# A Study Guide for *A Study Guide for the $l^2$-decoupling Inequality*

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# 1 Introduction

This article (second edition) serves as an informal study guide for [4], which in turn serves as a study guide for [3]. It gives a detailed proof for the decoupling inequality in the case $2 \le p \le \frac{2(n+1)}{n-1}$. It is self-contained if and only if combined with [2], [4], and [5].

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With the basic setting in [4] and the notations and conventions in the next section, our final goal is to prove the following decoupling theorem:

**Theorem 1.1.** *Let $n \ge 2$, $E \ge 100n$ and $2 \le p \le \frac{2(n+1)}{n-1}$. Then we have the following (local) decoupling inequality:*

$$\|Eg\|_{L^p(w_{B_R,E})} \lesssim_{\varepsilon,p,n,E} R^{\varepsilon} \left( \sum_{Q \in \mathcal{P}_{R^{-1/2}}([-\frac{1}{2},\frac{1}{2}]^{n-1})} \|E_Q g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}},$$

*for any $R \ge 1$.*

# 2 Notations and Conventions

1. $e(x) = \exp(2\pi i x)$ for $x \in \mathbb{C}$.

2. Cross-references in red with a hyper-link refer to an item in this article, while those in black refer to an item in the study guide by [Bourgain-Demeter, [4]]. We write [BD] for short in subsequent texts.

3. Unless otherwise specified, $B(x,R)$ denotes the axis parallel cube centred at $x$ with side length $R$. Given $E \ge 100n$ and any cube $B$ with side length $R$, we let $c_B$ denote the centre of $B$. We then define the weight function adapted to $B$ as $w_B(x) = w_{B,E}(x) = \left(1 + \frac{|x-c_B|}{R}\right)^{-E}$.

   We will also use the notation $B_R$ to denote any cube with side length $R$ and some centre $c_B$. For $C > 0$ and any geometric figure $F$ (e.g. rectangles, squares and balls) with centre $c$, we also use $CF$ to denote the dilation of $F$ with $c$ fixed.

4. By an absolute constant we mean a constant depending on the dimension $n$ (and possibly $E$) only. When writing a constant $C_{a,b,c,\text{etc}}$ or a parameter $\nu(a,b,c,\text{etc})$ depending on several parameters, we also implicitly assume that they depend on $n$ (and $E$) as well.

By definition, the decoupling constant $\mathrm{Dec}_n(\delta, p, E)$ depends on $n, p, \delta, E$. Unless we need to specify its dependence on $E$, we will drop it in the following texts. Sometimes we will also drop its dependence on $n$ or $p$ if it is not the main point of concern. (For example, the simplest notation of the decoupling constant will be $\mathrm{Dec}(\delta)$.)

5. Unless otherwise specified, every physical scale in this note and [BD] will be in $2^{\mathbb{N}}$, and every frequency scale will be in $2^{-\mathbb{N}}$. (We say the scales are dyadic in this case.) This makes all partitions of cubes real partitions instead of bounded overlapping covers. Nevertheless, sometimes we may prove more general theorems that help to deal with bounded overlapping cases.

6. By disjoint rectangles in $\mathbb{R}^k$ we mean rectangles whose interiors are disjoint. Hence they are disjoint a.e. in terms of $k$-dimensional Lebesgue measure. Unless otherwise specified, we will simply say they are disjoint.

7. We use $\mathbb{P}^{n-1}$ to denote the truncated paraboloid in the frequency space:
$$\mathbb{P}^{n-1} = \left\{ (\xi, |\xi|^2) : \xi \in \left[ -\frac{1}{2}, \frac{1}{2} \right]^{n-1} \right\}.$$

8. Instead of using the longer notation Part to mean partition of cubes, we simply denote it as $\mathcal{P}$.

# 3 Some Preliminary Technicalities

## 3.1 Inequality on weights, I

The scales in this subsection are not necessarily dyadic.

**Proposition 3.1.** *Let $1 \le R' \le R$. Let $\mathcal{B}$ be a finite overlapping covering of $B$ with $R'$-cubes $\Delta$ which intersect $B$. Here, by finite overlapping we mean:*
$$1_B(x) \le \sum_{\Delta \in \mathcal{B}} 1_\Delta(x) \le C_n \tag{3.1}$$
*for all $x$. Then*

1. $\#\mathcal{B} \lesssim (R/R')^n$.
2. $1_B \lesssim \sum_{\Delta \in \mathcal{B}} w_\Delta \lesssim w_B$.

*The implicit constants here depend on $E$ and $n$ only.*

*Proof.* 1. For all $\Delta \in \mathcal{B}$, $\Delta \cap B \neq \varnothing$ and $1 \le R' \le R$, so $\Delta \in 3B$. Hence by (3.1) we further have
$$\sum_{\Delta \in \mathcal{B}} 1_\Delta \le C_n 1_{3B}.$$
Integrating on both sides, we get $\#\mathcal{B}(R')^n \le C_n(3R)^n$, so $\#\mathcal{B} \lesssim (R/R')^n$.

2. "$1_B \lesssim \sum_{\Delta\in\mathcal{B}} w_\Delta$"

If $x \in B$, then there is $\Delta \in \mathcal{B}$ so that $x \in \Delta$. Then $|x - c_\Delta| \le \sqrt{n}R'/2$, whence

$$\sum_{\Delta\in\mathcal{B}} w_\Delta(x) \ge w_\Delta(x) = \frac{1}{\left(1+\frac{|x-c_\Delta|}{R'}\right)^E} \gtrsim 1.$$

"$\sum_{\Delta\in\mathcal{B}} w_\Delta \lesssim w_B$"

- If $x \notin 4\sqrt{n}B$, then $|x - c_B| \ge 4\sqrt{n}\cdot R/2 = 2\sqrt{n}R$. For each $\Delta$ intersecting $B$, we have $|c_B - c_\Delta| \le \sqrt{n}R/2 + \sqrt{n}R'/2 \le \sqrt{n}R$. Hence $|x - c_\Delta| \ge |x - c_B| - |c_B - c_\Delta| \ge |x - c_B| - \sqrt{n}R \ge \frac{1}{2}|x - c_B|$, where we have $\frac{1}{2}|x - c_B| \ge R \ge R'$. Hence

$$\sum_{\Delta\in\mathcal{B}} w_\Delta(x) = \sum_{\Delta\in\mathcal{B}} \frac{1}{\left(1+\frac{|x-c_\Delta|}{R'}\right)^E} \lesssim \#\mathcal{B}\cdot\frac{1}{\left(\frac{|x-c_B|}{R'}\right)^E} \lesssim \left(\frac{R}{R'}\right)^n\cdot\frac{R'^E}{|x-c_B|^E}.$$

On the other hand, as $|x - c_B| \ge 2\sqrt{n}R \ge R$,

$$w_B(x) = \frac{1}{\left(1+\frac{|x-c_B|}{R}\right)^E} \ge \frac{1}{\left(\frac{2|x-c_B|}{R}\right)^E} \sim \frac{R^E}{|x-c_B|^E}.$$

Since $E \ge 100n$ and $R \ge R'$, we have $\left(\frac{R}{R'}\right)^n \cdot \frac{R'^E}{|x-c_B|^E} \le \frac{R^E}{|x-c_B|^E}$.

- If $x \in 4\sqrt{n}B$, then $w_B(x) \sim 1$. We need to show that

$$\sum_{\Delta\in\mathcal{B}} w_\Delta(x) \lesssim 1. \tag{3.2}$$

To do this we need a lemma:

**Lemma 3.2.** *Let $x$ be a point in $\mathbb{R}^n$ and let $R > 0$, $K \in \mathbb{N}$. Let $\mathcal{B}$ be a collection of cubes $\Delta$ with the same side length $R$ satisfying the following property:*

*(a) They have bounded overlap:*

$$\sum_{\Delta\in\mathcal{B}} 1_\Delta \le C_n.$$

*(b) For each $\Delta \in \mathcal{B}$, $|c_\Delta - x| \le 2^K R$.*

*Then $\#\mathcal{B} \lesssim 2^{Kn}$.*

*Proof of Lemma 3.2.* All $\Delta \in \mathcal{B}$ are contained in the cube $B(x, 2^{K+2}R)$. Then

$$\#\mathcal{B}R^n = \#\mathcal{B}|\Delta| = \int_{B(x,2^{K+2}R)} \left(\sum_{\Delta\in\mathcal{B}} 1_\Delta\right) \le \int_{B(x,2^{K+2}R)} C_n \sim 2^{Kn}R^n. \tag{3.3}$$

Hence $\#\mathcal{B} \lesssim 2^{Kn}$. □

Now we return to the proof that $\sum_{\Delta\in\mathcal{B}} w_\Delta(x) \lesssim 1$. Fix $x \in 4\sqrt{n}B$. Partition $\mathcal{B} = \cup_{K=0}^{\infty}\mathcal{B}_K$ where

$$\mathcal{B}_K := \{\Delta \in \mathcal{B} : 2^{K-1}R' < |c_\Delta - x| \le 2^K R'\}, \qquad K \ge 1$$
$$\mathcal{B}_0 = \{\Delta \in \mathcal{B} : |c_\Delta - x| \le R'\}.$$

Since $\Delta$ has side length $R'$, $\mathcal{B}_0$ has at most $O(1)$ elements, in the same spirit as (3.3). Indeed, if $|c_\Delta - x| \le R'$, then $\Delta \in B(x, 3R')$. Hence by finite overlapping (3.1),

$$\sum_{\Delta\in\mathcal{B}_0} 1_\Delta(y) \le C_n 1_{B(x,3R')}(y).$$

Integrating over $\mathbb{R}^n$ on both sides, we get $\#\mathcal{B}_0 \lesssim 1$. Hence $\sum_{\Delta\in\mathcal{B}_0} w_\Delta(x) \lesssim 1$. For $K \ge 1$, $|c_\Delta - x| \sim 2^K R'$, so $w_\Delta(x) \sim 2^{-KE}$. Using Lemma 3.2 we have

$$\sum_{\Delta\in\mathcal{B}_K} w_\Delta(x) \le \#\mathcal{B}_K \cdot 2^{-KE} \lesssim 2^{-K(E-n)}.$$

Summing with respect to $K \ge 0$, we have $\sum_{\Delta\in\mathcal{B}} w_\Delta(x) \lesssim 1$.

□

**Remark:** The proof of $\sum_{\Delta\in\mathcal{B}} w_\Delta(x) \lesssim 1$ shows that it is generally true whenever the $\Delta$'s have finite overlap. It is independent of the geometric figure they are covering. This observation will be useful in some technical argument in parabolic rescaling.

**Proposition 3.3.** *Fix $\mathcal{B}$, a finite overlapping cover of $\mathbb{R}^n$ with $R$-cubes $B'$, and let $B$ be an arbitrary $R$-cube. Then*

1. $w_B(x) \lesssim \sum_{B'\in\mathcal{B}} 1_{B'}(x) w_B(c_{B'})$.
2. $\sum_{B'\in\mathcal{B}} w_{B'}(x) w_B(c_{B'}) \lesssim w_B(x)$.

*Proof.* Let $c$ denote the centre of $B$.

1. Let $x \in \mathbb{R}^n$. Then there is $B' \in \mathcal{B}$ such that $x \in B'$, so $|x - c_{B'}| \le \sqrt{n}R/2$. We have two cases.

   - $|c_{B'} - c| \ge \sqrt{n}R$.
     In this case,

     $$|c_{B'} - c| \le |c_{B'} - x| + |x - c| \le \frac{\sqrt{n}}{2}R + |x - c| \le \frac{1}{2}|c - c_{B'}| + |x - c|.$$

     Hence $|c - c_{B'}| \le 2|x - c|$, whence

     $$w_B(x) = \frac{1}{\left(1 + \frac{|x-c|}{R}\right)^E} \le \frac{1}{\left(1 + \frac{|c_{B'}-c|}{2R}\right)^E} \sim w_B(c_{B'}).$$

- $|c_{B'} - c| < \sqrt{n}R$.
  In this case, $w_B(c_{B'}) \sim 1$. Since $w_B(x) \leq 1$, the result follows.

2. Fix $x \in \mathbb{R}^n$. We partition the covering into $\mathcal{B}_1, \mathcal{B}_2$, where $\mathcal{B}_1 := \{B' \in \mathcal{B} : |x - c_{B'}| \geq 2|x-c|\}$, $\mathcal{B}_2 := \{B' \in \mathcal{B} : |x - c_{B'}| < 2|x-c|\}$.
   If $B' \in \mathcal{B}_1$, then $|c - c_{B'}| \geq |x - c_{B'}| - |x - c| \geq |x - c|$. Hence

$$\begin{aligned}\sum_{B' \in \mathcal{B}_1} w_{B'}(x) w_B(c_{B'}) &\leq \sum_{B' \in \mathcal{B}_1} w_{B'}(x) \cdot \frac{1}{\left(1 + \frac{|x-c|}{R}\right)^E} \\ &\leq \frac{1}{\left(1 + \frac{|x-c|}{R}\right)^E} \sum_{B' \in \mathcal{B}} w_{B'}(x) \\ &\lesssim \frac{1}{\left(1 + \frac{|x-c|}{R}\right)^E} = w_B(x),\end{aligned}$$

   since $\sum_{B' \in \mathcal{B}} w_{B'}(x) \lesssim 1$ in the same spirit as in (3.2).
   If $B' \in \mathcal{B}_2$, we consider the following cases:

   - If $x \in \sqrt{n}B$, then using the trivial bound $w_B \leq 1$:

$$\sum_{B' \in \mathcal{B}_2} w_{B'}(x) w_B(c_{B'}) \lesssim \sum_{B' \in \mathcal{B}} w_{B'}(x) \cdot 1 \lesssim 1 \sim w_B(x).$$

   - If $x \notin \sqrt{n}B$, then we consider the inequality defining $\mathcal{B}_2$: $|x - c_{B'}| < 2|x - c|$. This implies that the entire cube $B'$ is contained in $B(x, 3|x - c|)$: indeed, if $y \in B'$, then $|y - x| \leq |y - c_{B'}| + |c_{B'} - x| \leq \sqrt{n}R/2 + 2|x - c| \leq 3|x - c|$ as $x \notin \sqrt{n}B$.
     Then in the same spirit of (3.3), we have $\#\mathcal{B}_2 \lesssim \left(\frac{|x-c|}{R}\right)^n$. Thus

$$\sum_{B' \in \mathcal{B}_2} w_{B'}(x) w_B(c_{B'}) \lesssim \left(\frac{|x-c|}{R}\right)^n \leq \left(\frac{|x-c|}{R}\right)^E \sim w_B(x).$$

□

## 3.2 Use of reverse Minkowski in Remark 4

Write

$$A_i = \|f_i\|^p_{L^p(\alpha u + \beta v)}.$$

Then

$$A_i = \alpha \|f_i\|^p_{L^p(u)} + \beta \|f_i\|^p_{L^p(v)} := A_{i,1} + A_{i,2}.$$

Then

$$\begin{aligned}
O_2(\alpha u+\beta v) &= \left(\sum_i \|f_i\|^2_{L^p(\alpha u+\beta v)}\right)^{\frac{p}{2}} = \left(\sum_i A_i^{\frac{2}{p}}\right)^{\frac{p}{2}} \\
&= \left(\sum_i (A_{i,1}+A_{i,2})^{\frac{2}{p}}\right)^{\frac{p}{2}} = \|A_{i,1}+A_{i,2}\|_{l^{\frac{2}{p}}(i)} \\
&\geq \|A_{i,1}\|_{l^{\frac{2}{p}}(i)} + \|A_{i,2}\|_{l^{\frac{2}{p}}(i)} \\
&= \left(\sum_i A_{i,1}^{\frac{2}{p}}\right)^{\frac{p}{2}} + \left(\sum_i A_{i,2}^{\frac{2}{p}}\right)^{\frac{p}{2}} \\
&= \left(\sum_i \alpha^{\frac{2}{p}} \|f_i\|^2_{L^p(u)}\right)^{\frac{p}{2}} + \left(\sum_i \beta^{\frac{2}{p}} \|f_i\|^2_{L^p(v)}\right)^{\frac{p}{2}} \\
&= \alpha\left(\sum_i \|f_i\|^2_{L^p(u)}\right)^{\frac{p}{2}} + \beta\left(\sum_i \|f_i\|^2_{L^p(v)}\right)^{\frac{p}{2}} \\
&= \alpha O_2(u)+\beta O_2(v).
\end{aligned}$$

# 4 A Reverse Hölder's Inequality

## 4.1 A lemma on Schwartz functions

We start with the following technical lemma.

**Lemma 4.1.** *Let $1 \leq p < \infty$. There is a nonnegative Schwartz function $\eta$ on $\mathbb{R}^n$ such that $\eta(x) \geq 1$ on $B(0,1)$ and that the Fourier transform of $\eta^{1/p}$ is supported on $B(0,1)$.*

*Proof.* Let $\phi$ be a bump function supported on $B(0,\frac{1}{2})$ with, say, $\phi^\vee(0) = \int \phi = 2$ and such that $|\phi^\vee| \geq 1$ on $B(0,1)$. Take $\eta = (\phi^\vee \cdot \overline{\phi^\vee})^p$, so $\eta$ is positive, smooth, and $\eta(x) \geq 1$ on $B(0,1)$. By construction, $(\eta^{\frac{1}{p}})^\wedge = \phi(\cdot) * \phi(-\cdot)$, so it is supported on $B(0,1)$. □

**Definition 4.2.** *Let $1 \leq p < \infty$. Let $B \subseteq \mathbb{R}^n$. Given $\eta$ as in Lemma 4.1, we define*

$$\eta_B(x) = \eta_{B,p}(x) = \eta\left(\frac{x-c_B}{l(B)}\right). \tag{4.1}$$

*We call $\eta_B$ a Schwartz function adapted to $B$ with exponent $p$.*

Note that $\widehat{\eta_B^{1/p}}$ will be supported on $B(0, l(B)^{-1})$.

## 4.2 Computations related to extension operator

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, write $x' = (x_1, \ldots, x_{n-1})$. Write $e(t) = \exp(2\pi i t)$. The extension operator defined in BD is as follows:

$$Eg(x) = \int_{[-\frac{1}{2},\frac{1}{2}]^{n-1}} g(\xi) e(x' \cdot \xi + x_n |\xi|^2) d\xi.$$

With this notation, $g$ is defined directly on the frequency cube $[-\frac{1}{2},\frac{1}{2}]^{n-1}$ but not on the paraboloid; note also that the Jacobian is not present. Since we may assume that $g \in C_c^\infty([-\frac{1}{2},\frac{1}{2}]^{n-1})$, we could also view $g$ as a smooth function defined on $\mathbb{R}^{n-1}$ that vanishes outside $[-\frac{1}{2},\frac{1}{2}]^{n-1}$.

We rewrite $Eg$ as

$$Eg(x) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \delta_0(\xi_n - |\xi|^2) g(\xi) e(x' \cdot \xi + x_n \xi_n) d\xi_n d\xi.$$

Hence, formally,

$$\widehat{Eg}(\xi, \xi_n) = g(\xi)\delta_0(\xi_n - |\xi|^2).$$

Thus $\widehat{Eg}$ is a (tempered) distribution in $\mathbb{R}^n$ supported on the compact hypersurface $S$, in the sense that if $h \in C^\infty(\mathbb{R}^n)$, then

$$\begin{aligned}\widehat{Eg}(h) &= \int_{\mathbb{R}^n} g(\xi)\delta_0(|\xi|^2 - \xi_n) h(\xi, \xi_n) d\xi d\xi_n \\ &= \int_{\mathbb{R}^{n-1}} g(\xi) h(\xi, |\xi|^2) d\xi.\end{aligned}$$

Moreover, $\widehat{Eg}$ will not be a function unless the support of $g$ is disjoint from $S$. Hence it makes no sense to talk about $E^*Eg$ in this case.

