{B,E})}$ without necessarily changing the decoupling constant. (See also Section 4 of [2].) However, if we have η_B on the right hand side instead, then this is not obvious and may even fail.

We also have various neighbourhood versions of decoupling. The following one will be referred to as the global neighbourhood version.

Definition 3. *Let $\mathcal{P}_\delta$ be a partition of $[0,1]$. For $0 < p, q \leq \infty$ and $\tau > 0$, let $G^\phi_{l^q(L^p),\tau}(\mathcal{P}_\delta)$ be the best constant such that for any f with Fourier support in $\mathcal{N}^\phi_{[0,1]}(\tau\delta)$, we have*

$$\|f\|_{L^p(\mathbb{R}^2)} \leq G^\phi_{l^q(L^p),\tau}(\mathcal{P}_\delta)\left\|\|f_I\|_{L^p(\mathbb{R}^2)}\right\|_{l^q(I\in\mathcal{P}_\delta)}.$$

Similarly, we have the corresponding local neighbourhood version of decoupling.

Definition 4. *Let $\mathcal{P}_\delta$ be a partition of $[0,1]$. For $0 < p, q \leq \infty$ and $\tau > 0$, let $L^\phi_{l^q(L^p),\tau,E}(\mathcal{P}_\delta)$ be the best constant such that for any f with Fourier support in $\mathcal{N}^\phi_{[0,1]}(\tau\delta)$ and any square B with side length δ^{-1}, we have*

$$\|f\|_{L^p(B)} \leq L^\phi_{l^q(L^p),\tau,E}(\mathcal{P}_\delta)\left\|\|f_I\|_{L^p(w_{B,E})}\right\|_{l^q(I\in\mathcal{P}_\delta)}.$$

For a good Schwartz function η defined in the beginning of this section, we also define the constant $L^\phi_{l^q(L^p),\tau,\eta}(\mathcal{P}_\delta)$ in the same way as above, except that the $w_{B,E}$ on the right hand side is replaced by η_B.

Remark. By a modulation of f in the frequency space, in the local neighbourhood version we may always take B to be centred at 0. Also, similar to the remark after Definition 2, if $p \geq q$ and we have $w_{B,E}$ on the right hand side, then we may replace the left hand side by $\|f\|_{L^p(w_{B,E})}$ without necessarily changing the decoupling constant.

2 Relation between decoupling constants

The main goal of this article is to study the relations between the decoupling constants defined as above.

Let $\phi : [0,1] \to \mathbb{R}$ be C^2, $0 < \delta \leq 1$, $\mathcal{P}_\delta$ be a partition of $[0,1]$ depending on ϕ, δ. Also, let E, p, q, η be given as above. To make the notation simpler, we will do the following reductions. It turns out that in all the following propositions and theorems, the scale δ, the partition $\mathcal{P}_\delta$ and the Lebesgue exponents p, q

will be fixed at the beginning of the assumption and will not change in their proofs. Hence, we will drop $\mathcal{P}_\delta$ and the subscript $l^q(L^p)$ in the decoupling constants. For instance, $L^\phi_{l^q(L^p),C,E}(\mathcal{P}_\delta)$ will simply be reduced to $L^\phi_{C,E}$ and $D^\phi_{l^q(L^p),\eta}(\mathcal{P}_\delta)$ will become D^ϕ_η. Moreover, as we proceed to obtain more equivalence theorems in the following sections, we will keep dropping more subscripts and superscripts to make the notation even simpler.

Now we come to our first easy observation, namely, the decoupling constants are essentially the same when the thickness of the neighbourhood is multiplied by an absolute constant. The simple proof is given in Section 4.1.

Proposition 5. *For any $\tau_2 > \tau_1 > 0$ we have*

$$L^\phi_{\tau_1,E} \sim L^\phi_{\tau_2,E}, \quad G^\phi_{\tau_1} \sim G^\phi_{\tau_2}.$$

The implicit constants here depend only on $\tau_2/\tau_1, p, E$.

Reduction of notation. By this proposition, from now on we will drop the thickness τ in the decoupling constants $L^\phi_{\tau,E}$ and G^ϕ_τ (but not $L^\phi_{\tau,\eta}$) to make the notation even simpler. In this reduced notation we will implicitly assume $\tau = 1$.

The following is our first main theorem on the relation of the decoupling constants. The proof is given in Section 5.

Theorem 6. *If $q \le p$, $1 < p < \infty$[1] and every interval I in the partition $\mathcal{P}_\delta$ has length at least δ, then for any $\tau > 0$ we have*

$$D^\phi_E \lesssim D^\phi_\eta \lesssim L^\phi_{\tau,\eta} \lesssim G^\phi \lesssim L^\phi_E.$$

The implicit constants here only depend on $\phi, \tau, E, \eta, p, q$ but will never depend on δ and the actual choice of the partition $\mathcal{P}_\delta$ (as long as the assumption holds).

2.1 Generalisation to higher dimensions

In $\mathbb{R}^n$, one can similarly define all the decoupling constants for a surface given by a C^2-function ϕ over $[0,1]^{n-1}$, where $\mathcal{P}_\delta$ is a partition of $[0,1]^{n-1}$ into axis-parallel rectangles. If p, q are in the same range of Theorem 6 and all dimensions of each rectangle are at least δ, then it turns out that the proof of Theorem 6 can be easily modified to higher dimensional cases, yielding the same chain of inequalities as in Theorem 6. The details are left to the reader.

3 The main equivalence theorem

Our final goal in this text is to show that under some general conditions, all the above decoupling constants are equivalent. In view of Theorem 6, if we are able to show $L^\phi_E \lesssim D^\phi_E$, then we are done. For this purpose, we need some regularity on ϕ and the partition $\mathcal{P}_\delta$ of $[0,1]$. (Compare the formulation below to that of the sub-admissible partitions in [5].)