If $\phi \in \mathcal{S}(\mathbb{R}^n)$, then for $\eta \in \mathbb{R}^{n-1}$, $\eta_n \in \mathbb{R}$,

$$\begin{aligned}\widehat{\phi Eg}(\xi) = \widehat{Eg} * \phi(\xi) &= \int\int g(\eta)\delta_0(|\eta|^2 - \eta_n)\phi(\xi - \eta, \xi_n - \eta_n) d\eta d\eta_n \\ &= \int g(\eta)\phi(\xi - \eta, \xi_n - \eta^2) d\eta.\end{aligned}$$

which is a normal function in the frequency space. So if $g$ is supported on a cube $Q \subseteq \mathbb{R}^{n-1}$ with $l(Q) = R^{-1}$ and $\phi$ is supported on $B(0, R^{-1}/10)$, then $\widehat{Eg} * \phi(\xi)$ is supported on a tiny neighbourhood of the paraboloid over $Q$.

## 4.3 Locally constant property

The following proposition is a slight generalization to Corollary 4.3. This locally constant property will also be referred to as the uncertainty principle, a reverse Hölder's inequality or a Bernstein-type inequality in the following texts.

**Proposition 4.3.** *Let $1 \le p \le q \le \infty$. Let $R \ge 1$. Let $\eta_B = \eta_{B,p}$ be a Schwartz function adapted to a cube $B \subseteq \mathbb{R}^n$ with exponent $p$ (see (4.1)). Then for each cube $Q \subseteq [0,1]^{n-1}$ with $l(Q) = R^{-1}$ and each cube $B \subseteq \mathbb{R}^n$ with $l(B) \ge R$ we have*

$$\|E_Q g\|_{L^q(B)} \lesssim (R^{-n})^{\frac{1}{p}-\frac{1}{q}} \|E_Q g\|_{L^p(\eta_B)}. \tag{4.2}$$

*Thus we have, for all $E \ge 0$,*

$$\|E_Q g\|_{L^q_\#(B)} \lesssim \left(\frac{l(B)}{R}\right)^{\frac{n}{p}-\frac{n}{q}} \|E_Q g\|_{L^p_\#(w_{B,E})}, \tag{4.3}$$

This says that $E_Q g$ is roughly constant at scales $\geq R$.

*Proof.* By (4.1), $\widehat{\eta_B^{1/p}}$ is a function supported on $B(0, l(B)^{-1}) \subseteq B(0, R^{-1})$. As shown above, $\widehat{E_Q g}$ is a distribution supported on the paraboloid above $Q$ with $l(Q) = R^{-1}$, so in particular, it is supported on a rectangle $Q \times I$ where $|I| \leq 2(n-1)R^{-1}$. Using Minkowski sum, $\widehat{\eta_B^{1/p} E_Q g} = \widehat{\eta_B^{1/p}} * \widehat{E_Q g}$ is supported in a rectangle $3Q \times J$ with all dimensions $\leq c_n R^{-1}$ (using the map $Q \ni \xi \mapsto |\xi|^2$ and triangle inequality), which is in turn contained in some cube $Q' \subseteq \mathbb{R}^n$ with $l(Q') = c_n R^{-1}$. Then apply the usual Bernstein inequality to the function $\eta_B^{1/p} E_Q g$ to get

$$\left\| \eta_B^{1/p} E_Q g \right\|_{L^q(B)} \lesssim (R^{-n})^{\frac{1}{p} - \frac{1}{q}} \left\| \eta_B^{1/p} E_Q g \right\|_{L^p(B)}, \tag{4.4}$$

which then implies (4.2) as $\eta \geq 1$ in $B$. The implication from (4.2) to (4.3) is trivial. □

# 5 Thickening the Paraboloid (Section 5)

## 5.1 Inequality on weights, II

A single exponent $E$ will be fixed throughout the subsection.

**Proposition 5.1.** *Let $B$ be a cube with side length $R$. Then $w_B(y) \sim w_B(x)$ if $y \in B(x, R)$. That is, $w_B$ is roughly constant at scale $R$.*

*Proof.* Let $c_B$ be the centre of $B$. Let $y \in \mathbb{R}^n$. If $y \in B$, then $w_B(y) \sim 1$. Also,

$$|x - c_B| \leq |x - y| + |y - c_B| \leq \sqrt{n} R \lesssim R.$$

Hence $w_B(x) \gtrsim 1 \sim w_B(y)$. Switching the roles of $x$ and $y$ shows that $w_B(y) \sim w_B(x)$.
If $y \notin B$, then $|y - c_B| > \sqrt{n}R/2$. let $K$ be the unique natural number such that $2^{K-1}\sqrt{n}R/2 < |y - c_B| \leq 2^K \sqrt{n} R/2$. Then $w_B(y) \sim 2^{-KE}$. Also,

$$|x - c_B| \leq |x - y| + |y - c_B| \leq (1 + 2^K) \frac{\sqrt{n}}{2} R \lesssim 2^K R.$$

Thus $w_B(x) \gtrsim 2^{-KE} \sim w_B(y)$. Switching the roles of $x$ and $y$ shows that $w_B(y) \sim w_B(x)$. □

For future use, we will need a slightly more general proposition:

**Proposition 5.2.** *Let $B_R$ be a cube centred at the origin. If $|c_j| \leq 1$ for $j = 1, 2, \ldots, n$, then*

$$\frac{1}{R^n} \int w_{B_R}(y) w_{B_R}(x_1 - c_1 y_1, \cdots x_n - c_n y_n) dy_1 \cdots dy_n \lesssim w_{B_R}(x),$$

*where the implicit constant does not depend on $c_j$, $1 \leq j \leq n$.*

As a corollary, we have the following proposition.

**Proposition 5.3.** *If $R' \le R$ and $B_R$, $B_{R'}$ are centred at the origin, then we have*

$$w_{B_R} * \left(\frac{1}{(R')^n} w_{B_{R'},E}\right) \lesssim w_{B_R}.$$

*Proof of Proposition 5.2.* Cover $\mathbb{R}^n$ by the collection $\mathcal{B}$ of translates of $B_R$. Then by 1 of Proposition 3.3, it suffices to prove that

$$\sum_{B' \in \mathcal{B}} w_{B_R}(c_{B'}) \frac{1}{R^n} \int 1_{B'}(y) w_{B_R}(x_1 - c_1 y_1, \cdots x_n - c_n y_n) dy_1 \cdots dy_n \lesssim w_{B_R}(x).$$

Since $|c_j| \le 1$ for $j = 1, 2, \ldots, n$, by Proposition 5.1, we have

$$1_{B'}(y) w_{B_R}(x_1 - c_1 y_1, \ldots, x_n - c_n y_n) \sim 1_{B'}(y) w_{B_R}(x_1, \ldots, x_n),$$

where the implicit constant does not depend on $c_j$, $1 \le j \le n$. Hence we have

$$\begin{aligned}
&\sum_{B' \in \mathcal{B}} w_{B_R}(c_{B'}) \frac{1}{R^n} \int 1_{B'}(y) w_{B_R}(x_1 - c_1 y_1, \cdots x_n - c_n y_n) dy_1 \cdots dy_n \\
&\sim \sum_{B' \in \mathcal{B}} w_{B_R}(c_{B'}) \frac{1}{R^n} \int 1_{B'}(y) w_{B'_R}(x_1, \ldots, x_n) dy_1 \cdots dy_n \\
&= \sum_{B' \in \mathcal{B}} w_{B'}(x) w_{B_R}(c_{B'}) \lesssim w_{B_R}(x),
\end{aligned}$$

by 2 of Proposition 3.3. □

## 5.2 A slight generalisation of Theorem 5.1

For future use, we generalise Theorem 5.1 slightly by enlarging $N_{1/R}$ to $N_{C/R}$ where $C > 1$ is an absolute constant and the new $N_{C/R}$ extends to negative neighbourhoods: $-C/R \le \delta \le C/R$.

For each $Q \subseteq [-1/2, 1/2]^{n-1}$ and $\delta > 0$, we denote

$$N_{Q,\delta} = \{\xi = (\xi, \xi_n) : \xi \in Q, |\xi_n - |\xi|^2| \le \delta\}.$$

We also denote

$$N_\delta = N_{[-\frac{1}{2},\frac{1}{2}]^{n-1},\delta}.$$

For a Schwartz function $f : \mathbb{R}^n \to \mathbb{C}$ and $Q \subseteq [-1/2, 1/2]^{n-1}$ with Fourier support in $N_\delta$, we denote

$$f_{Q,\delta} = (\widehat{f} 1_{N_{Q,\delta}})^\vee = f * 1^\vee_{N_{Q,\delta}}.$$

In (5), we change the notation $g_j$ to $f_j$ to indicate the dependence of $f_j$ on $f$.

$$f_j(s) = \int_{-C/R}^{C/R} \widehat{f}(s, s^2 + t) \left(\frac{Rt}{2}\right)^j dt. \tag{5.1}$$

### 5.2.1 Reduction to the case $Q = [0, R^{-1/2}]$

It is easy to see why we can assume $B_R$ to be centred at the origin, but it is not really easy to see why we can just consider the case $Q = [0, R^{-1/2}]$.

**Proposition 5.4.** *Let $Q_0$ denote the interval $[0, R^{-1/2}]$. Assume*

$$\sup_j \|E_{Q_0} g_j\|_{L^p(w_{B_R}, F)} \lesssim \left\| g_{N_{C/R}(Q_0)} \right\|_{L^p(w_{B_R}, E)} \tag{5.2}$$

*holds for all functions $g$ with Fourier support in $N_{C/R}(Q_0)$. Then for all cubes $Q = [u, u + R^{-1/2}] \subseteq [0, 1]$, we have*

$$\sup_j \|E_Q f_j\|_{L^p(w_{B_R}, F)} \lesssim \left\| f_{N_{C/R}(Q)} \right\|_{L^p(w_{B_R}, E)} \tag{5.3}$$

*holds for all functions $f$ with Fourier support in $N_{C/R}(Q)$, where the implicit constant does not depend on $u$.*

*Proof.* By a change of variable (as in the proof of parabolic rescaling 7.1 in the following) we have

$$|E_Q f_j(x)| = \left| \int_{Q_0} f_j(u + s) e(x_1 s + 2x_2 us + x_2 s^2) ds \right|. \tag{5.4}$$

Let $T$ denote the following affine shear transformation:

$$T(\xi_1, \xi_2) := (\xi_1 + u, 2u\xi_1 + \xi_2 + u^2).$$

We have $|\det(T)| = 1$ and

$$T^{-1}(\eta_1, \eta_2) = (\eta_1 - u, \eta_2 - 2u\eta_1 + u^2).$$

Then let $g(x) = f(T^{-1}x)$, so $\widehat{g}(\xi) = \widehat{f}(T\xi)$, i.e. $\widehat{f}(\xi) = \widehat{g}(T^{-1}\xi)$. Thus by (5.1) and simple computation,

$$f_j(u + s) = \int_{-C/R}^{C/R} \widehat{g}(s, s^2 + t) \left( \frac{Rt}{2} \right)^j dt. \tag{5.5}$$

As (5.1) suggests, we will show $f_j(u+s) = g_j(s)$, which is true if $g$ also has Fourier support within $N_{C/R}([0, 1])$. One good thing about the parabola is that it interacts well with such affine shear transformation $T$. More precisely, not only does $T^{-1}$ map the parabola over $Q$ to the parabola over $Q_0$, but $T^{-1}(N_{C/R}(Q))$ is also exactly $N_{C/R}(Q_0)$. Too see this, note $(\eta_1 - u) - (\eta_1' - u) = \eta_1 - \eta_1'$ and

$$(\eta_2 - 2u\eta_1 + u^2) - (\eta_1 - u)^2 = \eta_2 - \eta_1^2.$$

Hence $g$ also has Fourier support $N_{C/R}([0, 1])$ (see Figure 1), and so $f_j(u + s) = g_j(s)$.

If we define

$$y = (y_1, y_2) = L(x_1, x_2) = (x_1 + 2ux_2, x_2),$$

Then (5.4) rewrites

$$|E_Q f_j(x)| = \left| \int_{Q_0} g_j(s) e(y_1 s + y_2 s^2) ds \right| = |E_{Q_0} g_j(y)|. \tag{5.6}$$

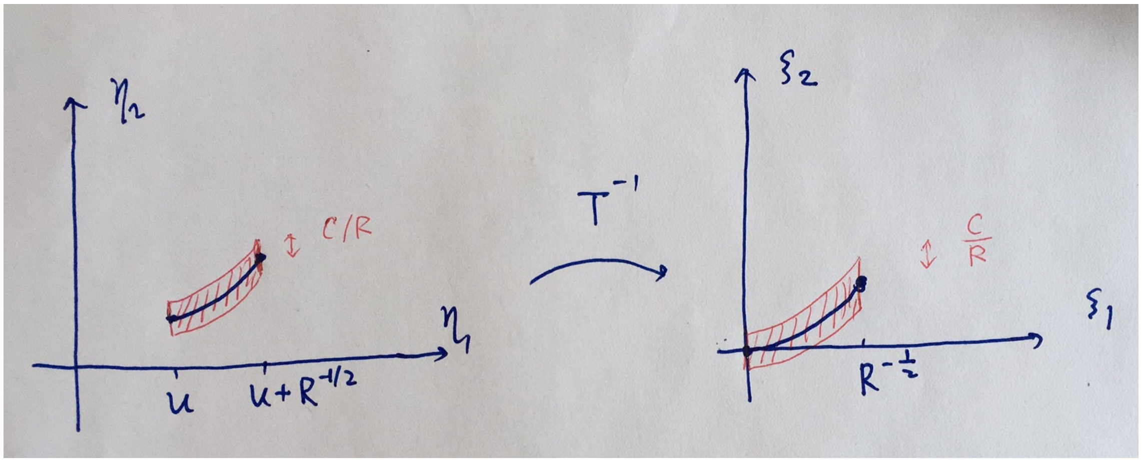


Figure 1: $T$ acts well with $N_{C/R}$

Thus we can apply (5.2) to $g_j$ to get

$$\|E_{Q_0} g_j\|_{L^p(w_{B_R},F)} \lesssim \left\| g_{N_{C/R}(Q_0)} \right\|_{L^p(w_{B_R},E)}. \tag{5.7}$$

Since $|\det(L)| = 1$, by (5.6), the left hand side of (5.7) is equal to

$$\|E_{Q_0} g_j\|_{L^p(w_{B_R},F)} = \left( \int |E_Q f_j(x)|^p w_{B_R}(Lx)dx \right)^{\frac{1}{p}}.$$

Recall we assumed $c(B_R) = 0$. Since $0 \le u < 1$, we have $|Lx| \sim |x|$ and hence $w_{B_R}(Lx) \sim w_{B_R}(x)$. Hence $\|E_{Q_0} g_j\|_{L^p(w_{B_R},F)} \sim \|E_Q f_j\|_{L^p(w_{B_R},F)}$.

For the right hand side of (5.7), we have

$$\begin{aligned} |g_{N_{C/R}(Q_0)}(x)| &= \left| \int_{N_{C/R}(Q_0)} \widehat{f}(T\xi) e(x_1\xi_1 + x_2\xi_2) d\xi_1 d\xi_2 \right| \\ &= \left| \int_{N_{C/R}(Q)} \widehat{f}(\eta) e((x_1 - 2ux_2)\eta_1 + x_2\eta_2) d\eta_1 d\eta_2 \right| \\ &= f_{N_{C/R}(Q)}(L^{-1}x). \end{aligned}$$

Hence similarly, $\left\| g_{N_{C/R}(Q_0)} \right\|_{L^p(w_{B_R},E)} \sim \left\| f_{N_{C/R}(Q)} \right\|_{L^p(w_{B_R},E)}$, and so we have (5.3). $\square$

From now on we write $Q = Q_0 = [0, R^{-1/2}]$.

#### 5.2.2 About cutoff functions

The second to last equality on Page 7 follows from Proposition 5.1 and Fubini's theorem:

$$\|E_Q f_j\|^p_{L^p(w_{B_R},F)} \sim \int \|E_Q f_j\|^p_{L^p_\#(B(y,R))} w_{B_R,F}(y) dy.$$

The last equality on Page 7 follows from (5.1) by writing back to product measure on $N_{C/R}(Q)$:

$$E_Q f_j(x) = \int_{N_{C/R}(Q)} \widehat{f}(\xi) \left( \frac{R(\xi_2 - \xi_1^2)}{2} \right)^j e((\xi_1^2 - \xi_2)x_2) e(\xi \cdot x) d\xi.$$

For the third inequality on Page 8, instead of proving it holds uniformly in $j$, we could allow a multiplicative factor $C^j$ on the right hand side.

The fourth equality on Page 8 now becomes

$$\int_{N_{C/R}(Q)} \widehat{f}(\xi) \left( \frac{R(\xi_2 - \xi_1^2)}{2} \right)^j e((\xi_1^2 - \xi_2)y_2) e(\xi \cdot x) d\xi = C^j \int \widehat{F}(\xi) m_j(\xi) e(\xi_1 x_1 + \xi_2(x_2 - y_2)) d\xi,$$

where

$$m_j(\xi) = m_{j,y_2}(\xi) = e(\xi_1^2 y_2) M_j \left( \frac{R(\xi_2 - \xi_1^2)}{2C} \right) \eta(R^{1/2}\xi_1) \eta(R\xi_2). \tag{5.8}$$

Here, $\eta \equiv 1$ on $[-2C, 2C]$ and supported on $[-3C, 3C]$ is a bump function. (Note this is where the reduction $Q = [0, R^{-1/2}]$ is used.) Also, $M_j$ is defined to be a bump function that agrees with $t^j$ on $[-1/2, 1/2]$ and that satisfies the derivative bound:

$$\sup_{j \geq 0} \left\| \frac{d^k}{dt^k} M_j \right\|_{L^\infty(\mathbb{R})} \lesssim_k 1, \tag{5.9}$$

for each $k \geq 0$. For example, we can just take $M_j(t) = \sigma(t) t^j$ where $\sigma$ is a bump function that equals 1 on $[-1/2, 1/2]$ and that is supported on $[-3/4, 3/4]$. We then check (5.9).

Let $k \geq 0$. By the Leibniz rule, for $j \geq k$ we have

$$\begin{aligned} \left\| \frac{d^k}{dt^k} M_j \right\|_{L^\infty(\mathbb{R})} &\leq \sum_{l=0}^{k} \binom{k}{l} \|\sigma^{(l)}\|_\infty j(j-1)\cdots(j-k+l+1) \left(\frac{3}{4}\right)^{j-k+l+1} \\ &\leq (k+1)! \sup_{0 \leq l \leq k} \|\sigma^{(l)}\|_\infty \left(\frac{3}{4}\right)^j j^k \\ &\lesssim_k 1, \end{aligned}$$

once we notice that $\sup_{j \geq k} (3/4)^j j^k \lesssim_k 1$ (say, using ratio test).

### 5.2.3 Reduction to weight inequality

The last inequality on Page 8 (note we directly defined $m_j$ so there is no $\tilde{m}_j$ in our notation:)

$$\begin{aligned} &\int \left\| \int \widehat{F}(\xi) m_j(\xi) e(\xi_1 x_1 + \xi_2(x_2 - y_2)) d\xi \right\|_{L^p_\#(B(y,R))}^p w_{B_R,F}(y) dy \\ &= \int \left\| \left( \widehat{F} m_j \right)^\vee (x_1, x_2 - y_2) \right\|_{L^p_\#(B(y,R))}^p w_{B_R,F}(y) dy \\ &= \int \|F * m_j^\vee (x_1, x_2 - y_2)\|_{L^p_\#(B(y,R))}^p w_{B_R,F}(y) dy \\ &= R^{-2} \int \int_{B(y,R)} |F * m_j^\vee(x)|^p \, dx \, w_{B_R,F}(y) dy. \end{aligned} \tag{5.10}$$

Now we show that $\left\|m_j^\vee\right\|_1 \lesssim 1$. Indeed, $m_j$ is a bump function with height $\sim 1$ and is supported on $N_{C/R}(Q)$, which is contained in a rectangle of dimensions $C'R^{-1/2} \times C'R^{-1}$. Hence we can write $m_j(\xi_1, \xi_2) = e(\xi_1^2 y_2)\psi(R^{1/2}\xi_1, R\xi_2)$ for some absolute bump function $\psi$. Thus $m_j^\vee(x_1, x_2) = R^{-3/2}\psi^\vee(R^{-1/2}x_1, R^{-1}x_2)$ will be a Schwartz function. Then it is easy to see that $\left\|m_j^\vee\right\|_1 \lesssim 1$.