Proposition 7. *Let $E \ge 100$, and suppose $\phi \in C^E([0,1])$. Let $0 < \delta \le 1$ and assume that for every $I \in \mathcal{P}_\delta$ and every $2 \le k \le E$ we have $|I| \ge \delta$ and*

$$\sup_{s \in I} |\phi^{(k)}(s)||I|^k \lesssim_\phi \delta. \tag{1}$$

Then for any $1 \le p < \infty$ we have

$$L^\phi_E \lesssim D^\phi_{2E+2},$$

where the implicit constant only depend on ϕ, E, p but will never depend on q, δ and the actual choice of the partition $\mathcal{P}_\delta$ (as long as the assumption holds).

[1]Note that the theorem does not cover the case $p = \infty$. But in practice, all decoupling constants given by the triangle inequality and Hölder will usually be sharp at $p = \infty$. Also, if $p < q$, all decoupling inequalities are trivial. Indeed, by triangle inequality and Hölder followed by an interpolation (see here) we get a trivial upper bound for the decoupling constants. On the other hand, by taking each f_I with sparse physical support, we get a lower bound which essentially coincides with the upper bound.

The proof will be given in Section 6. As a corollary, we finally arrive at our main equivalence theorem of decoupling constants.

Theorem 8 (The equivalence theorem)**.** *Assume the conditions of Proposition 7. Then for $E, F \geq 300$, $q \leq p$, $1 < p < \infty$ and $\tau > 0$ we have*

$$D_F^\phi \sim D_E^\phi \sim D_\eta^\phi \sim L_{\tau,\eta}^\phi \sim G^\phi \sim L_E^\phi \sim L_F^\phi.$$

The implicit constants here only depend on $\phi, \tau, E, F, \eta, p, q$ but will never depend on δ and the actual choice of the partition $\mathcal{P}_\delta$ (as long as the assumption holds).

Proof. By Theorem 6 and Proposition 7, for all $E \geq 100$ we have

$$L_E^\phi \lesssim D_{2E+2}^\phi \lesssim D_\eta^\phi \lesssim L_{\tau,\eta}^\phi \lesssim G^\phi \lesssim L_E^\phi,$$

and so all the above $\lesssim$ can be replaced by $\sim$. In particular, $L_E^\phi \sim G^\phi$ for all $E \geq 100$. Hence $L_E^\phi \sim L_F^\phi$ for $E, F \geq 100$. Similarly, $D_{2E+2}^\phi \sim G^\phi$, and so $D_F^\phi \sim D_E^\phi$ for $E, F \geq 300$. Thus we have the chain of equivalences. ☐

To familiarise the reader of the assumptions of Proposition 7, we have the following simple examples.

1. If $\phi \in C^E([0,1])$ and ϕ'' does not vanish on $[0,1]$, and $\mathcal{P}_\delta$ is given by the standard partition of $[0,1]$ into intervals of equal length $\delta^{1/2}$ where $\delta \in \mathbb{N}^{-2}$, then the assumptions of the proposition hold.
2. For $\phi(s) = s^r$, $r \geq 3$ and $\delta \in \mathbb{N}^{-2r}$, let
$$a_j = j^{\frac{2}{r}} \delta^{\frac{1}{r}},$$
and $\mathcal{P}_\delta := \{[a_{j-1}, a_j] : 1 \leq j \leq \delta^{-1/2}\}$. Then the assumptions of the proposition hold. Indeed, it suffices to note that the length of the interval $[a_{j-1}, a_j]$ obeys
$$a_j - a_{j-1} \sim j^{\frac{2}{r}-1} \delta^{\frac{1}{r}}.$$

It turns out that the assumptions of Proposition 7 hold for a large family of functions, including all polynomial functions of degree at most E.

We need a little more terminology to state the result in full generality. Let ϕ be an analytic function such that ϕ'' does not vanish identically on $[0,1]$. Then ϕ'' has at most finitely many zeros $z_i \in [0,1]$, and each zero is of some finite order n_i. The maximum of the n_i's will be called the *maximum order of vanishing of ϕ''*. (If ϕ'' does not vanish, we just set its maximum order of vanishing to be 0.) See also [1] and Section 12.6 of [3]). With this, we have

Lemma 9. *Let ϕ be an analytic function such that ϕ'' vanishes to the order at most $E - 2$ on $[0,1]$. Let $I \in \mathcal{P}_\delta$ and suppose that (1) holds for $k = 2$. Then (1) holds for all $2 \leq k \leq E$. Hence, if in addition we have $|I| \geq \delta$ for every $I \in \mathcal{P}_\delta$, then the assumptions of Proposition 7 hold.*

Proof. Let $\psi = \phi''$. Then there are finitely many closed intervals J partitioning $[0,1]$ such that on each J, ψ admits an infinite series expansion. Without loss of generality, each J contains at most one zero z of ψ of some order $l \leq E - 2$. Hence, on J we can write

$$\psi(s) = \sum_{j=l}^{\infty} a_j (s - z)^j,$$

where $a_l \neq 0$. Since ψ does not have any other zero on J, we have $|\psi(s)| \gtrsim |s - z|^l$ for all $s \in J$.

Let $I \in \mathcal{P}_\delta$, and without loss of generality, assume $I \subseteq J$. Thus, for $s \in J$ we have

$$\sup_{s \in I} |\phi''(s)| = \sup_{s \in I} |\psi(s)| \gtrsim |I|^l.$$

Using (1) for $k = 2$, we then have $|I| \lesssim \delta^{\frac{1}{l+2}} \leq \delta^{1/E}$.