As a consequence of Jensen's inequality, we have

$$\begin{aligned} \left|F * m_j^\vee(x_1, x_2 - y_2)\right|^p &= \left|\iint F(x_1 - z_1, x_2 - y_2 - z_2) m_j^\vee(z_1, z_2) dz_1 dz_2\right|^p \\ &\lesssim \iint |F(x_1 - z_1, x_2 - y_2 - z_2)|^p \, m_j^\vee(z_1, z_2) dz_1 dz_2 \\ &= |F|^p * \left|m_j^\vee\right| (x_1, x_2 - y_2). \end{aligned}$$

Continuing the computation in (5.10), we have

$$\begin{aligned} &R^{-2} \int \int_{B(y,R)} \left|F * m_j^\vee(x)\right|^p dx \, w_{B_R,F}(y) dy \\ &\lesssim R^{-2} \int \iint |F|^p * \left|m_j^\vee\right| (x_1, x_2 - y_2) 1_{B_R}(x_1 - y_1, x_2 - y_2) dx_1 dx_2 \, w_{B_R,F}(y) dy \\ &= R^{-2} \int \iint |F|^p * \left|m_j^\vee\right| (x) 1_{B_R}(x_1 - y_1, x_2) dx_1 dx_2 \, w_{B_R,F}(y) dy, \quad (x_2 - y_2 \mapsto x_2) \\ &= \int |F(x')|^p \left( \int \iint \left|m_j^\vee\right| (x - x') R^{-2} 1_{B_R}(x_1 - y_1, x_2) w_{B_R,F}(y) dx_1 dx_2 dy \right) dx'. \end{aligned}$$

Recall $F = f_{N_{C/R}(Q)}$. Hence it remains to show that

$$\int \int \left|m_j^\vee\right| (x - x') R^{-2} 1_{B_R}(x_1 - y_1, x_2) w_{B_R,F}(y) dx dy \lesssim w_{B_R,E}(x').$$

By symmetry of $1_{B_R}$, the left hand side is equal to

$$\int \left|m_j^\vee\right| * (R^{-2} 1_{B_R})(y_1 - x_1', -x_2') w_{B_R,F}(y) dy.$$

For simplicity of notations we write $x' = x$. Using

$$w_{B_R,E}(x) \gtrsim \left(1 + \frac{|x_1|}{R}\right)^{-E} \left(1 + \frac{|x_2|}{R}\right)^{-E},$$

it suffices to prove

$$\int \left|m_j^\vee\right| * (R^{-2} 1_{B_R})(y_1 - x_1, -x_2) w_{B_R,F}(y) dy \lesssim \left(1 + \frac{|x_1|}{R}\right)^{-E} \left(1 + \frac{|x_2|}{R}\right)^{-E},$$

which is a pure weight inequality independent of $f$ or $F$.

### 5.2.4 A derivative bound

We prove the following derivative bound:

$$\left\|\partial_{\xi_1}^{s_1}\partial_{\xi_2}^{s_2} m_j\right\|_\infty \lesssim_{s_1,s_2} \left(R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|\right)^{s_1} R^{s_2}. \tag{5.11}$$

We consider each factor in (5.8).

1. For the first term $e(\xi_1^2 y_2)$, we can show by induction that

$$\partial_{\xi_1}^{s_1} e(\xi_1^2 y_2) = \begin{cases} e(\xi_1^2 y_2) y_2^{(s_1+1)/2} \xi_1 \sum_{k=0}^{(s_1-1)/2} c_{s_1,k}(\xi_1^2 y_2)^k, & \text{if } s_1 \text{ is odd} \\ e(\xi_1^2 y_2) y_2^{s_1/2} \sum_{k=0}^{s_1/2} c_{s_1,k}(\xi_1^2 y_2)^k, & \text{if } s_1 \text{ is even.} \end{cases}$$

   We consider two cases: $|y_2| \le R$ or $|y_2| > R$.

   - If $|y_2| \le R$, then we have $|\xi_1^2 y_2| \le 1$ since $|\xi_1| \le R^{-1/2}$. Then $|y_1|^{1/2} \ge |y_1 \xi_1|$ and so in both cases of $s_1$, we have the bound
   $$\left\|\partial_{\xi_1}^{s_1} e(\xi_1^2 y_2)\right\|_\infty \lesssim_{s_1} |y_2|^{s_1/2}. \tag{5.12}$$
   - If $|y_2| > R$, then we have $|\xi_1^2 y_2| > 1$ and hence we have the bound
   $$\left|\sum_{k=0}^{K} c_{s_1,k}(\xi_1^2 y_2)^k\right| \lesssim_{s_1} |\xi_1^2 y_2|^K.$$
   So in both cases of $s_1$, we have the bound
   $$\left\|\partial_{\xi_1}^{s_1} e(\xi_1^2 y_2)\right\|_\infty \lesssim_{s_1} R^{-s_1/2}|y_2|^{s_1}. \tag{5.13}$$

   The first term has no contribution to $\partial_{\xi_2}^{s_2} m_j$.

2. The analysis for $\partial_{\xi_1}^{s_1}$ for the second term $M_j(R(\xi_2 - \xi_1^2)/(2C))$ is similar as the first term. We can show by induction that

$$\partial_{\xi_1}^{s_1} M_j\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right) = \begin{cases} R^{(s_1+1)/2}\xi_1 \sum_{k=0}^{(s_1-1)/2} c_{s_1,k} M_j^{(k+(s-1)/2)}\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)(R\xi_1^2)^k, & \text{if } s_1 \text{ is odd} \\ R^{s_1/2}\sum_{k=0}^{s_1/2} c_{s_1,k} M_j^{(k+s/2)}\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)(R\xi_1^2)^k, & \text{if } s_1 \text{ is even.} \end{cases}$$

   and hence

$$\partial_{\xi_2}^{s_2}\partial_{\xi_1}^{s_1} M_j\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right) = \begin{cases} R^{(s_1+1)/2+s_2}\xi_1 \sum_{k=0}^{(s_1-1)/2} c_{s_1,s_2,k} M_j^{(k+(s_1-1)/2+s_2)}\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)(R\xi_1^2)^k, & \text{if } s_1 \text{ is odd} \\ R^{s_1/2+s_2}\sum_{k=0}^{s_1/2} c_{s_1,s_2,k} M_j^{(k+s_1/2+s_2)}\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)(R\xi_1^2)^k, & \text{if } s_1 \text{ is even.} \end{cases}$$

   Since $|\xi_1| \le R^{-1/2}$, we always have $|R\xi_1^2| \le 1$, so using (5.9) we have the bound

$$\left\|\partial_{\xi_2}^{s_2}\partial_{\xi_1}^{s_1} M_j(R(\xi_2-\xi_1^2)/(2C))\right\|_\infty \lesssim_{s_1,s_2} R^{s_1/2+s_2}. \tag{5.14}$$

3. The third term is easy:
$$\left\|\partial_{\xi_1}^{s_1}\eta(R^{1/2}\xi_1)\right\|_\infty \lesssim_{s_1} R^{s_2/2}. \tag{5.15}$$
It has no contribution to $\partial_{\xi_2}^{s_2} m_j$.

4. The fourth term is easy:
$$\left\|\partial_{\xi_2}^{s_2}\eta(R\xi_2)\right\|_\infty \lesssim_{s_2} R^{s_2}. \tag{5.16}$$
It has no contribution to $\partial_{\xi_1}^{s_1} m_j$.

- If $|y_2| \le R$, then the bounds given by (5.12), (5.15) and (5.16) are all dominated by that of (5.14). Hence we have
$$\left\|\partial_{\xi_1}^{s_1}\partial_{\xi_2}^{s_2} m_j\right\|_\infty \lesssim_{s_1,s_2} R^{s_1/2+s_2}. \tag{5.17}$$

- If $|y_2| > R$, then the bound on $\partial_{\xi_1}^{s_1} m_j$ is dominated by (5.13) and the bound on $\partial_{\xi_2}^{s_2} m_j$ is dominated by (5.14). Hence we have
$$\left\|\partial_{\xi_1}^{s_1}\partial_{\xi_2}^{s_2} m_j\right\|_\infty \lesssim_{s_1,s_2} |y_2|^{s_1} R^{s_2-s_1/2}. \tag{5.18}$$
Combining (5.17) and (5.18), we get (5.11).

The rest are easy to follow; the key is to use Proposition (5.3) many times.

## 5.3 Higher dimensions

If $n \ge 3$, then Subsection 5.2.1 becomes a similar reduction to the case $[0, R^{-1/2}]^{n-1}$ using a similar affine shear transformation $T$. After such reduction, we need to perform $n-1$ Taylor expansions instead of just once (so we use $n$ Taylor expansions in total). The corresponding $m_j$ will then be (with $\xi = (\xi', \xi_n)$)

$$m_j(\xi) = e(|\xi'|^2 y_n) M_j\left(\frac{R(\xi_n - |\xi'|^2)}{2C}\right)\prod_{k=1}^{n-1}\eta(R^{1/2}\xi_k)\eta(R\xi_n).$$

The analogue of (7) is then

$$\int \left|m_j^\vee\right| * (R^{-n}1_{B_R})(y'-x', -x_n) w_{B_R,F}(y)dy \lesssim \left(1+\frac{|x'|}{R}\right)^{-E}\left(1+\frac{|x_n|}{R}\right)^{-E},$$

The derivate bound becomes

$$\left\|\partial_{\xi_1}^{s_1}\cdots\partial_{\xi_{n-1}}^{s_{n-1}}\partial_{\xi_n}^{s_n} m_j\right\|_\infty \lesssim_{s_1,\cdots,s_{n-1},s_n} \left(R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_n|\right)^{s_1+\cdots+s_{n-1}} R^{s_2},$$

and hence Equation (8) becomes

$$\phi_1(x') \lesssim_s R^{-(n-1)/2}\left(\frac{1}{1+\frac{|x'|}{R^{1/2}+R^{-1/2}|y_n|}}\right)^s.$$

Equation (10) becomes

$$\int \phi_1 * \left(R^{1-n} 1_{I_R}\right)(y' - x') w_{B_R,F}(y) dy \lesssim R \left(1 + \frac{|x'|}{R}\right)^{-E}.$$

We still have three cases in the end, and Proposition 5.3 is applicable in all dimensions. The general (4) we require is

$$w_{B_R,F}(x', x_n) \le \left(1 + \frac{|x'|}{R}\right)^{-E_1} \left(1 + \frac{|x_n|}{R}\right)^{-E_2}, \text{ if } E_1 + E_2 \le F,$$

and the corresponding assumptions on $F$ for all three cases becomes $F \ge E$, $F \ge E+n+1$, $F \ge 2E + 2$, respectively. Hence $\Gamma_n(E) = 2E + 2$ works for all dimensions.

# 6 $L^2$-decoupling

**Theorem 6.1.** *Let $Q$ be a cube with $l(Q) \ge R^{-1}$. Then for each cube $B_R$ with side length $R$ we have*

$$\|E_Q g\|_{L^2(w_{B_R})} \lesssim \left( \sum_{q \in \mathcal{P}_{R^{-1}}(Q)} \|E_q g\|^2_{L^2(w_{B_R})} \right)^{\frac{1}{2}}. \tag{6.1}$$

Note that the physical and frequency scales are exactly dual to each other in this case.

**Lemma 6.2.** *Let $f = \sum_n f_n$ and suppose* $\mathrm{supp} f_n := \{x \in \mathbb{R}^n : f_n(x) \neq 0\}$ *has finite overlap in the sense that*

$$\sum_n 1_{\mathrm{supp} f_n} \le C. \tag{6.2}$$

*Then $f_n$'s are almost orthogonal in the sense that for each $1 \le p < \infty$,*

$$\int |f|^p \lesssim \sum_n \int |f_n|^p.$$

*Proof.* We compute directly

$$\begin{aligned} \int |f|^p &= \int \left| \sum_n f_n \right|^p = \int \left| \sum_n f_n 1_{\mathrm{supp} f_n} \right|^p \\ &\le \int \left( \sum_n |f_n|^p \right)^{\frac{p}{p}} \left( \sum_n |1_{\mathrm{supp} f_n}|^{p'} \right)^{\frac{p}{p'}} \\ &\lesssim \int \sum_n |f_n|^p = \sum_n \int |f_n|^p, \end{aligned}$$

where the $\lesssim$ follows from (6.2). ∎

Hence to show (12), write, by Plancherel,

$$\|E_q g\|^2_{L^2(\eta_{B'})} = \int \left| \widehat{E_q g} * \widehat{\eta_{B'}^{1/2}} \right|^2 .$$

Then with $f_Q = \widehat{E_q g} * \widehat{\eta_{B'}^{1/2}}$, we see supp$f_Q$ is contained in a tiny neighbourhood of the paraboloid over $Q$, so supp$f_Q$ and supp$f_{Q'}$ overlap only if $Q$ and $Q'$ are adjacent. Hence each point in $\mathbb{R}^n$ lies in at most $2^n$ slightly enlarged cubes, so supp$f_Q$ has finite overlap. Using Lemma 6.2 with $f_Q$ and $\sum_{q \in \mathrm{Part}_{1/R}(Q)} E_q g = E_Q g$, we have

$$\int \left| \widehat{E_Q g} * \widehat{\eta_{B'}^{1/2}} \right|^2 \lesssim \int \sum_{q \in \mathrm{Part}_{1/R}(Q)} \left| \widehat{E_q g} * \widehat{\eta_{B'}^{1/2}} \right|^2 = \sum_{q \in \mathrm{Part}_{1/R}(Q)} \|E_q g\|^2_{L^2(\eta_{B'})}.$$

For the left hand side of the above equation, we have

$$\int \left| \widehat{E_Q g} * \widehat{\eta_{B'}^{1/2}} \right|^2 = \|E_Q g\|^2_{L^2(\eta_{B'})} \geq \|E_Q g\|^2_{L^2(B')}.$$

# 7 Parabolic Rescaling

**Proposition 7.1.** *Let $0 < \delta \leq \sigma \leq 1$ and $p \geq 2$. For each cube $Q \subseteq [0,1]^{n-1}$ with $l(Q) = \sigma^{1/2}$ and each cube $B \subseteq \mathbb{R}^n$ with $l(B) \geq \delta^{-1}$ we have*

$$\|E_Q g\|_{L^p(w_B)} \lesssim \mathrm{Dec}_n(p, \delta\sigma^{-1}) \left( \sum_{q \in \mathcal{P}_{\delta^{1/2}}(Q)} \|E_q g\|^2_{L^p(w_B)} \right)^{\frac{1}{2}} .$$

*Edited Aug 2023:* there was a problem with the weight inequality in the previous version, so I removed this part.

## 7.1 Trivial decoupling and trivial scaling

The following propositions are too trivial to be written in [BD], but they are often used. Note that in this subsection, we shall not assume that the constants $c$ and $C$ are dyadic.

**Proposition 7.2** (Trivial decoupling)**.** *If $c \leq 1$ is a constant, then* $\mathrm{Dec}(c) \lesssim_c 1$.

(Note that $\mathrm{Dec}(c, p)$ does not make sense if $c > 1$.)

*Proof.* If we cover $[0,1]^{n-1}$ by finitely overlapping $c$-cubes $Q$, then there are at most

$O(c^{1-n})$ such cubes. Hence for any cube $B$ with $l(B) \geq c^{-1}$,

$$\begin{aligned}\|Eg\|_{L^p(w_B)} &\leq \sum_Q \|E_Q g\|L^p(w_B) \\ &\leq \left(\sum_Q \|E_Q g\|^2_{L^p(w_B)}\right)^{\frac{1}{2}} \left(\sum_Q 1^2\right)^{\frac{1}{2}} \\ &\lesssim_c \left(\sum_Q \|E_Q g\|^2_{L^p(w_B)}\right)^{\frac{1}{2}}.\end{aligned}$$

Hence $\mathrm{Dec}(c) \lesssim_c 1$. □

**Proposition 7.3** (Trivial scaling)**.** *If $C \geq 1$ is a constant, we have* $\mathrm{Dec}(\delta) \lesssim_C \mathrm{Dec}(C\delta)$.

*Proof.* Consider $B = B_{\delta^{-1}} \subseteq \mathbb{R}^n$ and a finitely overlapping of $[0,1]^{n-1}$ using $C\delta$ cubes $Q$. By the remark in the last subsection above, decoupling with scale $C\delta$ can be also applied to a spacial cube $B$ of scale $\delta^{-1} \geq (C\delta)^{-1}$. Thus

$$\|Eg\|_{L^p(w_B)} \leq \mathrm{Dec}(C\delta) \left(\sum_{Q \in \mathcal{P}_{(C\delta)^{1/2}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p(w_B)}\right)^{\frac{1}{2}}. \tag{7.1}$$

Then we use cubes $q$ of scales $\delta$ to cover $Q$. Using parabolic rescaling proved just now to the cube $B$ with $l(B) = \delta^{-1}$, we have

$$\|E_Q g\|_{L^p(w_B)} \leq \mathrm{Dec}(C^{-1}) \left(\sum_{q \in \mathcal{P}_{\delta^{1/2}}(Q)} \|E_q g\|^2_{L^p(w_B)}\right)^{\frac{1}{2}}. \tag{7.2}$$

By trivial decoupling 7.2, we have $\mathrm{Dec}(C^{-1}) \lesssim_C 1$. Combining (7.1) and (7.2) gives

$$\|Eg\|_{L^p(w_B)} \lesssim_C \mathrm{Dec}(C\delta) \left(\sum_{q \in \mathcal{P}_{\delta^{1/2}}([0,1]^{n-1})} \|E_q g\|^2_{L^p(w_B)}\right)^{\frac{1}{2}}.$$

Hence $\mathrm{Dec}(\delta) \lesssim_C \mathrm{Dec}(C\delta)$. □

**Corollary 7.4** (Non-dyadic scales)**.** *To prove the decoupling theorem for all frequency scales $\delta \leq 1$, it suffices to prove it for all dyadic scales $\delta \in 2^{-\mathbb{N}}$.*

# 8 Reduction to Multilinear Decoupling

## 8.1 Remark 8.3

**Lemma 8.1.** *Let $P^i = (\xi^{(i)}, |\xi^{(i)}|^2)$, $1 \leq i \leq n$ be points on $\mathbb{P}^{n-1}$. Then the volume $V$ of the parallelepiped spanned by the (upward) unit normals $n(P^i)$ is comparable to the area $A$ of the $n$-simplex with vertices $\xi^{(i)}$, $1 \leq i \leq n$.*

*Proof.* By shoelace formula, the area $A$ of the $n$-simplex equals

$$\frac{1}{(n-1)!}\left|\det\begin{bmatrix}\xi_1^{(1)} & \xi_2^{(1)} & \cdots & \xi_{n-1}^{(1)} & 1\\ \xi_1^{(2)} & \xi_2^{(2)} & \cdots & \xi_{n-1}^{(2)} & 1\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ \xi_1^{(n)} & \xi_2^{(n)} & \cdots & \xi_{n-1}^{(n)} & 1\end{bmatrix}\right|$$

On the other hand, we have $n(P^i) = \frac{(-2\xi^{(i)},1)}{\sqrt{4|\xi^{(i)}|^2+1}}$. The volume $V$ of the parallelepiped formed by $P^i$, $1 \le i \le n$ equals

$$\left|\det\begin{bmatrix}-2\xi_1^{(1)} & -2\xi_2^{(1)} & \cdots & -2\xi_{n-1}^{(1)} & 1\\ -2\xi_1^{(2)} & -2\xi_2^{(2)} & \cdots & -2\xi_{n-1}^{(2)} & 1\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ -2\xi_1^{(n)} & -2\xi_2^{(n)} & \cdots & -2\xi_{n-1}^{(n)} & 1\end{bmatrix}\right| \prod_{i=1}^{n}(4|\xi^{(i)}|^2+1)^{-\frac{1}{2}} \sim A.$$

□

We change the definition of transversality, so that in the argument below, in all dimensions $n$, the corresponding cubes are $K^{-1}$-transverse. This makes the notations a little simpler.

**Definition 8.2.** *We say that cubes $Q_1, \ldots, Q_n \subseteq [0,1]^{n-1}$ are $\nu$-transverse if the volume of the parallelepiped spanned by unit normals $n(P^i)$ is greater than $\nu^{n-1}$, for each choice of $P^i = (\xi^{(i)}, |\xi^{(i)}|^2) \in \mathbb{P}^{n-1}$ with $\xi^{(i)} \in Q_i$.*

**Lemma 8.3.** *Let $n \ge 2$, $K \ge 1$. Then there is an absolute (dyadic) constant $C_1 = C_1(n)$ such that the following is true. For each $1 \le i \le n$, let $\alpha_i$ be cubes in $[0,1]^{n-1}$ with side length $K^{-1}$ and centres $c_i$. Suppose that*

1. $|c_1 - c_2| \ge C_1 K^{-1}$.
2. *For all $3 \le i \le n$, the hyperplane $H_i$ formed by $\{c_j : 1 \le j \le i-1\}$ is $(i-2)$-dimensional.*
3. *For all $3 \le i \le n$, the distance from $c_i$ to $H_i$ is $\ge C_1 K^{-1}$.*

*Then $\alpha_i$, $1 \le i \le n$ are $K^{-1}$-transverse.*

## 8.2 Proposition 8.4

The following is modified from Definition 8.1 in [BD].

**Definition 8.4** (Multilinear decoupling constant)**.** *Let $E \ge 100n$, $2 \le p < \infty$, $0 < \nu \le 1$. We define $\mathrm{MDec}_n(\delta, \nu, m) = \mathrm{MDec}_n(\delta, p, \nu, m, E)$ be the smallest constant such that the inequality*

$$\left[\sum_{\Delta \in \mathcal{P}_{\mu^{-1}}(B)} \prod_{i=1}^{n} \|E_{Q_i} g\|_{L^p(\Delta)}^{p/n}\right]^{\frac{1}{p}} \le \mathrm{MDec}_n(\delta, \nu, m) \prod_{i=1}^{n}\left(\sum_{q_i \in \mathcal{P}_{\delta^{1/2}}(Q_i)} \|E_{q_i} g\|_{L^p(w_B)}^2\right)^{\frac{1}{2n}}$$

*holds for each cube $B \subseteq \mathbb{R}^n$ with $l(B) = \delta^{-1}$, each $g : [0,1]^{n-1} \to \mathbb{C}$ and each $n$-tuple of $\nu$-transverse cubes $Q_i$ with equal side length $\mu \ge \delta^{2^{-m}}$.*

The parameter $m$ is introduced only due to some technicality in the final induction step, when we will take $m$ to be a large positive integer.

The following is Proposition 8.4. in [BD].

**Proposition 8.5.** *Let $E \geq 100n$ and $2 \leq p < \infty$. Assume we have the decoupling theorem for a lower dimensional (thickened) paraboloid:*

$$\mathrm{Dec}_2(\delta, p, \Gamma_2(E)) \lesssim_\varepsilon \delta^{-\varepsilon}.$$

*Then there is $C_\varepsilon > 0$ and an absolute constant $C > 0$ such that for all $K \geq 1$, all $m \geq 1$ and all $R \geq K^{2^m}$,*

$$\begin{aligned}\|Eg\|_{L^p(w_{B_R})} \leq C_\varepsilon K^\varepsilon & \left( \sum_{\alpha \in \mathcal{P}_{K^{-1}}([0,1]^2)} \|E_\alpha g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}} \\ & + C_\varepsilon K^\varepsilon \left( \sum_{\beta \in \mathcal{P}_{K^{-1/2}}([0,1]^2)} \|E_\beta g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}} \\ & + CK^4 \mathrm{MDec}_3(R^{-1}, p, K^{-1}, m, E) \left( \sum_{\Delta \in \mathcal{P}_{R^{-1}}([0,1]^2)} \|E_\Delta g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}}.\end{aligned}$$

### 8.2.1 Three scenarios

Let us consider a typical case $n = 3$ first. For each $\alpha \in \mathcal{P}_{K^{-1}}([0,1]^2)$, define

$$c_\alpha(B_K) = \left( \frac{1}{|B_K|} \int |E_\alpha g|^p \right)^{\frac{1}{p}} = \|E_\alpha g\|_{L^p_\#(B_K)}.$$

Let $\alpha^* = \alpha^*(K)$ be a cube that maximizes $c_\alpha(B_K)$.

We define

$$S_{\text{big}} = \{\alpha : c_\alpha(B_K) \geq K^{-2} c_{\alpha^*}(B_K)\}. \tag{8.1}$$

With the $C_1$ in lemma 8.3, we define

$$S_L = \{\xi \in \mathbb{R}^2 : d(\xi, L) \leq C_1 K^{-1}\}. \tag{8.2}$$

**Proposition 8.6.** *With notations above, we have*

$$\|Eg\|_{L^p_\#(B_K)} \lesssim \left[ c_{\alpha^*}(B_K) + K^4 \max_{\substack{\alpha_1,\alpha_2,\alpha_3 \\ K^{-1}-\text{transverse}}} \prod_{i=1}^3 c^{\frac{1}{3}}_{\alpha_i}(B_K) \right] + \left\| \sum_{\alpha \subseteq S_L} E_\alpha g \right\|_{L^p_\#(B_K)}. \tag{8.3}$$

*Proof.* The three scenarios are as follows (see Figure 2:)

1. In Case 1, by triangle inequality we have

$$\|Eg\|_{L^p_\#(B_K)} \leq \sum_\alpha \|E_\alpha g\|_{L^p_\#(B_K)} = \sum_\alpha c_\alpha(B_K) \lesssim_{C_1} c_{\alpha^*}(B_K).$$

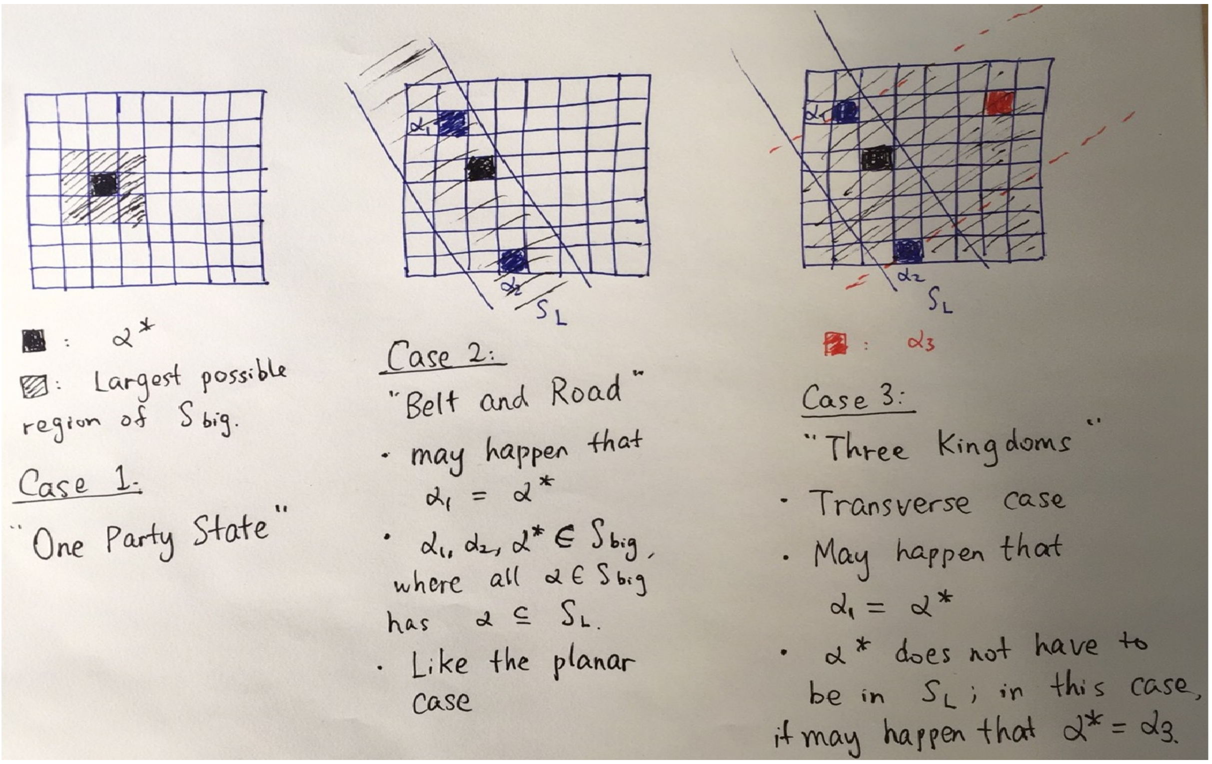


Figure 2: Three Different Scenarios

2. In Case 2, we have the bound

$$|Eg| \leq \left|\sum_{\alpha\subseteq S_L} E_\alpha g\right| + \left|\sum_{\alpha\nsubseteq S_L} E_\alpha g\right| \leq \left|\sum_{\alpha\subseteq S_L} E_\alpha g\right| + \sum_{\alpha\notin S_{\text{big}}} |E_\alpha g|,$$

and hence

$$\begin{aligned}\|Eg\|_{L^p_\#(B_K)} &\lesssim \left\|\sum_{\alpha\subseteq S_L} E_\alpha g\right\|_{L^p_\#(B_K)} + \sum_{\alpha\notin S_{\text{big}}} \|E_\alpha g\|_{L^p_\#(B_K)} \\ &\lesssim \left\|\sum_{\alpha\subseteq S_L} E_\alpha g\right\|_{L^p_\#(B_K)} + K^2K^{-2}c_{\alpha^*}(B_K),\end{aligned}$$

since there are $O(K^2)$ cubes $\alpha$.

3. In Case 3, we have the bound

$$\|Eg\|_{L^p_\#(B_K)} \leq \sum_\alpha \|E_\alpha g\|_{L^p_\#(B_K)} \leq K^2 c_{\alpha^*}(B_K) \leq K^2K^2c_{\alpha_i}(B_K),$$

for all $i = 1,2,3$. Taking geometric averages over $i = 1,2,3$ on both sides,

$$\|Eg\|_{L^p_\#(B_K)} \leq K^4 \prod_{i=1}^3 c_{\alpha_i}^{\frac{1}{3}}(B_K) \leq K^4 \max_{\substack{\alpha_1,\alpha_2,\alpha_3 \\ K^{-1}-\text{transverse}}} \prod_{i=1}^3 c_{\alpha_i}^{\frac{1}{3}}(B_K).$$

□

Using Proposition 8.6, the first and the second terms of (8.3) are easy to bound, using the definition of $c_\alpha(B_K)$:

$$c_{\alpha^*}(B_K) \le \|E_{\alpha^*}g\|_{L^p_\#(B_K)} \le \left(\sum_{\alpha\in \mathrm{Part}_{K^{-1}}([0,1]^{n-1})} \|E_\alpha g\|^2_{L^p_\#(B_K)}\right)^{\frac{1}{2}}, \tag{8.4}$$

Also,

$$K^4 \max_{\substack{\alpha_1,\alpha_2,\alpha_3\\ K^{-1}-\text{transverse}}} \prod_{i=1}^{3} c^{\frac{1}{3}}_{\alpha_i}(B_K) = K^4 \max_{\substack{\alpha_1,\alpha_2,\alpha_3\\ K^{-1}-\text{transverse}}} \prod_{i=1}^{3} \|E_{\alpha_i}g\|^{\frac{1}{3}}_{L^p_\#(B_K)}. \tag{8.5}$$

### 8.2.2 Analysis on the strip

For the term in Case 2 above, we define $E_{S_L}g = \sum_{\alpha\subseteq S_L} E_\alpha g$. By a translation and rotation we may assume that $L$ is the line $\xi_2 = 1$ (using $\xi_2 = 0$ will result in the following $\delta' \sim K^{-2}$, which is unrepresentative). We also assume the spacial cube is $[0,K]^3$. Then for each $x_2$, let $f_{x_2}(\xi_1,\xi_3) = (\mathcal{F}_{1,3}(e(-\cdot_3)E_{S_L}g))(\xi_1,\xi_3)$, that is,

$$\begin{aligned} f_{x_2}(\xi_1,\xi_3) &= \iint e(-x_3)E_{S_L}g(x_1,x_2,x_3)e(-x_1\xi_1 - x_3\xi_3)dx_1dx_3 \\ &= \iiiint (g1_{S_L})(\eta_1,\eta_2)e(x_1\eta_1 + x_2\eta_2 + x_3\eta_1^2 + x_3\eta_2^2 - x_3)d\eta_1 d\eta_2 e(-x_1\xi_1 - x_3\xi_3)dx_1dx_3 \\ &= \iint (g1_{S_L})(\eta_1,\eta_2)e(x_2\eta_2)\delta_0(\eta_1 - \xi_1)\delta_0(\eta_1^2+\eta_2^2 - 1 - \xi_3)d\eta_1 d\eta_2 \\ &= \int (g1_{S_L})(\xi_1,\eta_2)e(x_2\eta_2)\delta_0(\xi_1^2 + \eta_2^2 - 1 - \xi_3)d\eta_2. \end{aligned}$$

Next we perform a change of variables $u = \xi_1^2 + \eta_2^2 - \xi_3 - 1$, with $\frac{du}{d\eta_2} = 2\eta_2$. Then the above is equal to

$$\begin{aligned} &\int (g1_{S_L})\left(\xi_1, \sqrt{u+\xi_3-\xi_1^2+1}\right) e\left(x_2\sqrt{u+\xi_3-\xi_1^2+1}\right)\delta_0(u)\frac{1}{2\sqrt{u+\xi_3-\xi_1^2+1}}du \\ &= (g1_{S_L})\left(\xi_1,\sqrt{\xi_3-\xi_1^2+1}\right) e\left(x_2\sqrt{\xi_3-\xi_1^2+1}\right)\frac{1}{2\sqrt{\xi_3-\xi_1^2+1}}. \end{aligned} \tag{8.6}$$

Hence for each $x_2$, $f_{x_2}(\xi_1,\xi_3)$ is supported in the set $S = \left\{\left(\xi_1,\sqrt{\xi_3-\xi_1^2+1}\right)\in S_L\right\}$, that is,

$$S = \{(\xi_1,\xi_3) : 0 \le \xi_1 \le 1, \xi_1^2 - 2C_1K^{-1} + C_1^2K^{-2} \le \xi_3 \le \xi_1^2 + 2C_1K^{-1} + C_1^2K^{-2}\}. \tag{8.7}$$

Let $\delta' = 2C_1K^{-1} + C_1^2K^{-2}$ and assume $K$ is large enough so that $\delta' \sim C_1K^{-1}$. So $S$ is in turn contained in the $\delta'$-neighbourhood of the parabola $\xi_3 = \xi_1^2$ over $\xi \in [0,1]$. Hence we can apply (a slightly generalised version of) Theorem 5.1 with $n = 2$, $f = e(-x_3)E_{S_L}$ and $B_R = [0, K]^2$ to get

$$\|E_{S_L} g\|_{L^p_{x_1,x_3}(w_{[0,K]^2,E})}(x_2) \lesssim \mathrm{Dec}_2(K^{-1}, p, \Gamma_2(E)) \left( \sum_{I \in \mathrm{Part}_{K^{-1/2}}[0,1]_{\xi_1}} \left\| (e(-\cdot_3)E_{S_L} g)_{N_{\delta'}(I)} \right\|^2_{L^p_{x_1,x_3}(w_{[0,K]^2,E})} \right)^{\frac{1}{2}} (x_2), \tag{8.8}$$

where $N_{\delta'}(I)$ denotes the $\delta'$-neighbourhood of the parabola $\xi_3 = \xi_1^2$ over $\xi_1 \in I$. Note the implicit constant is independent of $x_2$.

We cover $S_L$ by disjoint rectangles $U$ of dimensions $C_1 K^{-1}$ and $K^{-1/2}$ (see Figure 3). The scale $K^{-1/2}$ is chosen to be the same as the scale of partitioning intervals $I$ in (8.8). Note $\sum_{I \in \mathrm{Part}_{K^{-1/2}}[0,1]_{\xi_1}}$ can be replaced by $\sum_U$. Moreover,

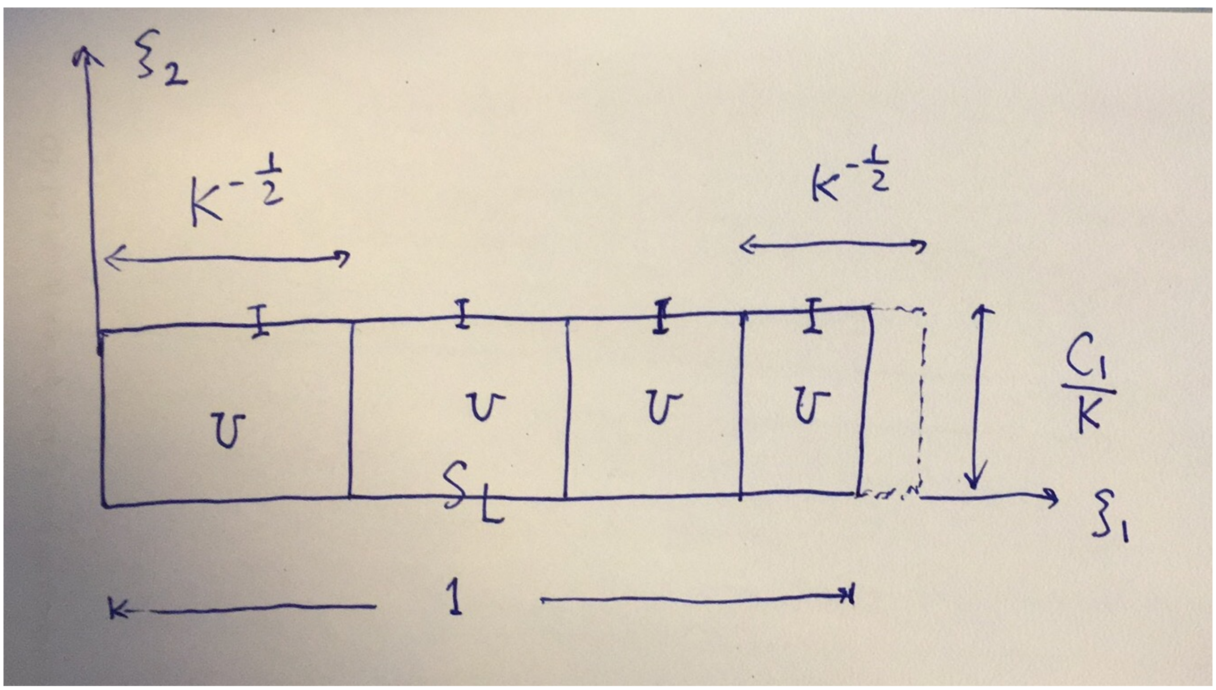


Figure 3: Dividing $S_L$ by rectangles $U$

$$\begin{aligned}
&(e(-\cdot_3)E_{S_L} g)_{N_{\delta'}(I)}(x_1, x_3) \\
&= \int_{N_{\delta'}(I)} (\mathcal{F}_{1,3}(e(-\cdot_3)E_{S_L} g))(\xi_1, \xi_3) e(x_1\xi_1 + x_3\xi_3) d\xi_1 d\xi_3 \\
&\overset{(8.6)}{=} \int_{N_{\delta'}(I)} (g 1_{S_L}) \left(\xi_1, \sqrt{\xi_3 - \xi_1^2 + 1}\right) e\left(x_2 \sqrt{\xi_3 - \xi_1^2 + 1}\right) \frac{1}{2\sqrt{\xi_3 - \xi_1^2 + 1}} e(x_1\xi_1 + x_3\xi_3) d\xi_1 d\xi_3.
\end{aligned}$$

Then in view of (8.7) we observe

$$1_{N_{\delta'}(I) \cap S_L}(\xi_1, \xi_3) = 1_U \left(\xi_1, \sqrt{\xi_3 - \xi_1^2 + 1}\right).$$

Hence

$$
\begin{aligned}
&(e(-\cdot_3)E_{S_L}g)_{N_{\delta'}(I)}(x_1,x_3)\\
&=\int (g1_U)\left(\xi_1,\sqrt{\xi_3-\xi_1^2+1}\right)e\left(x_2\sqrt{\xi_3-\xi_1^2+1}\right)\frac{1}{2\sqrt{\xi_3-\xi_1^2+1}}e(x_1\xi_1+x_3\xi_3)d\xi_1 d\xi_3.
\end{aligned}
$$