Now let $2 \leq k \leq E$. By the series representing ψ, we also have $|\psi^{(k-2)}(s)| \lesssim |s - z|^{l-k+2}$, from which we have

$$\sup_{s \in I} |\phi^{(k)}(s)| = \sup_{s \in I} |\psi^{(k-2)}(s)| \lesssim |I|^{l-k+2}.$$

Using $|I| \lesssim \delta^{1/E}$ we just obtained and $l \leq E - 2$, we thus have (1) for k. ☐

3.1 Generalisation to higher dimensions

One may ask if Proposition 7 generalises to $\mathbb{R}^n$. Indeed, suppose in addition that we are in the special case where

$$\phi(s_1, \dots, s_{n-1}) = \phi_1(s_1) + \cdots + \phi_{n-1}(s_{n-1})$$

and $\phi_i \in C^E([0,1])$. Given a partition $\mathcal{P}_\delta$ of $[0,1]^{n-1}$ given by the Cartesian product of $n-1$ partitions $\mathcal{P}_\delta^i$ each satisfying the assumptions in Proposition 7. Then for any $1 \le p < \infty$ we again have

$$L_E^\phi \lesssim D_F^\phi,$$

for some suitable $F = F(E)$. The detail is left to the reader.

The case for a general ϕ is much more subtle, so we leave it open.

4 Two preliminary equivalences

In this section we prove two easy equivalences of decoupling constants. First, we give a proof of Proposition 5 on the equivalence of comparable thickness of the neighbourhoods. Next we prove another equivalence theorem on the decoupling constants, namely, adding to ϕ a linear function with bounded slope does not essentially affect some decoupling constants. This fact will also be used in the proof of Proposition 7.

4.1 Changing thickness of neighbourhoods

Here we give the simple proof of Proposition 5.

Proof. We only prove the equivalence for $L_{\tau,E}^\phi$ as the global counterpart is even easier. We also only need to show $L_{\tau_1,E}^\phi \gtrsim L_{\tau_2,E}^\phi$ as the other side is trivial.

The proof uses a very trivial scaling argument. Let f have Fourier support on $\mathcal{N}_I^\phi(\tau_2\delta)$, and let B be the square centred at 0 with side length δ^{-1}. Let $g(x,y) = f(x, \frac{\tau_1}{\tau_2}y)$, so that g has Fourier support on $\mathcal{N}_I^\phi(\tau_1\delta)$. (This is why we choose vertical neighbourhoods.) Then we can apply the definition of $L_{\tau_1,E}^\phi$ to get

$$\|g\|_{L^p(B)} \le L_{\tau_1,E}^\phi \Big\| \|g_I\|_{L^p(w_{B,E})} \Big\|_{l^q(I \in \mathcal{P}_\delta)}.$$

But then $\|g\|_{L^p(B)} \sim \|f\|_{L^p(T)}$ where T is the axis-parallel rectangle centred at 0 with base δ^{-1} and height $\frac{\tau_2}{\tau_1}\delta^{-1}$, so in particular $\|f\|_{L^p(T)} \ge \|f\|_{L^p(B)}$. On the right hand side, $\|g_I\|_{L^p(w_{B,E})} \sim \|f_I\|_{L^p(w_T)} \sim \|f_I\|_{L^p(w_{B,E})}$ since $w_T \sim w_{B,E}$. Thus we also have

$$\|f\|_{L^p(B)} \lesssim L_{\tau_1,E}^\phi \Big\| \|f_I\|_{L^p(w_{B,E})} \Big\|_{l^q(I \in \mathcal{P}_\delta)}.$$

Thus we have $L_{\tau_2,E}^\phi \lesssim L_{\tau_1,E}^\phi$. □

Now we can drop again the thickness τ in the notation as we did after Proposition 5.

4.2 Adding a linear function

In this section we prove the following proposition.

Proposition 10. *Let $\psi(s) = \phi(s) + as + b$ where $|a| \le C$. Then we have*

$$D_E^\phi \sim D_E^\psi, \quad G^\phi = G^\psi, \quad L_E^\phi \sim L_E^\psi.$$

The implicit constants here depend on p, E only.

Proof. We give a proof of the case $L_E^\phi \sim L_E^\psi$ as the proof of $D_E^\phi \sim D_E^\psi$ is similar and the proof of $G^\phi = G^\psi$ is easier. We will only prove the $\lesssim$ direction and the other side follows by symmetry.
Let f have Fourier support in $\mathcal{N}^\phi_{[0,1]}(\delta)$ and B be a square of side length δ^{-1}. We will show that

$$\|f\|_{L^p(B)} \lesssim L_E^\psi \Big\| \|f_I\|_{L^p(w_{B,E})} \Big\|_{l^q(I\in\mathcal{P}_\delta)}.$$

Let g be defined by the relation

$$\hat{g}(s,t) = \hat{f}(s, t - as - b),$$

which is supported in $\mathcal{N}^\psi_{[0,1]}(\delta)$. We also have for all $I \in \mathcal{P}_\delta$

$$g_I(x,y) = e(by) f_I(x + ay, y).$$

Then we can apply the definition of L_E^ψ to get

$$\|g\|_{L^p(B)} \lesssim L_{C,E}^\psi \Big\| \|g_I\|_{L^p(w_{cB,E})} \Big\|_{l^q(I\in\mathcal{P}_\delta)}.$$

Since $|a| \le C$, we have $\|g\|_{L^p(B)} \ge \|f\|_{L^p(C^{-1}B)}$. On the right hand side, we have $\|g_I\|_{L^p(w_{B,E})} \gtrsim \|f_I\|_{L^p(w_{B,E})}$. Thus

$$\|f\|_{L^p(C^{-1}B)} \lesssim L_E^\psi \Big\| \|g_I\|_{L^p(w_{B,E})} \Big\|_{l^q(I\in\mathcal{P}_\delta)}.$$

But the above holds for all B. Covering B by $O(1)$ many $C^{-1}B$'s, we are done. □

Reduction of notation. After this proposition, the phase functions ϕ will no longer change, and so we will drop the superscript ϕ from the decoupling constants below.

5 Local-global equivalence, easier relations

In this section, we will prove Theorem 6 step by step.

Proposition 11. *For $q \le p \le \infty$ and any $E \ge 100$ we have*

$$G \lesssim L_E.$$

The implicit constant here depends on p, E only.

Proof. Cover $\mathbb{R}^2$ by a tiling of squares B of side length δ^{-1}. We then have the weight inequality $\sum_B w_{B,E} \lesssim 1$. Hence, for $q \le p < \infty$ we have

$$\begin{aligned}
\|f\|_{L^p(\mathbb{R}^2)} &= \left(\sum_B \|f\|^p_{L^p(B)} \right)^{\frac{1}{p}} \\
&\le L_E \left(\sum_B \left(\sum_{I\in\mathcal{P}_\delta} \|f_I\|^q_{L^p(w_{B,E})} \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \\
\text{(by Minkowski)} &\le L_E \left(\sum_{I\in\mathcal{P}_\delta} \left(\sum_B \|f_I\|^p_{L^p(w_{B,E})} \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&\lesssim L_E \left(\sum_{I\in\mathcal{P}_\delta} \|f_I\|^q_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{q}}.
\end{aligned}$$

The case $p = \infty$ is immediate, since then $L^\infty(\mathbb{R}^2)$ equals $L^\infty(w_{B,E})$ (as w is everywhere positive). □

Proposition 12. *For any $E \geq 100$, we have $D_E \lesssim D_\eta$, where the implicit constant depends on E and η only. Also, if $1 < p < \infty$ and each $I \in \mathcal{P}_\delta$ has length at least δ, then for any $\tau > 0$, we have $D_\eta \lesssim L_{\tau,\eta}$, where the implicit constant depends on p, q, η, τ.*

Proof. The relation $D_E \lesssim D_\eta$ is trivial since $\eta_B \lesssim_E w_{B,E}$. To prove $D_\eta \lesssim L_{\tau,\eta}$, we remove the first and the last intervals of $\mathcal{P}_\delta$ and call the resulting collection $\mathcal{P}'_\delta$. By the triangle and Hölder's inequalities, it then suffices to decouple with respect to $\mathcal{P}'_\delta$ only, namely, it suffices to prove that for any $g \in L^1([0,1])$ and any square B with side length δ^{-1} we have

$$\|\mathcal{E}g'\|_{L^p(B)} \lesssim L_{\tau,\eta}\Big\|\|\mathcal{E}g_I\|_{L^p(\eta_B)}\Big\|_{l^q(I\in\mathcal{P}_\delta)},$$

where $g' = g1_{\cup\mathcal{P}'_\delta}$.

Consider the function $\mathcal{E}g'\eta_B$, whose Fourier transform equals a function $\widehat{\mathcal{E}g'} * \widehat{\eta_B}$. By the assumption on the partition, the left endpoint of $\cup\mathcal{P}'_\delta$ will be bounded away from 0 by at least δ, and the same holds for the right endpoint. Hence, by the support condition of η, $\widehat{\mathcal{E}g'} * \widehat{\eta_B}$ is a function supported within the strip $0 \leq s \leq 1$. Moreover, if the absolute constant C' in the definition of η is chosen small enough (depending on τ), then $\widehat{\mathcal{E}g'} * \widehat{\eta_B}$ is supported on $\mathcal{N}^{\phi}_{[0,1]}(\tau\delta)$. Hence, we may apply the definition of $L_{\tau,\eta}$ to get

$$\|\mathcal{E}g'\eta_B\|_{L^p(B)} \leq L_{\tau,\eta}\Big\|\|(\mathcal{E}g'\eta_B)_I\|_{L^p(\eta_B)}\Big\|_{l^q(I\in\mathcal{P}'_\delta)}.$$

For the left hand side, we have $\|\mathcal{E}g'\eta_B\|_{L^p(B)} \geq \|\mathcal{E}g'\|_{L^p(B)}$.

The right hand side is trickier. By definition, for each $I \in \mathcal{P}'_\delta$,

$$((\mathcal{E}g'\eta_B)_I)^\wedge(s,t) = \widehat{\mathcal{E}g'\eta_B}(s,t)1_I(s) = \widehat{\mathcal{E}g'} * \widehat{\eta_B}(s,t)1_I(s).$$

Using the support of $\widehat{\eta_B}$ and the assumption on the partition, we have

$$\widehat{\mathcal{E}g'} * \widehat{\eta_B}(s,t)1_I(s) = \widehat{\mathcal{E}g_{\tilde{I}}} * \widehat{\eta_B}(s,t)1_I(s)$$

where $\tilde{I}$ is the union of I and all intervals in $\mathcal{P}_\delta$ adjacent to I. Here we also used the fact that $\widehat{\mathcal{E}g_I}$ is a distribution supported on the graph of ϕ above I for all $I \subseteq [0,1]$. Thus

$$(\mathcal{E}g'\eta_B)_I = (\mathcal{E}g_{\tilde{I}}\eta_B)_I,$$

from which we have

$$\|(\mathcal{E}g'\eta_B)_I\|_{L^p(\eta_B)} = \|(\mathcal{E}g_{\tilde{I}}\eta_B)_I\|_{L^p(\eta_B)} \lesssim \|(\mathcal{E}g_{\tilde{I}}\eta_B)_I\|_{L^p(\mathbb{R}^2)} \lesssim \|\mathcal{E}g_{\tilde{I}}\eta_B\|_{L^p(\mathbb{R}^2)},$$

where in the last inequality we have used the boundedness of the Fourier multiplier $(s,t) \mapsto 1_I(s)$ and the assumption that $1 < p < \infty$. Thus we have

$$\begin{aligned}\Big\|\|(\mathcal{E}g'\eta_B)_I\|_{L^p(\eta_B)}\Big\|_{l^q(I\in\mathcal{P}'_\delta)} &\lesssim \Big\|\|\mathcal{E}g_{\tilde{I}}\eta_B\|_{L^p(\mathbb{R}^2)}\Big\|_{l^q(I\in\mathcal{P}'_\delta)} \\ &\lesssim \Big\|\|\mathcal{E}g_I\eta_B\|_{L^p(\mathbb{R}^2)}\Big\|_{l^q(I\in\mathcal{P}_\delta)} \\ &\lesssim \Big\|\|\mathcal{E}g_I\|_{L^p(\eta_B)}\Big\|_{l^q(I\in\mathcal{P}_\delta)},\end{aligned}$$

where in the second inequality we have used the triangle and Hölder's inequalities. $\square$