Changing variables back $v=\sqrt{\xi_3-\xi_1^2+1}$, the above is equal to

$$e(-x_3)E_U g(x_1,x_2,x_3).$$

Hence (8.8) becomes

$$\|E_{S_L}g\|_{L^p_{x_1,x_3}(w_{[0,K]^2,E})}(x_2)\lesssim \mathrm{Dec}_2(K^{-1},p,\Gamma_2(E))\left(\sum_U \|E_U g\|^2_{L^p_{x_1,x_3}(w_{[0,K]^2,E})}\right)^{\frac12}(x_2)$$

Raising both sides to the power $p$ and integrating over $x_2\in[0,K]$, using Minkowski's inequality we have

$$\|E_{S_L}g\|_{L^p(B_K,E)}\lesssim \mathrm{Dec}_2(K^{-1},p,\Gamma_2(E))\left(\sum_U \|E_U g\|^2_{L^p(w_{B_K},E)}\right)^{\frac12}. \tag{8.9}$$

We next show how to use triangle inequality to bound the term above on the right. For each $U$, enlarge it to become a square $U'$ of dimension $l(U')=K^{-1/2}$. Then $V:=U'\backslash U\subseteq S_L^c$, and $E_U g=E_{U'}g-E_V g$ (see Figure 4). Using triangle inequality, we have

$$\left(\sum_U \|E_U g\|^2_{L^p(w_{B_K})}\right)^{\frac12}\lesssim\left(\sum_{U'} \|E_{U'} g\|^2_{L^p(w_{B_K})}\right)^{\frac12}+\left(\sum_V \|E_V g\|^2_{L^p(w_{B_K})}\right)^{\frac12}. \tag{8.10}$$

Since each $U'$ is a square within $[0,1]^{n-1}$ with scale $K^{-1/2}$, we can bound the $\sum_{U'}$ term by:

$$\left(\sum_{\beta\in \mathrm{Part}_{K^{-1/2}}([0,1]^2)} \|E_\beta g\|^2_{L^p(w_{B_K})}\right)^{\frac12}. \tag{8.11}$$

For the $\sum_V$ term, we further split $V$ into $K^{-1}$ cubes $\alpha$. (They are the same $\alpha$'s that partitions $[0,1]^{n-1}$ at the beginning. See Figure 4 again.) Each such $\alpha$ lies outside $S_L$, so it has small contribution in the second scenario (see Figure 2 again):

$$\|E_\alpha g\|_{L^p(w_{B_K})}\le K^{-2}|B_K|^{\frac1p}c_{\alpha^*}(B_K)=K^{-2}\|E_{\alpha^*}g\|_{L^p(w_{B_K})}.$$

Since there are around $(K^{1/2})^2=K$ such cubes $\alpha$ in a single $V$ and we have around $K^{1/2}$

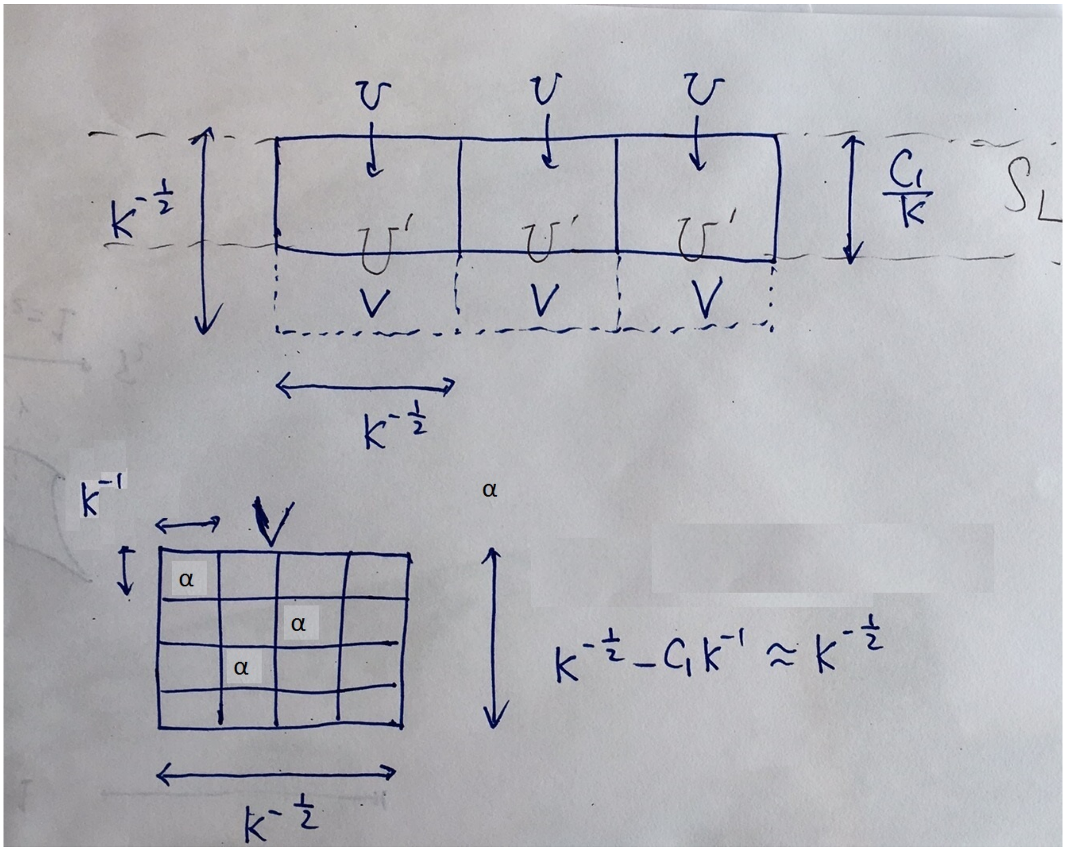


Figure 4: Enlarging $U$ to a square $U'$

such $V$'s,

$$\begin{aligned}
\left(\sum_V \|E_V g\|^2_{L^p(w_{B_K})}\right)^{\frac{1}{2}} &\le \left(\sum_V \left(\sum_{\alpha\subseteq V} \|E_\alpha g\|_{L^p(w_{B_K})}\right)^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_V \left(K^{-1}\|E_{\alpha^*} g\|_{L^p(w_{B_K})}\right)^2\right)^{\frac{1}{2}} \\
&\lesssim K^{-3/4}\|E_{\alpha^*} g\|_{L^p(w_{B_K})} \\
&\le K^{-3/4}\left(\sum_{\alpha\in \mathrm{Part}_{K^{-1}}([0,1]^2)} \|E_\alpha g\|^2_{L^p(w_{B_K})}\right)^{\frac{1}{2}}.
\end{aligned}$$

#### 8.2.3 Summation

Combining (8.3), (8.4), (8.5), (8.9), (8.10), (8.11) and the estimate right above and using the assumption that $\mathrm{Dec}_2(K^{-1}, p, \Gamma_2(E)) \lesssim_\varepsilon K^\varepsilon$, we get

$$\begin{aligned}\|Eg\|_{L^p(B_K)} \lesssim_\varepsilon &\left(\sum_{\alpha\in\mathrm{Part}_{K^{-1}}([0,1]^2)} \|E_\alpha g\|^2_{L^p(B_K)}\right)^{\frac12}\\ &+ K^\varepsilon \left(\sum_{\beta\in\mathrm{Part}_{K^{-1/2}}([0,1]^2)} \|E_\beta g\|^2_{L^p(w_{B_K})}\right)^{\frac12}\\ &+ K^\varepsilon K^{-3/4} \left(\sum_{\alpha\in\mathrm{Part}_{K^{-1}}([0,1]^2)} \|E_\alpha g\|^2_{L^p(w_{B_K})}\right)^{\frac12}\\ &+ K^4 \max_{\substack{\alpha_1,\alpha_2,\alpha_3\\ K^{-1}-\text{transverse}}} \prod_{i=1}^3 \|E_{\alpha_i} g\|^{\frac13}_{L^p(B_K)},\end{aligned}$$

Combining the terms above and using the trivial inequality $1_{B_K} \lesssim w_{B_K}$ except for the last term, we have

$$\begin{aligned}\|Eg\|_{L^p(B_K)} \lesssim_\varepsilon K^\varepsilon &\Bigg[\left(\sum_{\alpha\in\mathrm{Part}_{K^{-1}}([0,1]^2)} \|E_\alpha g\|^2_{L^p(w_{B_K})}\right)^{\frac12}\\ &+ \left(\sum_{\beta\in\mathrm{Part}_{K^{-1/2}}([0,1]^2)} \|E_\beta g\|^2_{L^p(w_{B_K})}\right)^{\frac12}\Bigg]\\ &+ K^4 \max_{\substack{\alpha_1,\alpha_2,\alpha_3\\ K^{-1}-\text{transverse}}} \prod_{i=1}^3 \|E_{\alpha_i} g\|^{\frac13}_{L^p(B_K)}.\end{aligned}$$

Then we raise both sides to the power $p$, sum over $B_K \in \mathcal{P}_K(B_R)$ and then raise both sides to the power $1/p$. We also use Minkowski's inequality with exponents $p \geq 2$ and the inequality $\sum_{B_K} w_{B_K} \lesssim w_R$ which follows from Proposition 3.1. The left hand side becomes $\|Eg\|_{L^p(B_R)}$. The first two terms on the right become

$$\left(\sum_{\alpha\in\mathrm{Part}_{K^{-1}}([0,1]^2)} \|E_\alpha g\|^2_{L^p(w_{B_R})}\right)^{\frac12} \text{ and } \left(\sum_{\beta\in\mathrm{Part}_{K^{-1/2}}([0,1]^2)} \|E_\beta g\|^2_{L^p(w_{B_R})}\right)^{\frac12},$$

respectively.

To deal with the last term, we note that for each $B_K$, the maximum is attained at some $K^{-1}$-transverse triples $\alpha_1, \alpha_2, \alpha_3$ dependent on each individual $B_K$. Thus

$$\max_{\substack{\alpha_1,\alpha_2,\alpha_3\\ K^{-1}-\text{transverse}}} \prod_{i=1}^3 \|E_{\alpha_i} g\|^{\frac13}_{L^p(B_K)} = \prod_{i=1}^3 \|E_{\alpha_i(B_K)} g\|^{\frac13}_{L^p(B_K)}.$$

Thus the above process gives

$$K^4 \left( \sum_{B_K \in \mathcal{P}_K(B_R)} \max_{\substack{\alpha_1,\alpha_2,\alpha_3 \\ K^{-1}-\text{transverse}}} \prod_{i=1}^{3} \|E_{\alpha_i} g\|_{L^p(B_K)}^{\frac{p}{3}} \right)^{\frac{1}{p}}$$

$$= K^4 \left( \sum_{B_K \in \mathcal{P}_K(B_R)} \prod_{i=1}^{3} \|E_{\alpha_i(B_K)} g\|_{L^p(B_K)}^{\frac{p}{3}} \right)^{\frac{1}{p}}$$

$$\leq K^4 \, \mathrm{MDec}_3(R^{-1}, p, K^{-1}, m, E) \left( \prod_{i=1}^{3} \sum_{q_i \in \mathcal{P}_{R^{-1/2}}(\alpha_i)} \|E_{q_i} g\|_{L^p(w_{B_R})}^2 \right)^{\frac{1}{6}},$$

where, in the notations of Defintion 8.4, $l(B) = R = \delta^{-1}$, $l(\alpha_i) = K^{-1} \geq \delta^{2^{-m}}$ since $R \geq K^{2^m}$.

For each $1 \leq i \leq 3$, we further use the trivial bound

$$\sum_{q_i \in \mathcal{P}_{R^{-1/2}}(\alpha_i)} \|E_{q_i} g\|_{L^p(w_{B_R})}^2 \leq \sum_{\Delta \in \mathcal{P}_{R^{-1/2}}([0,1]^2)} \|E_{\Delta} g\|_{L^p(w_{B_R})}^2.$$

Then the geometric mean is also bounded by $\sum_{\Delta \in \mathcal{P}_{R^{-1/2}}([0,1]^2)} \|E_\Delta g\|_{L^p(w_{B_R})}^2$.

Combining all the above computations, we get

$$\begin{aligned} \|Eg\|_{L^p(B_R)} \leq\, & C_\varepsilon K^\varepsilon \left( \sum_{\alpha \in \mathcal{P}_{K^{-1}}([0,1]^2)} \|E_\alpha g\|_{L^p(w_{B_R})}^2 \right)^{\frac{1}{2}} \\ & + C_\varepsilon K^\varepsilon \left( \sum_{\beta \in \mathcal{P}_{K^{-1/2}}([0,1]^2)} \|E_\beta g\|_{L^p(w_{B_R})}^2 \right)^{\frac{1}{2}} \\ & + C K^4 \mathrm{MDec}_3(R^{-1}, p, K^{-1}, m, E) \left( \sum_{\Delta \in \mathcal{P}_{R^{-1}}([0,1]^2)} \|E_\Delta g\|_{L^p(w_{B_R})}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Lastly, we can replace $L^p(B_R)$ on the left hand side by $L^p(w_{B_R})$ as usual. This proves Proposition 8.5.

## 8.3 Parabolic rescaling

**Proposition 8.7** (Parabolic rescaling)**.** *Let $m \geq 1$. Let $\tau \subseteq [0,1]^2$ be a square with side length $\delta \geq R^{-1/2} K^{2^{m-1}}$. Assume*

$$\mathrm{Dec}_2(\delta', p, \Gamma_2(E)) \lesssim_\varepsilon \delta'^{-\varepsilon}.$$

*Then for all $K \geq 1$ and all $R \geq K^{2^m}$,*

$$\|E_\tau g\|_{L^p(w_{B_R})} \leq C_\varepsilon K^\varepsilon \left( \sum_{\alpha \in \mathcal{P}_{\delta K^{-1}}(\tau)} \|E_\alpha g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}} + C_\varepsilon K^\varepsilon \left( \sum_{\beta \in \mathcal{P}_{\delta K^{-1/2}}(\tau)} \|E_\beta g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}} + K^4 \mathrm{MDec}_3(\delta^{-2}R^{-1}, K^{-1}, m) \left( \sum_{\Delta \in \mathcal{P}_{R^{-1}}(\tau)} \|E_\Delta g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}}.$$

*Proof.* Refer to the notations in previous parabolic rescaling. We change variables as before. Cover $T(B_R)$ by $\delta^2 R$ cubes $B'$. Applying Proposition 8.5 to each $B'$ with the same $K \geq 1$ and $\delta^2 R \geq K^{2^m}$, we have

$$\|EG\|_{L^p(B')} \lesssim_\varepsilon K^\varepsilon \left[ \left( \sum_{\alpha \in \mathrm{Part}_{K^{-1}}([0,1]^2)} \|E_\alpha G\|^2_{L^p(w_{B'})} \right)^{\frac{1}{2}} + \left( \sum_{\beta \in \mathrm{Part}_{K^{-1/2}}([0,1]^2)} \|E_\beta G\|^2_{L^p(w_{B'})} \right)^{\frac{1}{2}} \right] + CK^4 \mathrm{MDec}_3(\delta^{-2}R^{-1}, K^{-1}, m) \sum_{\Delta \in \mathcal{P}_{\delta^{-1}R^{-1/2}}([0,1]^2)} \|E_\Delta G\|^2_{L^p(w_{B'})}.$$

Change variables back. The first and the second terms are bounded above by

$$K^\varepsilon \left( \sum_{\alpha \in \mathrm{Part}_{\delta K^{-1}}(\tau)} \|E_\alpha g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}} \text{ and } K^\varepsilon \left( \sum_{\beta \in \mathrm{Part}_{\delta K^{-1/2}}(\tau)} \|E_\beta g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}},$$

respectively. For the third term, the factor $\delta$ in $T$ leads to the cancellation: $\delta\delta^{-1}R^{-1/2} = R^{-1/2}$, so it becomes

$$CK^4 \mathrm{MDec}_3(\delta^{-2}R^{-1}, K^{-1}, m) \left( \sum_{\Delta \in \mathcal{P}_{R^{-1/2}}(\tau)} \|E_\Delta g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}}.$$

□

## 8.4 Induction on scales

We prove the 3D-case first.

**Theorem 8.8.** *Fix $p \in [2,\infty)$ and $E \geq 300$. Then for each $0 < \nu \leq 1$, each $m \geq 1$ and each $\varepsilon > 0$, there is $C_{\nu,m} = C_{\nu,p,m,E}$ and $\eta_\varepsilon(\nu) = \eta_\varepsilon(\nu, p, E)$ with $\lim_{\nu\to 0^+} \eta_\varepsilon(\nu) = 0$ such that for all $R \geq \nu^{-2^m}$ we have*

$$\mathrm{Dec}_3(R^{-1}) \leq C_{\nu,m} R^{\eta_\varepsilon(\nu)+\varepsilon} \sup_{1\leq R'\leq R} \mathrm{MDec}_3(R'^{-1}, p, \nu, m, E).$$

We abbreviate $\mathrm{MDec}_3(\delta, p, \nu, m, E)$ as $\mathrm{MDec}(\delta)$ for this section as all the other parameters will be fixed.

*Proof.* Let $K = \nu^{-1}$, so $R \geq K^{2^m}$. By Proposition 8.5 applied to $\delta = l([0,1]^{n-1}) = 1$,

$$\|Eg\|_{L^p(w_{B_R})} \leq C_\varepsilon K^\varepsilon I_1 + C_\varepsilon K^\varepsilon I_2 + CK^4 \mathrm{MDec}(R^{-1}) I_3, \tag{8.12}$$

where

$$I_1 = \left( \sum_{Q_1 \in \mathcal{P}_{K^{-1}}([0,1]^2)} \|E_{Q_1} g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}},$$

$$I_2 = \left( \sum_{Q_2 \in \mathcal{P}_{K^{-1/2}}([0,1]^2)} \|E_{Q_2} g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}},$$

$$I_3 = \left( \sum_{Q_3 \in \mathcal{P}_{R^{-1/2}}([0,1]^2)} \|E_{Q_3} g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}},$$

and the constant $C_\varepsilon$ will be fixed throughout the iteration.

Note that $I_3$ is exactly the main term on the right hand side of the original decoupling inequality.

For each $Q_1$ in the expression $I_1$, if $l(Q_1) = K^{-1} \geq R^{-1/2} K^{2^{m-1}}$, then we can further use Proposition 8.7 with the same $K$, $R$ and $m$ but with $\delta = l(\tau) = l(Q_1) = K^{-1}$ to get

$$\|E_{Q_1} g\|_{L^p(w_{B_R})} \leq C_\varepsilon K^\varepsilon I_{1,1}(Q_1) + C_\varepsilon K^\varepsilon I_{1,2}(Q_1) + CK^4 \mathrm{MDec}(K^2 R^{-1}) I_{1,3}(Q_1),$$

where, similarly,

$$I_{1,1}(Q_1) = \left( \sum_{Q_{1,1} \in \mathcal{P}_{K^{-2}}(Q_1)} \|E_{Q_{1,1}} g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}}$$

$$I_{1,2}(Q_1) = \left( \sum_{Q_{1,2} \in \mathcal{P}_{K^{-3/2}}(Q_1)} \|E_{Q_{1,2}} g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}}$$

$$I_{1,3}(Q_1) = \left( \sum_{Q_{1,3} \in \mathcal{P}_{R^{-1/2}}(Q_1)} \|E_{Q_{1,3}} g\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2}}.$$

We can do the same for $I_2$ and $I_3$. We will also abbreviate $I_{i_1,i_2}([0,1]^2)$ simply as $I_{i_1,i_2}$ for $1 \le i_1, i_2 \le 3$.