Proposition 13. *If $1 < p < \infty$, $\tau > 0$ and each $I \in \mathcal{P}_\delta$ has length at least δ, then we have $L_{\tau,\eta} \lesssim G$, where the implicit constant depends on $\|\phi'\|_\infty$ as well as p, q, η, τ.*

Proof. Let f have Fourier support in $\mathcal{N}^{\phi}_{[0,1]}(\tau\delta)$, and let B be a square of side length δ^{-1}. Remove the first and the last intervals of $\mathcal{P}_\delta$ and call the resulting collection $\mathcal{P}'_\delta$. Let f' be the Fourier restriction of f to $\cup\mathcal{P}'_\delta \times \mathbb{R}$. It suffices to prove

$$\|f'\|_{L^p(B)} \lesssim G\Big\|\|f_I\|_{L^p(\eta_B)}\Big\|_{l^q(I\in\mathcal{P}_\delta)}.$$

We start from the left hand side. Since $\|f'\|_{L^p(B)} \lesssim \|f'\eta_B\|_{L^p(\mathbb{R}^2)}$, it suffices to study $f'\eta_B$. Note that $\widehat{f'\eta_B}$ is supported on the Minkowski sum of a square with side length δ and $\mathcal{N}^{\phi}_{\cup\mathcal{P}'_\delta}(\tau\delta)$, which is contained in $\mathcal{N}^{\phi}_{[0,1]}(\tau'\delta)$ where $\tau' = \tau + 1 + \|\phi'\|_\infty$. Using the definition of G, we have

$$\|f'\|_{L^p(B)} \lesssim \|f'\eta_B\|_{L^p(\mathbb{R}^2)} \leq G\Big\|\|(f'\eta_B)_I\|_{L^p(\mathbb{R}^2)}\Big\|_{l^q(I\in\mathcal{P}'_\delta)}.$$

By similar argument as in the proof of Proposition 12, we have for all $I \in \mathcal{P}'_\delta$

$$(f'\eta_B)_I = (f_{\tilde{I}}\eta_B)_I,$$

where $\tilde{I}$ is the union of I and all intervals in $\mathcal{P}_\delta$ adjacent to I. Thus, as $1 < p < \infty$, we also have

$$\|(f'\eta_B)_I\|_{L^p(\mathbb{R}^2)} = \|(f_{\tilde{I}}\eta_B)_I\|_{L^p(\mathbb{R}^2)} \lesssim \|f_{\tilde{I}}\eta_B\|_{L^p(\mathbb{R}^2)} \lesssim \|f_{\tilde{I}}\|_{L^p(\eta_B)}.$$

The result then follows from the triangle and Hölder's inequalities. $\square$

6 Local-global equivalence, continued

In this section we prove Proposition 7, by modifying the tedious Fourier analytic proof of Section 5 of [2].

Proof of Proposition 7. Let $E \geq 100$ and ϕ, $\mathcal{P}_\delta$ be given such that for every $I \in \mathcal{P}_\delta$ and every $2 \leq k \leq E$ we have $|I| \geq \delta$ and

$$\sup_{s\in I} |\phi^{(k)}(s)||I|^k \lesssim_\phi \delta. \tag{2}$$

Our goal is to show that for any $1 \leq p < \infty$ we have $L_E \lesssim D_{2E+2}$, that is, for any f with Fourier support in $\mathcal{N}^{\phi}_{[0,1]}(\delta/4)$ and any square B of side length δ^{-1} centred at 0, we have

$$\|f\|_{L^p(B)} \lesssim D_{2E+2}\Big\|\|f_I\|_{L^p(w_{B,E})}\Big\|_{l^q(I\in\mathcal{P}_\delta)}.$$

Let $F = 2E + 2$. We may assume without loss of generality that $\hat{f}$ is smooth. By the Fourier support of f, we can write

$$f(x,y) = \int_0^1 \int_{-\delta/4}^{\delta/4} \hat{f}(s, \phi(s) + t)e(xs + y\phi(s) + yt)dtds.$$

6.1 Taylor expansions

The term $e(yt)$ prevents us from writing the right hand side in the form of $\mathcal{E}^\phi g$ for some g. But since $|y| \leq \delta^{-1}$ and $|t| \leq \delta/4$, we know that $e(yt)$ is "negligible". To make it rigorous, we use Taylor expansion to write

$$e(yt) = \sum_{j=0}^{\infty} \frac{(2\pi i y t)^j}{j!} = \sum_{j=0}^{\infty} \frac{(2\pi i y\delta)^j}{j!} t^j \delta^{-j}.$$

But for $(x, y) \in B$ we have $|y\delta| \leq 1$. Hence

$$\begin{aligned}
|f(x,y)| &\leq \sum_{j=0}^{\infty} \frac{(2\pi)^j}{j!} \left| \int_0^1 \int_{-\delta/4}^{\delta/4} \hat{f}(s, \phi(s) + t)e(xs + y\phi(s))t^j\delta^{-j} dsdt \right| \\
&= \sum_{j=0}^{\infty} \frac{(2\pi)^j}{j!} |\mathcal{E}g_j(x,y)|,
\end{aligned}$$

where $\mathcal{E}g := \mathcal{E}^\phi g$ for simplicity and

$$g_j(s) = \int_{-\delta/4}^{\delta/4} \hat{f}(s, \phi(s) + t)t^j\delta^{-j} dt.$$