For a typical term $I_{1,3}$, observe that

$$\left(\sum_{Q_1\in\mathcal{P}_{K^{-1}}([0,1]^2)} I_{1,3}^2(Q_1)\right)^{\frac{1}{2}} = \left(\sum_{Q_1\in\mathcal{P}_{K^{-1}}([0,1]^2)}\sum_{Q_{1,3}\in\mathcal{P}_{R^{-1/2}}(Q_1)} \left\|E_{Q_{1,3}}g\right\|^2_{L^p(w_{B_R})}\right)^{\frac{1}{2}}$$
$$= \left(\sum_{Q_{1,3}\in\mathcal{P}_{R^{-1/2}}([0,1]^2)} \left\|E_{Q_{1,3}}g\right\|^2_{L^p(w_{B_R})}\right)^{\frac{1}{2}} = I_3.$$

Similar argument can be used to get, for example,

$$\left(\sum_{Q_1\in\mathcal{P}_{K^{-1}}([0,1]^2)} I_{1,1}^2(Q_1)\right)^{\frac{1}{2}} = \left(\sum_{Q_{1,1}\in\mathcal{P}_{K^{-2}}([0,1]^2)} \left\|E_{Q_{1,1}}g\right\|^2_{L^p(w_{B_R})}\right)^{\frac{1}{2}} = I_{1,1}$$
$$\left(\sum_{Q_1\in\mathcal{P}_{K^{-1}}([0,1]^2)} I_{1,2}^2(Q_1)\right)^{\frac{1}{2}} = \left(\sum_{Q_{1,2}\in\mathcal{P}_{K^{-3/2}}([0,1]^2)} \left\|E_{Q_{1,2}}g\right\|^2_{L^p(w_{B_R})}\right)^{\frac{1}{2}} = I_{1,2}.$$

Hence we have (combining all absolute constants into a single $C$)

$$I_1 = \left(\sum_{Q_1\in\mathcal{P}_{K^{-1}}([0,1]^2)} \|E_{Q_1}g\|^2_{L^p(w_{B_R})}\right)^{\frac{1}{2}} \le CC_\varepsilon K^\varepsilon(I_{1,1}+I_{1,2}) + CK^4\mathrm{MDec}(K^2R^{-1})I_3,$$

Similarly, as $l(Q_2) = K^{-1/2}$, using Proposition 8.7 with $K^{-1/2} \ge R^{-1}K^{2^m}$ we can get

$$I_2 \le CC_\varepsilon K^\varepsilon(I_{2,1}+I_{2,2}) + CK^4\mathrm{MDec}(KR^{-1})I_3.$$

Hence

$$\|Eg\|_{L^p(w_{B_R})} \le C\left(C_\varepsilon K^\varepsilon(I_{1,1}+I_{1,2}+I_{2,1}+I_{2,2}) + M\right),$$

where

$$M := K^4 \sup_{1\le R'\le R} \mathrm{MDec}(R'^{-1})I_3. \tag{8.13}$$

We can continue in this fashion to get $I_{\vec{j}}$ for $\vec{j} \in \{1,2\}^m$ provided $\delta \ge R^{-1/2}K^{2^{m-1}}$. For example, consider a typical term $I_{2,1}$. For each $Q_{2,1} \in \mathcal{P}_{K^{-3/2}}(Q_2)$, using Proposition 8.7, it splits into 3 terms:

$$\left\|E_{Q_{2,1}}g\right\|_{L^p(w_{B_R})} \le C_\varepsilon K^\varepsilon I_{2,1,1}(Q_{2,1}) + C_\varepsilon K^\varepsilon I_{2,1,2}(Q_{2,1}) + CK^4\mathrm{MDec}(K^3R^{-1})I_{2,1,3}(Q_{2,1}).$$

The scales of partitioning cubes in the terms $I_{2,1,1}$, $I_{2,1,2}$ and $I_{2,1,3}$ are $K^{-5/2}$, $K^{-2}$, and $R^{-1/2}$, respectively.

We define

$$N := \max\{N' \ge 1 : K^{N'-1}R^{-1} \le K^{-2^m}\}. \tag{8.14}$$

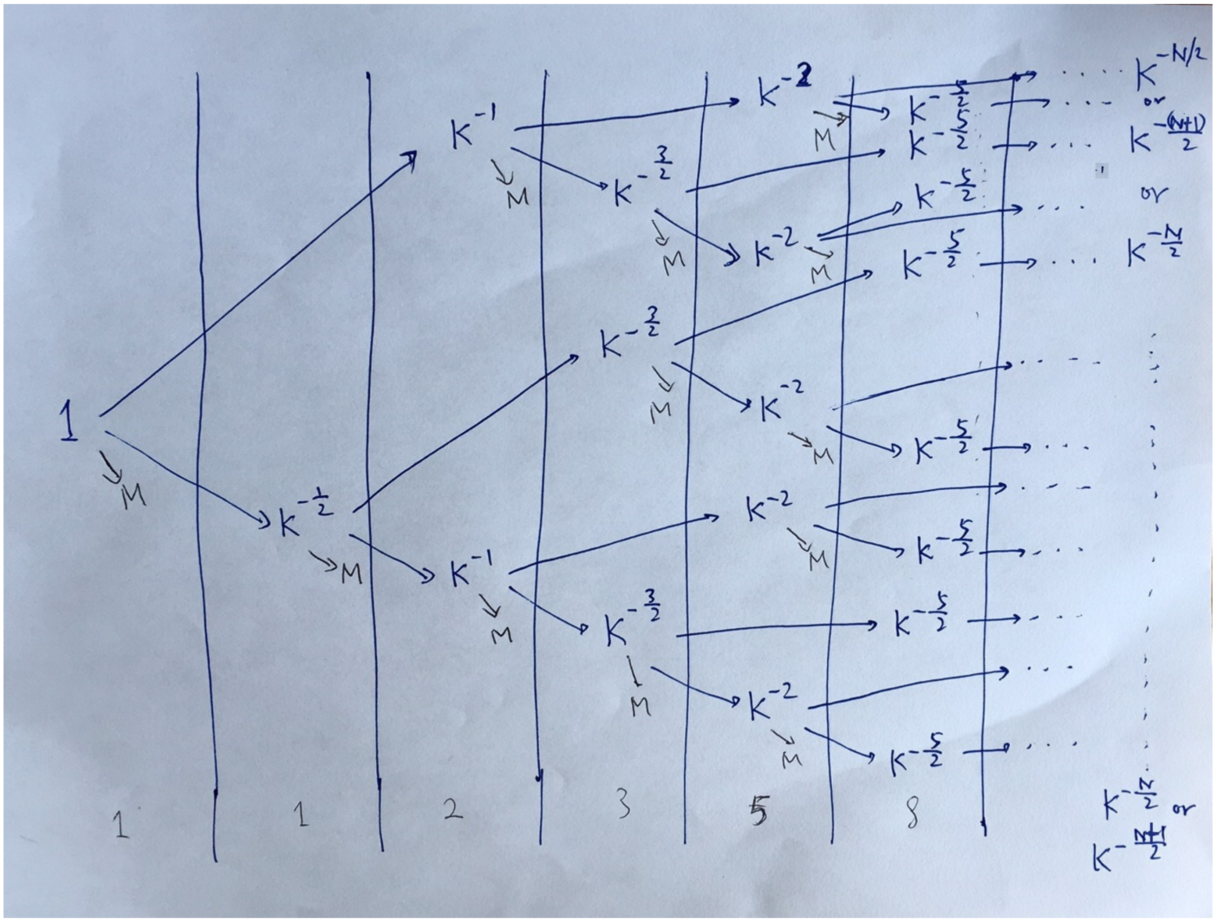


Figure 5: The iteration process

Then the final partitioning cubes should have length $R^{-1/2}$ (given by the term $I_3$), $K^{-N/2}$ or $K^{-(N-1)/2}$ (given by iterations of $I_1$ and $I_2$). See Figure 5.

Each arrow in blue in Figure 5 generates a multiplicative factor bounded by $A := CC_\varepsilon K^\varepsilon$ to the intermediate terms $I_{\vec{j}}$ with $j \in \{0,1,2\}^N$, where a 0 entry in the index means there is no further division (for example, the number of iterations along the top line is $\leq N/2$).

Each iteration also contributes to a multiplicative factor $C \leq A$ to the main term $M$. The bottom line $1 \to K^{-1/2} \to K^{-1} \to \cdots$ contributes to the largest number of multiplicative factors, which is bounded by

$$1 + A + A^2 + \cdots A^N \leq 2NA^N.$$

Since we iterate for at most $N$ times, the total number of multiplicative factors created is bounded by .

$$1+1+2+3+5+8+13+\cdots+F_N \leq 1+2+4+8+\cdots+2^N \leq 2^{N+1}.$$

If we bound all intermediate factors also by the trivial bound $2NA^N$, then

$$\|Eg\|_{L^p(w_{B_R})} \leq A^N \sum_{\vec{j}\in\{0,1,2\}^N} I_{\vec{j}} + 2NA^N 2^{N+1} M. \tag{8.15}$$

But each $I_{\vec{j}}$ is a term of the form

$$\left(\sum_{\alpha\in\mathcal{P}_{\delta'}([0,1]^2)} \|E_\alpha g\|^2_{L^p(w_{B_R})}\right)^{\frac{1}{2}},$$

where $\alpha$ is a cube of size $\delta' = K^{-N/2}$ or $K^{-(N-1)/2}$.

By (8.14), $\delta' \geq R^{-1/2}$. We can use the trivial triangle inequality and Cauchy Schwarz inequality to get

$$\begin{aligned}\|E_\alpha g\|_{L^p(w_{B_R})} &\leq \left(\left(\delta' R^{\frac{1}{2}}\right)^2 \sum_{\Delta\in\mathcal{P}_{R^{-1/2}}(\alpha)} \|E_\Delta g\|^2_{L^p(w_{B_R})}\right)^{\frac{1}{2}} \\ &\leq K^{2^{m-1}} \left(\sum_{\Delta\in\mathcal{P}_{R^{-1/2}}(\alpha)} \|E_\Delta g\|^2_{L^p(w_{B_R})}\right)^{\frac{1}{2}}.\end{aligned}$$

Squaring both sides and summing with respect to $\alpha$ shows that $I_{\vec{j}} \leq K^{2^{m-1}} I_3$. As we have $\leq 2^{N+1}$ many indices $\vec{j}$, by (8.13) and (8.15) we finally get

$$\begin{aligned}\|Eg\|_{L^p(w_{B_R})} &\leq A^N 2^{N+1} K^{2^{m-1}} I_3 + 2NA^N 2^{N+1} M \\ &\leq A^{2N} K^{2^{m+1}} \sup_{1\leq R'\leq R} \mathrm{MDec}(R'^{-1}) I_3,\end{aligned}$$

as we can assume $A \geq 100$ and we have $\mathrm{MDec}(R^{-1}) \geq 1$ (sharpness of the decoupling inequality). Lastly, by (8.14), $N \leq (\log R/\log K) - 2^m + 1 \leq \log R/\log K$, so

$$A^{2N} \leq A^{\frac{2\log R}{\log K}} \leq R^{2\varepsilon + \frac{2\log(CC_\varepsilon)}{\log K}}.$$

Recall $\nu = K^{-1}$. Thus taking

$$\eta_\varepsilon(\nu) = \frac{2\log(CC_\varepsilon)}{\log K}, \quad C_{\nu,m} = K^{2^{m+1}},$$

we are done. □

## 8.5 Other dimensions

### 8.5.1 The planar case

In the case $n = 2$, we are in either Case 1 or Case 3 of Figure 2. With an obvious reduction of lemma 8.1 to the planar case, Proposition 8.5 reduces to

$$\|Eg\|_{L^p(B_R)} \lesssim \left(\sum_{\alpha\in\mathcal{P}_{K^{-1}}([0,1])} \|E_\alpha g\|^2_{L^p(w_{B_R})}\right)^{\frac{1}{2}} + K^2 \max_{\substack{\alpha_1,\alpha_2 \\ K^{-1}-\text{transverse}}} \prod_{i=1}^{2} \|E_{\alpha_i} g\|^{\frac{1}{2}}_{L^p(w_{B_R})}$$

Note that without the lower dimensional term, we do not have the term $C_\varepsilon K^\varepsilon$, and the induction step is easier. This is the first case in Theorem 8.9.

### 8.5.2 Higher dimensions

Things get more complicated in the case $n \geq 4$. Let us take $n = 4$ as an example. In this case, Figure 2 gives 4 cases, namely,

1. The large terms are concentrated around a $CK^{-1}$-neighbourhood;
2. The large terms are concentrated around a tube with cross-section scale $CK^{-1}$;
3. The large terms are concentrated around a plate with thickness $CK^{-1}$;
4. There are 4 cubes with large contribution that are $CK^{-1}$-separated.

The first and the fourth cases are easy to deal with. The second term is bounded essentially by the $\mathrm{Dec}_2(K^{-1})$ times a term with frequency scales of partition $K^{-1/2}$, which can be proved following similar argument as in (8.2.2) by fixing 2 variables and taking the Fourier transform with respect to the remaining 2 variables. Similarly, the third term is bounded essentially by $\mathrm{Dec}_3(K^{-1})$ times a term with frequency scales of partition $K^{-1/2}$ (see Figure 6).

Now in each step of parabolic rescaling (Proposition 8.7), each term of scale $\delta$ splits into 4 terms with scales $K^{-1}\delta$, $K^{-1/2}\delta$, $K^{-1/2}\delta$ and $R^{-1/2}$ and multiplicative factors $O(1)$, $O(\mathrm{Dec}_2(K^{-1}))$, $O(\mathrm{Dec}_3(K^{-1}))$ and $O(K^{O(1)}\mathrm{MDec}_4(R^{-1}\delta^{-2}))$, respectively. By the induction hypothesis, both $\mathrm{Dec}_2(K^{-1})$ and $\mathrm{Dec}_3(K^{-1})$ are bounded above by $C_\varepsilon K^\varepsilon$, so similar argument as in Subsection 8.4 proves the following theorem in full generality (Theorem 8.2 with slight modifications):

**Theorem 8.9** (Multilinear decoupling dominates decoupling)**.** *Fix $n \geq 2$, $p \in [2, \infty)$ and $E \geq 100n$. Then we have the following:*

1. *If $n = 2$, then for each $0 < \nu \leq 1$ and $m \geq 1$, there is $C_{\nu,m} = C_{\nu,p,m,E}$ and $\eta(\nu) = \eta(\nu, p, E)$ such that $\lim_{\nu\to 0^+} \eta(\nu) = 0$ and for each $R \geq \nu^{-2^m}$, we have*

$$\mathrm{Dec}_2(R^{-1}, p, E) \leq C_{\nu,m} R^{\eta(\nu)} \sup_{1\leq R'\leq R} \mathrm{MDec}_2(R'^{-1}, p, \nu, m, E). \tag{8.16}$$

2. *If $n \geq 3$, then we have the following implication (a) $\implies$ (b), where*
   - (a) *For each $\varepsilon > 0$, there is $C_\varepsilon = C_{\varepsilon,p,n,E} > 0$ such that $\mathrm{Dec}_{n-1}(R, p, \Gamma_{n-1}(E)) \leq C_\varepsilon R^\varepsilon$ for all $R \geq 1$.*
   - (b) *For each $\varepsilon > 0$, $0 < \nu \leq 1$ and $m \geq 1$, there is $C_{\nu,m} = C_{n,\nu,p,m,E}$ and $\eta_\varepsilon(\nu) = \eta(\nu, \varepsilon, p, n, E)$ such that $\lim_{\nu\to 0^+} \eta_\varepsilon(\nu) = 0$ and for each $R \geq \nu^{-2^m}$, we have*

$$\mathrm{Dec}_n(R^{-1}, p, E) \leq C_{\nu,m} R^{\eta_\varepsilon(\nu)+\varepsilon} \sup_{1\leq R'\leq R} \mathrm{MDec}_n(R'^{-1}, p, \nu, m, E). \tag{8.17}$$

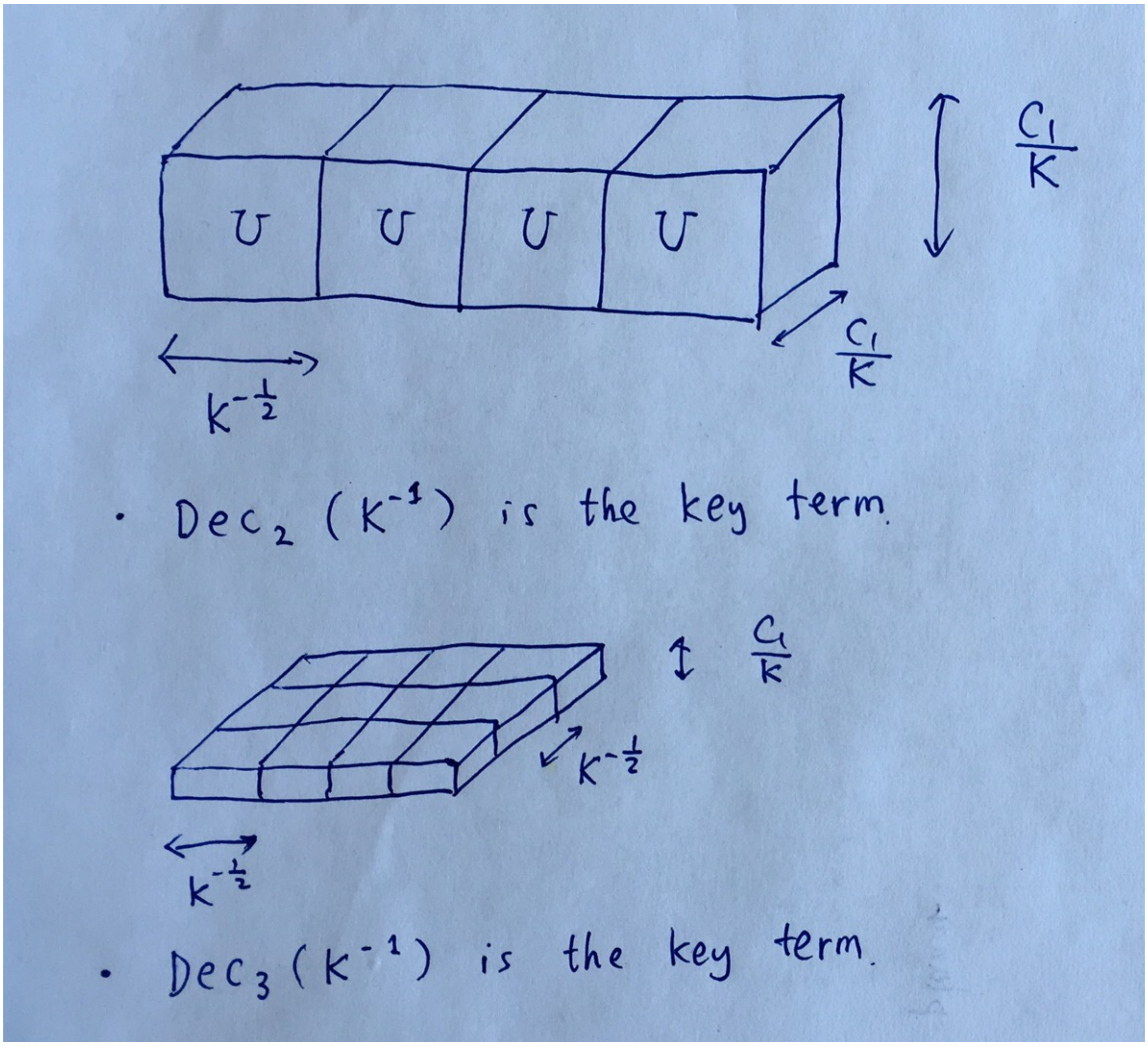


Figure 6: Reduction to lower dimensions when $n = 4$

# 9 Applying Multilinear Kakeya Inequality

Let $q = \frac{p(n-1)}{n}$. We rewrite the main inequality in Theorem 9.2 as

$$\left( \frac{1}{\#\mathcal{B}} \sum_{\Delta \in \mathcal{B}} \prod_{i=1}^{n} \left( \sum_{Q_{i,1} \in \mathcal{P}_\delta(Q_i)} \|E_{Q_{i,1}} g\|^2_{L^q_\#(w_\Delta)} \right)^{\frac{p}{2n}} \right)^{\frac{1}{p}} \tag{9.1}$$

$$\lesssim_{\varepsilon,\nu} \delta^{-\varepsilon} \prod_{i=1}^{n} \left( \sum_{Q_{i,1} \in \mathcal{P}_\delta(Q_i)} \|E_{Q_{i,1}} g\|^2_{L^q_\#(w_B)} \right)^{\frac{1}{2n}}, \tag{9.2}$$

where $p \geq \frac{2n}{n-1}$, i.e. $q \geq 2$. Here, $l(Q_i) = \mu \geq \delta$ for all $1 \leq i \leq n$. The implicit constant above will be independent of the positions and the size of $Q_i$, $1 \leq i \leq n$.