Hence, by the triangle inequality, we have

$$\begin{aligned}\|f\|_{L^p(B)} &\le \sum_{j=0}^{\infty} \frac{(2\pi)^j}{j!} \|\mathcal{E}g_j\|_{L^p(B)} \\ &\le \sum_{j=0}^{\infty} \frac{(2\pi)^j}{j!} D_F \Big\| \|\mathcal{E}(g_j)_I\|_{L^p(w_{B,F})} \Big\|_{l^q(I\in\mathcal{P}_\delta)},\end{aligned}$$

where $(g_j)_I(s) = g_j(s)1_I(s)$. As a result, it suffices to prove that for all $I \in \mathcal{P}_\delta$,

$$\sup_j \|\mathcal{E}(g_j)_I\|_{L^p(w_{B,F})} \lesssim \|f_I\|_{L^p(w_{B,E})}.$$

Since $p < \infty$ we can write

$$\|\mathcal{E}(g_j)_I\|^p_{L^p(w_{B,F})} \sim \iint \|\mathcal{E}(g_j)_I\|^p_{L^p(B((u,v),\delta^{-1}))} \delta^2 w_{B,F}(u,v)dudv,$$

where $B((u,v),\delta^{-1})$ is the square centred at (u,v) with side length δ^{-1}.

Now let $(x,y) \in B((u,v),\delta^{-1})$. We use a change of variable to compute

$$\begin{aligned}&\mathcal{E}(g_j)_I(x,y) \\ &= \int_0^1 \int_{-\delta/4}^{\delta/4} \hat{f}(s,\phi(s)+t)e(xs+y\phi(s))t^j\delta^{-j}dsdt \\ &= \iint \hat{f}(\xi_1,\xi_2)(\xi_2-\phi(\xi_1))^j\delta^{-j}e((\phi(\xi_1)-\xi_2)y)e(x\xi_1+y\xi_2)d\xi_1 d\xi_2,\end{aligned}$$

using the Fourier support condition of f. Then we write

$$(\phi(\xi_1)-\xi_2)y = (\phi(\xi_1)-\xi_2)v + (\phi(\xi_1)-\xi_2)(y-v).$$

Apply another Taylor expansion to $e((\phi(\xi_1)-\xi_2)(y-v))$ to get

$$e((\phi(\xi_1)-\xi_2)(y-v)) = \sum_{k=0}^{\infty} \frac{(2\pi i)^k}{k!} (\xi_2-\phi(\xi_1))^k(v-y)^k.$$

Since $(x,y) \in B((u,v),\delta^{-1})$, we have $|v-y| \le \delta^{-1}$. Hence

$$\begin{aligned}&|\mathcal{E}(g_j)_I(x,y)| \le \sum_{k=0}^{\infty} \frac{(2\pi)^k}{k!} \\ &\cdot \left| \int_I \int \hat{f}(\xi_1,\xi_2)(\xi_2-\phi(\xi_1))^{j+k}\delta^{-j-k}e((\phi(\xi_1)-\xi_2)v)e(x\xi_1+y\xi_2)d\xi_2 d\xi_1 \right|.\end{aligned}$$

Renaming $j+k$ to j, it now remains to prove

$$\begin{aligned}&\iint \left\| \int_I \int \hat{f}(\xi_1,\xi_2)(\xi_2-\phi(\xi_1))^j\delta^{-j}e((\phi(\xi_1)-\xi_2)v)e(x\xi_1+y\xi_2)d\xi_2 d\xi_1 \right\|^p_{L^p(B((u,v),\delta^{-1}))} \\ &\cdot \delta^2 w_{B,F}(u,v)dudv \lesssim \|f_I\|^p_{L^p(w_{B,E})},\end{aligned}$$

uniformly in j.

6.2 Introducing a Fourier multiplier

In this subsection we will use the case $k=2$ of the assumption (2).

Let $\sigma : [-1/2,1/2] \to \mathbb{R}$ be a smooth function that equals 1 on $[-1/4,1/4]$. Write $I = [h,h+l]$. Then we have $\sigma(\frac{s-h}{4l}) = 1$ for all $s \in I$.

Let $M_j(t) = t^j\sigma(t)$ so that M_j is smooth, equals t^j on $[-1/4, 1/4]$ and is supported on $[-1/2, 1/2]$. Moreover, M_j satisfies the derivative bound

$$\sup_{j\geq 0}\left\|M_j^{(k)}\right\|_{L^\infty(\mathbb{R})} \lesssim_k 1. \tag{3}$$

Now since $|\xi_2 - \phi(\xi_1)| < \delta/4$, if we define $m_v = m_{v,j}$ by

$$m_v(\xi_1, \xi_2) = M_j(\delta^{-1}(\xi_2 - \phi(\xi_1)))\sigma\left(\frac{\xi_1 - h}{4l}\right)e(v\phi(\xi_1))$$

which is a smooth function with compact support, then we have

$$\begin{aligned}&\int_I\int \hat{f}(\xi_1,\xi_2)(\xi_2-\phi(\xi_1))^j\delta^{-j}e((\phi(\xi_1)-\xi_2)v)e(x\xi_1+y\xi_2)d\xi_2 d\xi_1\\ &= \iint \hat{f}_I(\xi_1,\xi_2)m_v(\xi_1,\xi_2)e((x\xi_1+(y-v)\xi_2)d\xi_2 d\xi_1\\ &= f_I * m_v^\vee(x, y-v).\end{aligned}$$

Using (2) for $k = 2$, it is easy to see that for each $I \in \mathcal{P}_\delta$, $\mathcal{N}_I^\phi(\delta/4)$ is contained in a rectangle T of dimensions $\sim l \times \delta$ (with sides being tangent and normal, respectively, to the graph of ϕ at some point in I). Thus $|m_v^\vee|$ is roughly a constant $l\delta$ on the dual rectangle T^* of dimensions $\sim l^{-1}\times\delta^{-1}$. As a result, we have $\|m_v^\vee\|_{L^1(\mathbb{R}^2)} \sim 1$.