## 9.1 Heuristics: the axis-parallel case

We use a heuristic argument to see why the condition $q \geq 2$ is necessary.

For clarity, we slightly change notations. We will denote $Q_{i,j}$, $1 \leq j \leq (\mu/\delta)^{n-1} := J$ to be the partitioning cubes for $Q_i$, $1 \leq i \leq n$. Since $Q_1, \dots, Q_n$ are transverse, we have $Q_{1,j_1}, \dots, Q_{n,j_n}$'s are transverse for any $n$-tuple $(j_1, \dots, j_n)$.

To use heuristics to find necessary conditions on the exponent $p$, let us assume we are in the best case, in which all normal directions on $\mathbb{P}^{n-1}$ over $Q_{i,j}$ are exactly parallel to $e_i$ (however, this is always false in the rigourous sense, as the most separated two subcubes from the same cube would be almost transverse.) Hence for each $Q_{i,j}$, by wave packet decomposition, we may write

$$E_{Q_{i,j}} g \approx \sum_{T_i \in \mathbb{T}_i} c_{T_{i,j}} 1_{T_i},$$

where $T_i$ is a $(\delta^{-1})^{n-1} \times \delta^{-2}$-tube with the longest side nearly parallel to $e_i$. (The family $\mathbb{T}_i$ does not depends on $j$ since we made the simplification assuming the perfectly axis-parallel case.) Thus we have around $\delta^{-2n}/\delta^{-n-1} = \delta^{1-n}$ tubes $T_{i,j}$ contained in $B$. We change notations and index those cubes with $l = 1, 2, \dots, \delta^{1-n}$. Thus

$$E_{Q_{i,j}} g 1_B \approx \sum_l c_{i,j,l} 1_{T_{i,l}}.$$

An important observation is that for each $\Delta \in \mathcal{B}$,

$$\Delta = \bigcap_{i=1}^n F_i(\Delta),$$

where $F_i(\Delta)$ is a rectangle with dimensions $(\delta^{-2})^{n-1} \times \delta^{-1}$, with the shortest side parallel to $e_i$. Moreover, this relation is bijective: for each $n$-tuple $(F_1, \dots, F_n)$ such that $F_i \subseteq B$ for all $i$, the intersection

$$\bigcap_{i=1}^n F_i$$

uniquely determines a $\delta^{-1}$-cube $\Delta \subseteq B$. Hence we may rewrite the averaged summation $(\#\mathcal{B})^{-1} \sum_{\Delta \in \mathcal{B}}$ in (9.1) into

$$\left( \frac{1}{\#F_1} \sum_{F_1 \subseteq B} \right) \cdots \left( \frac{1}{\#F_n} \sum_{F_n \subseteq B} \right) = \left( \delta \sum_{k_1} \right) \cdots \left( \delta \sum_{k_n} \right),$$

where we used the observation that $\#\mathcal{B} = \delta^{-n}$ and $\#F_i = \delta^{-1}$ for each $i$, and indexed $F_i \subseteq B$ by $k_i = 1, 2, \dots, \delta^{-1}$.

Fix $1 \leq i \leq n$ and $j$. We can thus view $\left\| E_{Q_{i,j}} g \right\|_{L^q_\#(\Delta)}$ as a function of $F_i$, $1 \leq i \leq n$. But another important observation is that as $E_{Q_{i,j}} g$ is roughly a constant on $T_{i,l}$ for each $l$, $\left\| E_{Q_{i,j}} g \right\|_{L^q_\#(\Delta)}$ can be viewed as depending on

$$k_i' := (k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_n)$$

only. Hence if $\Delta = \cap_{i=1}^n F_i$, we may write

$$\left\| E_{Q_{i,j}} g \right\|_{L^q_{\#}(\Delta)} \approx c_j(k_i').$$

Hence (9.1) can be written as

$$\left( \left( \delta \sum_{k_1} \right) \cdots \left( \delta \sum_{k_n} \right) \prod_{i=1}^n \left( \sum_j c_j^2(k_i') \right)^{\frac{p}{2n}} \right)^{\frac{1}{p}}.$$

We further denote

$$f_i(k_i') = \left( \sum_j c_j^2(k_i') \right)^{\frac{p}{2n}}.$$

We will consider the Loomis-Whitney inequality, which is the prototype of the multilinear Kakeya inequality:

**Theorem 9.1** (Loomis-Whitney). *Let $n \geq 2$. Let $(X_i, \mu_i)$, $1 \leq i \leq n$ be measure spaces. Write $x = (x_1, \dots, x_n)$ and for each $1 \leq i \leq n$, write*

$$x_i' = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X_i' := X_1 \times \cdots X_{i-1} \times X_{i+1} \times \cdots X_n.$$

*Let $f_i$ be nonnegative measurable functions defined on $X_i'$. Then we have the following inequality:*

$$\int_{X_1} \cdots \int_{X_n} \prod_{i=1}^n f_i(x_i') d\mu_1(x_1) \cdots \mu_n(x_n) \leq \prod_{i=1}^n \|f_i\|_{L^{n-1}(X_i')}.$$

*Moreover, equality holds if and only if there are nonnegative measurable functions $g_i : \mathbb{R} \to \mathbb{R}$, $1 \leq i \leq n$ such that*

$$f_i(x_i') = \prod_{j \neq i} g_j(x_j).$$

In $n = 2$ this theorem is trivial. For $n \geq 3$, we can prove this by induction via a Hölder's inequality, a Minkowski's inequality followed by another Hölder's inequality.

Let us assume the case where the above inequality is almost an equality.

Observe that while $k_i'$ ranges through all possible $(n-1)$-tuples, at the same time $l_i$ ranges through all cubes $T_{i,l}$. Hence $c_j(k_i') = c_{i,j,l_i}$.

Now we use the theorem with the integrals taken to be $\delta \sum_{k_i}$ and with $f_i(x_i') = f_i(k_i')$. Applying the theorem assuming we approximately have an equality, we have

$$(9.1) \approx \left( \prod_{i=1}^n \|f_i\|_{l^{n-1}_{\#}(k_i')} \right)^{\frac{1}{p}}.$$

Now we analyse $\|f_i\|_{l_{\#}^{n-1}(k_i')}$. Fix $i$. Hence we can write

$$\begin{aligned}
\|f_i\|_{l_{\#}^{n-1}(k_i')} &= \left(\delta \sum_{k_i'} \left(\sum_j c_j^2(k_i')\right)^{\frac{p(n-1)}{2n}}\right)^{\frac{1}{n-1}} \\
&= \left(\delta \sum_{l_i} \left(\sum_j c_{i,j,l_i}^2\right)^{\frac{p(n-1)}{2n}}\right)^{\frac{1}{n-1}} \\
&= \left\| \|c_{i,j,l_i}\|_{l^2(j)} \right\|_{l_{\#}^{p(n-1)/n}(l_i)}^{p/n},
\end{aligned}$$

so

$$(9.1) \approx \prod_{i=1}^n \left\| \|c_{i,j,l_i}\|_{l^2(j)} \right\|_{l_{\#}^{p(n-1)/n}(l_i)}^{1/n} = \prod_{i=1}^n \left\| \|c_{i,j,l}\|_{l^2(j)} \right\|_{l_{\#}^q(l)}^{1/n},$$

since $q = p(n-1)/n$.

Now we come to the right hand side. For each $1 \leq i \leq n$, we may partition $B$ as

$$B = \bigcup_{T_i \subseteq B} T_i = \bigcup_{l_i} T_{i,l_i}.$$

Hence

$$\left\|E_{Q_{i,1}} g\right\|_{L_{\#}^q(B)} \approx \left(\frac{1}{|B|} \sum_l \int_{T_{i,l}} c_{i,j,l}^q\right)^{\frac{1}{q}} = \delta^{\frac{n-1}{q}} \left(\sum_{l_i} c_{i,j,l_i}^q\right)^{\frac{1}{q}} = \|c_{i,j,l}\|_{l_{\#}^q(l)}.$$

Hence ignoring the $\varepsilon$-loss, we have

$$(9.2) = \prod_{i=1}^n \left(\sum_j \|c_{i,j,l}\|_{l_{\#}^q(l)}^2\right)^{\frac{1}{2n}} = \prod_{i=1}^n \left\| \|c_{i,j,l}\|_{l_{\#}^q(l)} \right\|_{l^2(j)}^{1/n}.$$

Then we see that if we assume $q \geq 2$, then (9.1) $\lesssim$ (9.2) follows from Minkowski's inequality.

## 9.2 Proof of Theorem 9.2

We have two approaches to the rigorous proof. The first one is to use dyadic pigeonholing, which is how [BD] does it. The second approach is to generalise the multilinear Kakeya inequality using multilinear interpolation applied to the endpoints $(1, \frac{n}{n-1})$ and $(\infty, \infty)$, and it is much more succinct. We will give both arguments.

### 9.2.1 The first approach: dyadic partition

For each $i$, we partition $\mathcal{P}_\delta(Q_i) = \cup_{k=1}^{K+1}\mathcal{P}_{i,k}$, where

$$K = \min\{k \geq 1 : 2^k \geq \delta^{-E}\}, (E \text{ is a large constant, say } 100n) \tag{9.3}$$

$$Q_{i,1}^* \text{ is such that } \max_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)} \left\|E_{Q_{i,1}}g\right\|_{L^q_\#(w_B)} \text{ is attained,} \tag{9.4}$$

$$\mathcal{P}_{i,k} = \left\{Q_{i,1} \in \mathcal{P}_\delta(Q_i) : 2^{-k}\left\|E_{Q_{i,1}^*}g\right\|_{L^q_\#(w_B)} < \left\|E_{Q_{i,1}}g\right\|_{L^q_\#(w_B)} \leq 2^{-k+1}\left\|E_{Q_{i,1}^*}g\right\|_{L^q_\#(w_B)}\right\}, \quad 1 \leq k \leq K, \tag{9.5}$$

$$\mathcal{P}_{i,K+1} = \left\{Q_{i,1} \in \mathcal{P}_\delta(Q_i) : \left\|E_{Q_{i,1}}g\right\|_{L^q_\#(w_B)} \leq 2^{-K}\left\|E^*_{Q_{i,1}}g\right\|_{L^q_\#(w_B)}\right\}. \tag{9.6}$$

Thus (9.1) can be computed as

$$\begin{aligned}
&\left(\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^n\left(\sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)}\|E_{Q_{i,1}}g\|^2_{L^q_\#(w_\Delta)}\right)^{\frac{p}{2n}}\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^n\left(\sum_{k=1}^{K+1}\sum_{Q_{i,1}\in\mathcal{P}_{i,k}}\|E_{Q_{i,1}}g\|^2_{L^q_\#(w_\Delta)}\right)^{\frac{p}{2n}}\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\left(\sum_{k_1=1}^{K+1}\cdots\sum_{k_n=1}^{K+1}\prod_{i=1}^n\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\|E_{Q_{i,1}}g\|^2_{L^q_\#(w_\Delta)}\right)^{\frac{p}{2n}}\right)^{\frac{1}{p}} \\
\left(\frac{1}{2n}\leq 1\right) &\leq \left(\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\left(\sum_{k_1=1}^{K+1}\cdots\sum_{k_n=1}^{K+1}\left(\prod_{i=1}^n\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\|E_{Q_{i,1}}g\|^2_{L^q_\#(w_\Delta)}\right)^{\frac{1}{2n}}\right)^p\right)^{\frac{1}{p}} \\
(p\geq 1) &\leq \sum_{k_1=1}^{K+1}\cdots\sum_{k_n=1}^{K+1}\left(\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^n\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\|E_{Q_{i,1}}g\|^2_{L^q_\#(w_\Delta)}\right)^{\frac{p}{2n}}\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $K = O(\log(\delta^{-1}))$ by (9.3), there are at most $O(\log(\delta^{-1})^n)$ $n$-tuples $(k_1, \ldots, k_n)$. Since we allow an $\varepsilon$-loss here, to prove Theorem 9.2, it suffices to prove that for any $n$-tuple $(k_1, \ldots, k_n)$, we have

$$\left(\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^n\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\|E_{Q_{i,1}}g\|^2_{L^q_\#(w_\Delta)}\right)^{\frac{p}{2n}}\right)^{\frac{1}{p}} \lesssim_{\varepsilon,\nu} \delta^{-\varepsilon}\prod_{i=1}^n\left(\sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)}\|E_{Q_{i,1}}g\|^2_{L^q_\#(w_B)}\right)^{\frac{1}{2n}}. \tag{9.7}$$

By Hölder's inequality (as $q \geq 2$), we have

$$\left(\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^{n}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\left\|E_{Q_{i,1}}g\right\|^2_{L^q_\#(w_\Delta)}\right)^{\frac{p}{2n}}\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^{n}\left(\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\left\|E_{Q_{i,1}}g\right\|^q_{L^q_\#(w_\Delta)}\right)^{\frac{2}{q}}\#\mathcal{P}_{i,k_i}^{1-\frac{2}{q}}\right)^{\frac{p}{2n}}\right)^{\frac{1}{p}}$$

$$= \left(\left(\prod_{i=1}^{n}\#\mathcal{P}_{i,k_i}^{\frac{p}{2n}-\frac{1}{n-1}}\right)\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^{n}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\left\|E_{Q_{i,1}}g\right\|^q_{L^q_\#(w_\Delta)}\right)^{\frac{1}{n-1}}\right)^{\frac{1}{p}}.$$

**Proposition 9.2.** *For any $n$-tuple $(k_1,\ldots,k_n)$, we have*

$$\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^{n}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\left\|E_{Q_{i,1}}g\right\|^q_{L^q_\#(w_\Delta)}\right)^{\frac{1}{n-1}} \lesssim_{\varepsilon,\nu} \delta^{-\varepsilon}\prod_{i=1}^{n}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\left\|E_{Q_{i,1}}g\right\|^q_{L^q_\#(w_B)}\right)^{\frac{1}{n-1}}. \tag{9.8}$$

*Proof of Theorem 9.2 Assuming Proposition 9.2.* By Proposition 9.2, we have

$$\left(\left(\prod_{i=1}^{n}\#\mathcal{P}_{i,k_i}^{\frac{p}{2n}-\frac{1}{n-1}}\right)\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^{n}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\left\|E_{Q_{i,1}}g\right\|^q_{L^q_\#(w_\Delta)}\right)^{\frac{1}{n-1}}\right)^{\frac{1}{p}}$$

$$\lesssim_{\varepsilon,\nu} \delta^{-\varepsilon}\prod_{i=1}^{n}\#\mathcal{P}_{i,k_i}^{\frac{1}{n}\left(\frac{1}{2}-\frac{1}{q}\right)}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\left\|E_{Q_{i,1}}g\right\|^q_{L^q_\#(w_B)}\right)^{\frac{1}{qn}}.$$

Next we fix an $1 \leq i \leq n$. We have two cases: $1 \leq k_i \leq K$ or $k_i = K+1$.

- If $k_i = K+1$, then by (9.6) we have

$$\#\mathcal{P}_{i,k_i}^{\frac{1}{n}\left(\frac{1}{2}-\frac{1}{q}\right)}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\left\|E_{Q_{i,1}}g\right\|^q_{L^q_\#(w_B)}\right)^{\frac{1}{qn}} \leq \delta^{-\frac{n-1}{n}\left(\frac{1}{2}-\frac{1}{q}\right)}\left(\delta^{-(n-1)}2^{-Kq}\right)^{\frac{1}{qn}}\left\|E_{Q^*_{i,1}}g\right\|^{\frac{1}{n}}_{L^q_\#(w_B)}$$

$$\leq \delta^{-\frac{n-1}{2n}}\delta^{\frac{E}{n}}\left(\sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)}\left\|E_{Q_{i,1}}g\right\|^2_{L^q_\#(w_B)}\right)^{\frac{1}{2n}}$$

$$(E = 100n) \leq \left(\sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)}\left\|E_{Q_{i,1}}g\right\|^2_{L^q_\#(w_B)}\right)^{\frac{1}{2n}}.$$

- If $1 \leq k_i \leq K$, then by (9.5), each term $\left\|E_{Q_{i,1}}g\right\|^q_{L^q_\#(w_B)}$ is comparable, so we have the reverse Hölder's inequality:

$$\#\mathcal{P}_{i,k_i}^{\frac{1}{n}\left(\frac{1}{2}-\frac{1}{q}\right)}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\left\|E_{Q_{i,1}}g\right\|^q_{L^q_\#(w_B)}\right)^{\frac{1}{qn}} \lesssim \left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\left\|E_{Q_{i,1}}g\right\|^2_{L^q_\#(w_B)}\right)^{\frac{1}{2n}}$$
$$\leq \left(\sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)}\left\|E_{Q_{i,1}}g\right\|^2_{L^q_\#(w_B)}\right)^{\frac{1}{2n}}.$$

Hence we have

$$\prod_{i=1}^n \#\mathcal{P}_{i,k_i}^{\frac{1}{n}\left(\frac{1}{2}-\frac{1}{q}\right)}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\left\|E_{Q_{i,1}}g\right\|^q_{L^q_\#(w_B)}\right)^{\frac{1}{qn}} \lesssim \prod_{i=1}^n\left(\sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)}\left\|E_{Q_{i,1}}g\right\|^2_{L^q_\#(w_B)}\right)^{\frac{1}{2n}},$$

which finishes the proof of Theorem 9.2. □

### 9.2.2 Proof of Proposition 9.2

Fix a cube $Q = Q_{i,1}$. Cover $\mathbb{R}^n$ by a family $\mathcal{F}_Q$ of pairwise disjoint, mutually parallel tiles $T_Q$ pointing in the direction of the normal $N(c_Q)$ to $\mathbb{P}^{n-1}$ (the wave packets). We let them have the longer side $n\delta^{-2}$ and shorter sides $\sqrt{n}\delta^{-1}$ (so the scale is not dyadic, but it does not matter). We could also let 0 be the centre of some $T_Q \in \mathcal{F}_Q$.

For all $x \in B$, $x \in \Delta$ for some $\Delta \in \mathcal{B}$. There is also a unique $T_Q \in \mathcal{F}_Q$ that contains $x$, which we denote as $T_Q(x)$. We claim that $\Delta \subseteq 4T_Q(x)$. Indeed, since $x \in T_Q(x)$, in each shorter direction $v$, $|(x - c_{T(Q)(x)}) \cdot v| \leq \frac{1}{2}\sqrt{n}\delta^{-1}$. If $y \in \Delta$, then $|(y-x)\cdot v| \leq |y-x| \leq \sqrt{n}\delta^{-1}$, so $|(y - c_{T(Q)(x)}) \cdot v| \leq \frac{3}{2}\sqrt{n}\delta^{-1} \leq 2\sqrt{n}\delta^{-1}$. For the longer direction $v'$, similarly we also have $|(y - c_{T(Q)(x)}) \cdot v'| \leq 2n\delta^{-2}$. Hence $y \in 4T_Q(x)$.

Note that if $x \in B$ and $T_Q(x) \cap B \neq \varnothing$, then $T_Q(x) \subseteq 4nB$, by similar argument as above. Hence $4T_Q(x) \subseteq 16nB$ for all $x \in B$. Hence although $\mathcal{F}_Q$ is a cover for the whole $\mathbb{R}^n$, we only care about those cubes $T_Q \in \mathcal{F}_Q$ that intersects $B$ and such that $4T_Q$ fully lies in $16nB$.

(The analysis above could be more refined but only by a factor of $O(1)$, so this is not necessary.) See Figure 7 below.