Now since $p \geq 1$, we use Hölder's and the fact that $\|m_v^\vee\|_{L^1(\mathbb{R}^2)} \lesssim 1$ to estimate

$$\begin{aligned}&\iint\left\|\int_I\int \hat{f}(\xi_1,\xi_2)(\xi_2-\phi(\xi_1))^j\delta^{-j}e((\phi(\xi_1)-\xi_2)v)e(x\xi_1+y\xi_2)d\xi_2 d\xi_1\right\|^p_{L^p(B((u,v),\delta^{-1}))}\\ &\cdot\delta^2 w_{B,F}(u,v)dudv\\ &= \iint\iint \|f_I * m_v^\vee(x,y-v)\|^p_{L^p_{(x,y)}(B((u,v),\delta^{-1}))}\delta^2 w_{B,F}(u,v)dudv\\ &\lesssim \iint |f_I|^p * m_v^\vee(x,y')1_B(x-u,y')dxdy'\delta^2 w_{B,F}(u,v)dudv\\ &= \iint |f_I(\alpha,\beta)|^p\iint\iint |m_v^\vee(x-\alpha,y'-\beta)|\,\delta^2 1_B(x-u,y')dxdy'\\ &\cdot w_{B,F}(u,v)dudvd\alpha d\beta\\ &= \iint |f_I(\alpha,\beta)|^p\iint |m_v^\vee| * \delta^2 1_B(u-\alpha,-\beta)w_{B,F}(u,v)dudvd\alpha d\beta.\end{aligned}$$

Therefore, it remains to show the following pure weight inequality:

$$\iint |m_v^\vee| * \delta^2 1_B(u-\alpha,-\beta)w_{B,F}(u,v)dudv \lesssim w_{B,E}(\alpha,\beta).$$

In fact, we can show a slightly stronger inequality:

$$\iint |m_v^\vee| * \delta^2 1_B(u-\alpha,-\beta)w_{B,F}(u,v)dudv \lesssim (1+\delta|\alpha|)^{-E}(1+\delta|\beta|)^{-E}. \tag{4}$$

6.3 Derivative bound of m_v

In this subsection we are going to use the assumptions (2) up to order E. Recall that

$$m_v(\xi_1, \xi_2) = M_j(\delta^{-1}(\xi_2 - \phi(\xi_1)))\sigma\left(\frac{\xi_1 - h}{4l}\right)e(v\phi(\xi_1)). \tag{5}$$

Now we give a derivative estimate on m_v.

Lemma 14. *For each* $0 \leq s_1 \leq E$ *and* $s_2 \geq 0$ *we have*

$$\left\| \partial_{\xi_1}^{s_1} \partial_{\xi_2}^{s_2} m_v \right\|_\infty \lesssim_{s_1,s_2} l^{-s_1} \left(1 + |v|\delta\right)^{s_1} \delta^{-s_2}, \tag{6}$$

where the implicit constant does not depend on $j, |v|, \delta$.

Proof of lemma. First we use a translation and Proposition 10 to reduce to the case $h = 0$, $\phi(0) = 0$ and $\phi'(0) = 0$, whence (2) also holds for $k = 0, 1$. We also have $I = [0, l]$.

The part $\partial_{\xi_2}^{s_2}$ is an easy consequence of the derivative bounds of M_j (3). For the part $\partial_{\xi_1}^{s_1}$, we estimate respectively the three terms

$$M_j(\delta^{-1}(\xi_2 - \phi(\xi_1))), \quad \sigma\left(\frac{\xi_1}{4l}\right), \quad e(v\phi(\xi_1)).$$

For the first term, using Faà di Bruno's formula and (3), we see that the k-th derivative of $M_j(\delta^{-1}(\xi_2 - \phi(\xi_1)))$ is essentially bounded above by

$$\sum \prod_{j=1}^{k} \left\| \delta^{-1} \phi^{(j)} \right\|_\infty^{m_j},$$

where the sum is taken over all k-tuples of nonnegative integers $m_1, \ldots, m_k$ satisfying

$$m_1 + 2m_2 + \cdots + km_k = k. \tag{7}$$

Thus, using the assumptions (2), we have

$$\begin{aligned} \sum \prod_{j=1}^{k} \left\| \delta^{-1} \phi^{(j)} \right\|_\infty^{m_j} &\lesssim \sum \delta^{-(m_1 + \cdots + m_k)} \prod_{j=1}^{k} (l^{-j}\delta)^{m_j} \\ &= \sum l^{-(m_1 + 2m_2 + \cdots + km_k)} \\ &\sim l^{-k}, \end{aligned}$$

since the number of such k-tuples depends on k and in turn depends on E only.

The second term $\sigma(\frac{\xi}{4l})$ also gives a factor l^{-k} after differentiating k times. The third term $e(v\phi(\xi))$ can be estimated in a similar way as in the first term:

$$\left| \frac{d^k}{d\xi_1^k} e(v\phi(\xi_1)) \right| \lesssim l^{-k} \sum (|v|\delta)^{m_1 + \cdots + m_k}.$$

But for all k-tuples $m_1, \ldots, m_k$ satisfying (7), the largest possible value of $m_1 + \cdots + m_k$ is k, attained when $m_1 = k$ and $m_j = 0$ for all $2 \leq j \leq k$. Hence the term $e(v\phi(\xi))$ gives a factor $|v|^k l^{-k} \delta^k$ after differentiating k times.

Combining the three estimates and using Leibniz rule complete the proof of the Lemma. □

6.4 Proof of the weight inequality

Now we prove (4). The proof is almost the same as the last part of Section 5 of [2], but for completeness, we also give it here.

In this part we will use the assumption $l = |I| \geq \delta$ and the choice $F = 2E + 2$. We will also use the weight inequalities (5.1) and (5.2) of [2], namely,

$$\left(1 + \delta|\cdot|\right)^{-E} * \delta' \left(1 + \delta'|\cdot|\right)^{-E}(x) \lesssim \left(1 + \delta|x|\right)^{-E}, \text{ if } \delta \leq \delta' \tag{8}$$

and

$$w_{B,2E}(x, y) \leq \left(1 + \delta|x|\right)^{-E} \left(1 + \delta|y|\right)^{-E}. \tag{9}$$

Since m_v is supported on a constant multiple of the rectangle $[0,l]\times[-\delta,\delta]$, using Lemma 14 and repeated integration by parts shows that

$$|m_v^\vee(x,y)| \le \tau_1(x)\tau_2(y),$$

where

$$\tau_1(x) \lesssim_{s_1} l\left(1+\frac{|x|}{l^{-1}(1+|v|\delta)}\right)^{-s_1}, \quad \tau_2(y) \lesssim_{s_2} \delta\,(1+\delta|y|)^{-s_2},$$

for any $0\le s_1\le E$, $s_2\ge 0$. In particular, both the above hold for $s_1=s_2=E$. Also, τ_2 does not depend on v. Now we split

$$1_B(u-\alpha,-\beta)=1_J(u-\alpha)1_J(-\beta)$$

where $J=[-\delta^{-1}/2,\delta^{-1}/2]$. Thus, taking $s_2=E$ and using (8),

$$\begin{aligned}
&\iint |m_v^\vee| * \delta^2 1_B(u-\alpha,-\beta)w_{B,F}(u,v)dudv\\
&\le \iint \tau_1 * \delta 1_J(u-\alpha)\ \tau_2 * \delta 1_J(-\beta) w_{B,F}(u,v)dudv\\
&\lesssim \delta\,(1+\delta|\beta|)^{-E}\iint \tau_1 * \delta 1_J(u-\alpha) w_{B,F}(u,v)dudv.
\end{aligned}$$

Hence, all that remains is to show

$$\iint \tau_1 * \delta 1_J(u-\alpha)w_{B,F}(u,v)dudv \lesssim \delta^{-1}\,(1+\delta|\alpha|)^{-E}. \tag{10}$$

Let the integrand be denoted by $H(u,v)$. We split the analysis according to the range of $|v|$.

(a) If $|v|\le\delta^{-1}$, then we have

$$\tau_1(x)\lesssim_E l\,(1+l|x|)^{-E}.$$

Using $\delta\le l$ and (8), we have

$$\tau_1 * \delta 1_J(u-\alpha)\lesssim \delta\,(1+\delta|u-\alpha|)^{-E}.$$

Use (9) to split

$$w_{B,F}(u,v)\lesssim (1+\delta|u|)^{-E}\,(1+\delta|v|)^{-E-2},$$

and thus using (8) again we have

$$\begin{aligned}
\iint_{|v|\le\delta^{-1}} H(u,v)dudv &\lesssim (1+\delta|\alpha|)^{-E}\int_{|v|\le\delta^{-1}}(1+\delta|v|)^{-E-2}\,dv\\
&\lesssim \delta^{-1}(1+\delta|\alpha|)^{-E}.
\end{aligned}$$

(b) If $|v|\sim K\delta^{-1}$ where $1\le K\le l\delta^{-1}$, then

$$\tau_1(x)\lesssim_E l\left(1+\frac{l|x|}{|v|\delta}\right)^{-E}.$$

Using $|v|\delta/l\le\delta^{-1}$ and (8), we have

$$\tau_1 * \delta 1_J(u-\alpha)\lesssim |v|\delta^2\,(1+\delta|u-\alpha|)^{-E}.$$

Thus, similarly we have

$$\begin{aligned}
&\iint_{|v|\sim K\delta^{-1}} H(u,v)dudv\\
&\lesssim \delta(1+\delta|\alpha|)^{-E}\int_{|v|\sim K\delta^{-1}}|v|(1+\delta|v|)^{-E-2}dv\\
&\lesssim \delta^{-1}K^{-E}(1+\delta|\alpha|)^{-E}.
\end{aligned}$$

Summing with respect to $K\in 2^{\mathbb{N}}$ gives the required bound for the integral of H over $\delta^{-1}\le|v|\le l\delta^{-2}$.

(c) If $|v| \sim Kl\delta^{-2}$ where $K \geq 1$, we also use

$$\tau_1(x) \lesssim_E l \left(1 + \frac{l|x|}{|v|\delta}\right)^{-E}.$$

This time, using $|v|\delta/l \geq \delta^{-1}$ and (8), we have

$$\tau_1 * \delta 1_J(u-\alpha) \lesssim l \left(1 + \frac{l|x|}{|v|\delta}\right)^{-E}.$$

Thus, using (8), (9) again we have

$$\begin{aligned}
&\iint_{|v|\sim Kl\delta^{-2}} H(u,v)dudv \\
&\lesssim \int_{|v|\sim Kl\delta^{-2}} \delta^{-1} l \left(1 + \frac{l|\alpha|}{|v|\delta}\right)^{-E} (1+\delta|v|)^{-E-2} dv \\
&\lesssim \delta^{-1} l \int_{|v|\sim Kl\delta^{-2}} (1+\delta|\alpha|)^{-E} (|v|\delta^2 l^{-1})^E (1+\delta|v|)^{-E-2} dv \\
&\lesssim \delta^{-1}(1+\delta|\alpha|)^{-E} l\delta^{-2}(\delta\lambda^{-1})^E \int_{|v|\sim Kl\delta^{-2}} |v|^{-2} dv \\
&\sim K^{-1}\delta^{-1}(1+\delta|\alpha|)^{-E},
\end{aligned}$$

using $\delta \leq l$. Summing with respect to $K \in 2^{\mathbb{N}}$ gives the required bound for the integral of H over $|v| \geq l\delta^{-2}$.

Combining the three bounds above, we thus have

$$\iint H(u,v)dudv \lesssim \delta^{-1}(1+\delta|\alpha|)^{-E},$$

as required. This finishes the proof of (10) and hence the proof of Proposition 7. □

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