We define

$$F_Q(x) = \sum_{T_Q\in\mathcal{F}_Q} \sup_{y\in 4T_Q(x)} \|E_Q g\|_{L^q_\#(w_{B(y,\delta^{-1})})} 1_{T_Q}(x), \tag{9.9}$$

which is constant on each tile $T_Q$. (Note that $F_Q$ is defined on all of $\mathbb{R}^n$, and is independent of $B$. The reason why we defined $F_Q$ instead of the following more natural summation

$$\sum_{T_Q\in\mathcal{F}_Q} \|E_Q g\|_{L^q_\#(\Delta)} 1_{T_Q}(x)$$

is that the term $\|E_Q g\|_{L^q_\#(w_\Delta)}$ depends on individual $\Delta$'s, but $F_Q$ does not.) Since $\Delta \subseteq 4T_Q(x)$, this implies that

$$\|E_Q g\|_{L^q_\#(w_\Delta)} 1_\Delta(x) \leq F_Q(x) 1_\Delta(x). \tag{9.10}$$

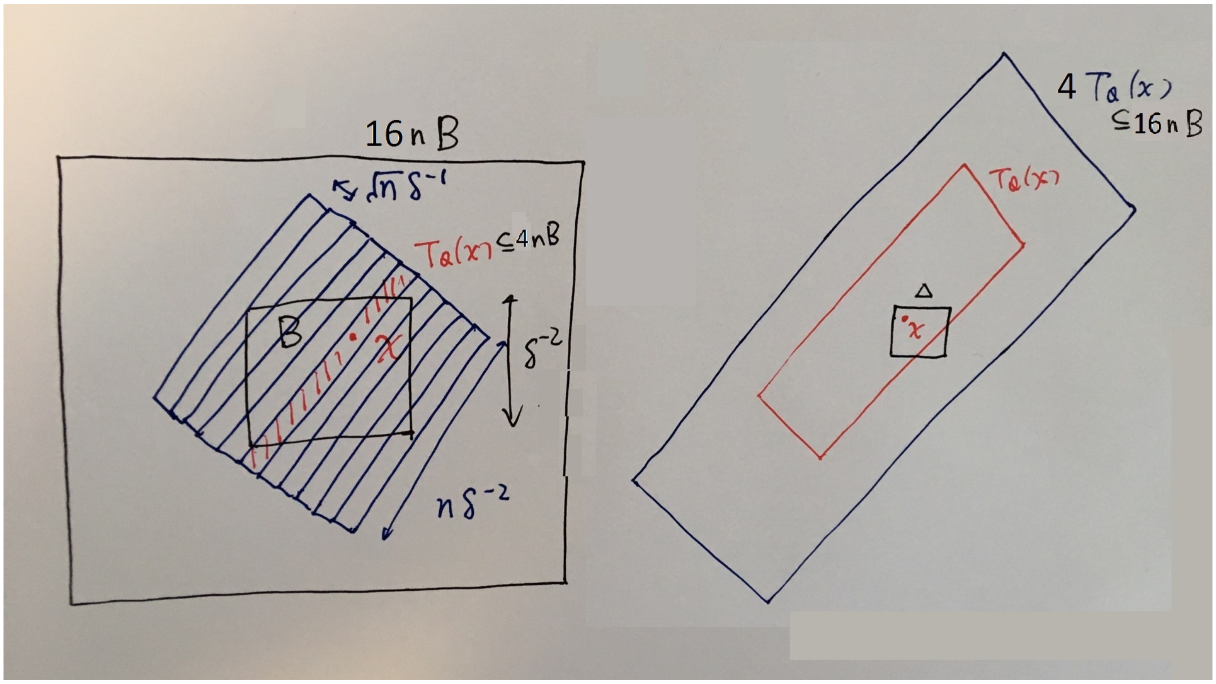


Figure 7: The wave packets $T_Q$

**Lemma 9.3.** *Let $A_{i,\Delta}$ be any sequence indexed by $i$ and $\Delta$ where $\Delta$'s are disjoint. Then we have the following equality:*

$$\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^{n} A_{i,\Delta} = \frac{1}{|\Delta|}\int_{\mathbb{R}^n}\prod_{i=1}^{n}\left(\sum_{\Delta\in\mathcal{B}} A_{i,\Delta}1_\Delta(x)\right)dx.$$

*Proof of lemma.* Consider the right hand side. By the distributive law,

$$\prod_{i=1}^{n}\left(\sum_{\Delta\in\mathcal{B}} A_{i,\Delta}1_\Delta(x)\right) = \sum_{\Delta_1\in\mathcal{B}}\cdots\sum_{\Delta_n\in\mathcal{B}}\prod_{i=1}^{n} A_{i,\Delta_i}1_{\Delta_i}(x).$$

However, recall the $\Delta$'s are disjoint. Hence $\prod_{i=1}^{n} A_{i,\Delta_i}1_{\Delta_i}(x) \neq 0$ only if $\Delta_1 = \cdots = \Delta_n$. Thus we have

$$\sum_{\Delta_1\in\mathcal{B}}\cdots\sum_{\Delta_n\in\mathcal{B}}\prod_{i=1}^{n} A_{i,\Delta_i}1_{\Delta_i}(x) = \sum_{\Delta\in\mathcal{B}}\prod_{i=1}^{n} A_{i,\Delta}1_\Delta(x).$$

Taking integrals over $\mathbb{R}^n$, we get

$$\begin{aligned}\frac{1}{|\Delta|}\int_{\mathbb{R}^n}\prod_{i=1}^{n}\left(\sum_{\Delta\in\mathcal{B}} A_{i,\Delta}1_\Delta(x)\right)dx &= \frac{1}{|\Delta|}\int_{\mathbb{R}^n}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^{n} A_{i,\Delta}1_\Delta(x)dx\\ &= \frac{1}{|\Delta|}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^{n} A_{i,\Delta}\int_{\mathbb{R}^n}1_\Delta(x)dx\\ &= \sum_{\Delta\in\mathcal{B}}\prod_{i=1}^{n} A_{i,\Delta}.\end{aligned}$$

□

Recall $Q = Q_{i,1}$. Applying Lemma 9.3 with $A_{i,\Delta} = \left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}} \|E_{Q_{i,1}}g\|^q_{L^q_\#(w_\Delta)}\right)^{\frac{1}{n-1}}$, we have

$$\begin{aligned}
&\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^n\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\|E_{Q_{i,1}}g\|^q_{L^q_\#(w_\Delta)}\right)^{\frac{1}{n-1}}\\
&=\frac{1}{\#\mathcal{B}}\frac{1}{|\Delta|}\int_{\mathbb{R}^n}\prod_{i=1}^n\sum_{\Delta\in\mathcal{B}}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\|E_{Q_{i,1}}g\|^q_{L^q_\#(w_\Delta)}\right)^{\frac{1}{n-1}}1_\Delta(x)dx\\
&=\delta^{2n}\int_{16nB}\prod_{i=1}^n\sum_{\Delta\in\mathcal{B}}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\left(\|E_{Q_{i,1}}g\|_{L^q_\#(w_\Delta)}1_\Delta(x)\right)^q\right)^{\frac{1}{n-1}}dx\\
\text{(by (9.10))}&\leq\delta^{2n}\int_{16nB}\prod_{i=1}^n\sum_{\Delta\in\mathcal{B}}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}F^q_{Q_{i,1}}(x)1_\Delta(x)\right)^{\frac{1}{n-1}}dx\\
&=\delta^{2n}\int_{16nB}\prod_{i=1}^n\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}F^q_{Q_{i,1}}(x)\right)^{\frac{1}{n-1}}\left(\sum_{\Delta\in\mathcal{B}}1_\Delta(x)\right)dx\\
&\lesssim\frac{1}{|16nB|}\int_{16nB}\prod_{i=1}^n\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}F^q_{Q_{i,1}}(x)\right)^{\frac{1}{n-1}}dx.
\end{aligned}$$

Now we are ready to use the following version of the multilinear Kakeya inequality (slightly modified from a combination of Corollary 5 and Corollary 6 of [5]): for any cube $B_R$ of side length $R \geq n\delta^{-2}$ and any family of functions $F_i$ of the form

$$F_i = \sum_{P\in\mathcal{P}_i} c_P 1_P,$$

where $\mathcal{P}_i$, $1 \leq i \leq n$ are $\nu$-transverse families of $(R^{1/2})^{n-1}\times R$-tiles, we have

$$\frac{1}{|B_R|}\int_{B_R}\prod_{i=1}^n|F_i|^{\frac{1}{n-1}} \lesssim_{\varepsilon,\nu} R^\varepsilon\prod_{i=1}^n\left(\frac{1}{|B_R|}\int_{B_R}|F_i|\right)^{\frac{1}{n-1}}. \tag{9.11}$$

(The main difference from Theorem 9.1 is that we do not restrict the tiles $P \subseteq B_R$ here; we restrict the domain of integration instead.) We let $R = 16n\delta^{-2} = l(16nB)$ and

$$F_i = \sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}} F^q_{Q_{i,1}} = \sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\sum_{T_Q\in\mathcal{F}_{Q_{i,1}}}\sup_{y\in 4T_Q(x)}\|E_Qg\|^q_{L^q_\#(w_{B(y,\delta^{-1})})}1_{T_Q}(x).$$

Let $\mathcal{P}_i = \cup_{Q_{i,1}\in\mathcal{P}_{i,k_i}}\mathcal{F}_{Q_{i,1}}$. Since $\{Q_i : 1 \leq i \leq n\}$ are $\nu$-transverse, so are any family of sub-cubes $\{Q_{i,1} \subseteq Q_i : 1 \leq i \leq n\}$, and thus any choice of tiles $T_Q$ from each one of the families $\mathcal{F}_{Q_{i,1}}, 1 \leq i \leq n$ also have $\nu$-transverse directions. Hence we can apply (9.11) with the above $R$ and $F_i$ to get

$$\begin{aligned}
&\frac{1}{|16nB|}\int_{16nB}\prod_{i=1}^{n}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}} F^q_{Q_{i,1}}(x)\right)^{\frac{1}{n-1}} dx\\
(\text{by } (9.11)) \lesssim_{\varepsilon,\nu}\ &\delta^{-\varepsilon}\prod_{i=1}^{n}\left(\frac{1}{|16nB|}\int_{16nB}\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}} F^q_{Q_{i,1}}(x)dx\right)^{\frac{1}{n-1}}\\
&=\delta^{-\varepsilon}\prod_{i=1}^{n}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}} \|F_{Q_{i,1}}\|^q_{L^q_{\#}(16nB)}\right)^{\frac{1}{n-1}}.
\end{aligned}$$

**Proposition 9.4.** *For each $q \geq 1$, $Q = Q_{i,1}$ and any ball $B$ with radius $\delta^{-2}$,*

$$\|F_Q\|_{L^q_{\#}(16nB)} \lesssim \|E_Q g\|_{L^q_{\#}(w_B)},$$

where $F_Q$ is defined in (9.9). If the proposition is true, then we have

$$\prod_{i=1}^{n}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}} \|F_{Q_{i,1}}\|^q_{L^q_{\#}(16nB)}\right)^{\frac{1}{n-1}} \lesssim \prod_{i=1}^{n}\left(\sum_{Q_{i,1}\in\mathcal{P}_{i,k_i}} \|E_Q g\|^q_{L^q_{\#}(w_B)}\right)^{\frac{1}{n-1}},$$

which implies Proposition 9.2. Hence all that is left is the proof of Proposition 9.4.

### 9.2.3 The second approach: multilinear interpolation

Using the observation and terminology in Subsection 9.2.2, we shall compute (9.1) directly. Our goal is to use (9.10). First, we use Lemma 9.3 with

$$A_{i,\Delta} = \left(\sum_{Q_{i,1}\in\mathcal{P}_{\delta}(Q_i)} \|E_{Q_{i,1}} g\|^2_{L^q_{\#}(w_\Delta)}\right)^{\frac{p}{2n}}$$

and get

$$\begin{aligned}
(9.1)^p &= \frac{1}{\#\mathcal{B}}\frac{1}{|\Delta|}\int_{\mathbb{R}^n}\prod_{i=1}^{n}\sum_{\Delta\in\mathcal{B}}\left(\sum_{Q_{i,1}\in\mathcal{P}_{\delta}(Q_i)} \|E_{Q_{i,1}} g\|^2_{L^q_{\#}(w_\Delta)}\right)^{\frac{p}{2n}} 1_\Delta(x)dx\\
(\text{by } (9.10)) &\leq \delta^{2n}\int_{\mathbb{R}^n}\prod_{i=1}^{n}\sum_{\Delta\in\mathcal{B}}\left(\sum_{Q_{i,1}\in\mathcal{P}_{\delta}(Q_i)} F_{Q_{i,1}}(x)^2\right)^{\frac{p}{2n}} 1_\Delta(x)dx\\
&\lesssim \frac{1}{16nB}\int_{16nB}\prod_{i=1}^{n}\left(\sum_{Q_{i,1}\in\mathcal{P}_{\delta}(Q_i)} F_{Q_{i,1}}(x)^2\right)^{\frac{p}{2n}} dx.
\end{aligned}$$

We let, similar as above,

$$F_i = \sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)} F_{Q_{i,1}}^2 = \sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)} \sum_{T_Q\in\mathcal{F}_{Q_{i,1}}} \sup_{y\in 4T_Q(x)} \|E_Q g\|^q_{L^2_\#(w_{B(y,\delta^{-1})})} 1_{T_Q}(x).$$

Now we interpolate between (9.11) which is rewritten as:

$$\left\| \prod_{i=1}^n |F_i|^{\frac{1}{n}} \right\|_{L_\#^{\frac{n}{n-1}}(B_R)} \lesssim_{\varepsilon,\nu} R^\varepsilon \prod_{i=1}^n \|F_i\|^{\frac{1}{n}}_{L^1_\#(B_R)}$$

and the following trivial $(\infty,\infty)$-bound:

$$\left\| \prod_{i=1}^n |F_i|^{\frac{1}{n}} \right\|_{L_\#^{\infty}(B_R)} \le \prod_{i=1}^n \|F_i\|^{\frac{1}{n}}_{L^\infty_\#(B_R)}$$

to get

$$\left\| \prod_{i=1}^n |F_i|^{\frac{1}{n}} \right\|_{L_\#^{t}(B_R)} \lesssim_{\varepsilon,\nu} R^\varepsilon \prod_{i=1}^n \|F_i\|^{\frac{1}{n}}_{L^s_\#(B_R)}, \tag{9.12}$$

for all $n/(n-1) \le t \le \infty$ and $s = t(n-1)/n \ge 1$. (See Theorem 1.15 of [2].)

Now if $p \ge 2n/(n-1)$, we can apply (9.12) with $R = 16n\delta^{-2}$, $t = p/2 \ge n/(n-1)$ and $s = t(n-1)/n = q/2 \ge 1$ to get

$$\begin{aligned}(9.1) &\lesssim \left( \frac{1}{16nB} \int_{16nB} \prod_{i=1}^n \left( \sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)} F_{Q_{i,1}}(x)^2 \right)^{\frac{p}{2n}} dx \right)^{\frac{1}{p}} \\ &\lesssim_{\varepsilon,\nu} \delta^{-\varepsilon} \prod_{i=1}^n \left\| \sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)} F_{Q_{i,1}}^2 \right\|^{\frac{1}{2n}}_{L_\#^{\frac{q}{2}}(16nB)} \\ &\le \delta^{-\varepsilon} \prod_{i=1}^n \left( \sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)} \|F_{Q_{i,1}}\|^2_{L^q_\#(16nB)} \right)^{\frac{1}{n}},\end{aligned}$$

by the triangle inequality. Now using Proposition 9.4, we have

$$\begin{aligned}&\delta^{-\varepsilon} \prod_{i=1}^n \left( \sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)} \|F_{Q_{i,1}}\|^2_{L^q_\#(16nB)} \right)^{\frac{1}{n}} \\ &\lesssim \delta^{-\varepsilon} \prod_{i=1}^n \left( \sum_{Q_{i,1}\in\mathcal{P}_\delta(Q_i)} \|E_{Q_{i,1}} g\|^2_{L^q_\#(w_B)} \right)^{\frac{1}{n}},\end{aligned}$$

which is (9.2).

## 9.3 Proof of Proposition 9.4

Heuristically, Proposition 9.4 holds as by the uncertainty principle, $E_Q g$ is locally constant in each tile $T_Q$ and thus it will still be roughly constant in each smaller cube $B(y, \delta^{-1})$ near $4T_Q$. The rigorous proof is based on Fourier analytic techniques, which is similar to the proof in Subsection 5.2 of Section 5.

### 9.3.1 Reduction to the case $Q = Q_0$

First we show why we can reduce to the case $Q = Q_0 = [-\delta/2, \delta/2]^{n-1}$. (Note $Q_0$ is actually not a subset of $[0,1]^{n-1}$, but we can of course extend the definition of $E_Q g$ for such $Q$'s. The symmetry here will simplify the notations a bit.)

The proof will be similar to the argument in Subsection 5.2.1 in Section 5.2. For each $Q = Q_{i,1}$, write $Q = Q_0 + \sigma$ and $\xi = \eta + \sigma$ where $\xi \in Q$ and $\eta \in Q_0$. Write $G(u) = g(u+\sigma)$ and

$$z = Lu = (u_1 + 2u_n\sigma_1, \ldots, u_{n-1} + 2u_n\sigma_{n-1}, u_n),$$

so $|\det(L)| = 1$. With this, we can compute

$$\int |E_Q g(u)|^q w_{B(y,\delta^{-1})}(u) du = \int |E_{Q_0} G(z)|^q w_{B(y,\delta^{-1})}(L^{-1}z) dz. \tag{9.13}$$

We then prove a weight inequality:

$$w_{B(y,\delta^{-1})}(L^{-1}z) \sim w_{B(Ly,\delta^{-1})}(z). \tag{9.14}$$

Indeed, since $\sigma \in [-1,1]^{n-1}$, the mapping $L$ is bilipschitz with constant $O(1)$. Thus $|Ly - z| \sim |y - L^{-1}z|$, and hence we have (9.14). Thus

$$\int |E_{Q_0} G(z)|^q w_{B(y,\delta^{-1})}(L^{-1}z) dz \sim \int |E_{Q_0} G(z)|^q w_{B(Ly,\delta^{-1})}(z) dz,$$

which implies $\|E_Q g\|_{L^q_\#(w_{B(y,\delta^{-1})})} \sim \|E_{Q_0} G\|_{L^q_\#(w_{B(Ly,\delta^{-1})})}$.

Clearly, $L$ a bijection between each $T_Q \in \mathcal{F}_Q$ and some parallelepiped $L(T_Q)$. We claim that $L(T_Q) \subseteq 4T_{Q_0}$ for some $T_{Q_0} \in \mathcal{F}_{Q_0}$, where $\mathcal{F}_{Q_0}$ is defined in the same way as $\mathcal{F}_Q$ was defined (we can also assume 0 is the centre of some $T_{Q_0}$). Indeed, if $c_Q$ is the centre of $Q$, then $c_Q = \sigma + c_{Q_0} = \sigma$. Thus

$$N(c_Q) = \frac{(-2\sigma_1, \ldots, -2\sigma_{n-1}, 1)}{\sqrt{1 + 4\sigma_1^2 + \cdots + 4\sigma_{n-1}^2}}.$$

Recalling the definition of $L$, we have $L(N(c_Q)) \parallel (0, \ldots, 0, 1) = N(c_{Q_0})$. But $L$ is a shear transformation, so $L(T_Q)$ is a parallelepiped with the longest side parallel to $N(c_{Q_0})$. However, as $\sigma \in [-1,1]^{n-1}$, the distortion should not be too much.) Hence $L(T_Q) \subseteq 4T_{Q_0}$, and thus $L(4T_Q(x)) \subseteq 16T_{Q_0}(Lx)$.

Recalling the definition of $F_Q$ (9.9) and changing $Ly$ to $y$, we have

$$F_Q(x) \lesssim F_0(Lx) := \sum_{T_{Q_0} \in \mathcal{F}_{Q_0}} \sup_{y \in 16T_{Q_0}(Lx)} \|E_{Q_0} G\|_{L^q_\#(w_{B(y,\delta^{-1})})} 1_{4T_{Q_0}}(Lx).$$

The $Q = Q_0$ case is given by the following proposition:

**Proposition 9.5.** *We have, for all spatial cubes $B'$ of side length $\delta^{-2}$ and all $G$,*

$$\|F_0\|_{L^q_\#(64nB')} \lesssim \|E_{Q_0}G\|_{L^q_\#(w_{B'})}.$$

*Proof of Proposition 9.4 assuming Proposition 9.5.* Since $F_Q(x) \lesssim F_0(Lx)$, letting $x' = Lx$, by the bilipschitz property of $L$,

$$\int_{16nB} |F_Q(x)|^q dx \lesssim \int_{L^{-1}(16nB)} |F_0(x')|^q dx' \leq \int_{64nB'} |F_0(x')|^q dx',$$

for some $B'$ with $c_{B'} = L(c_B)$ and $l(B') = \delta^{-2}$. Thus $\|F_Q\|_{L^q_\#(16nB)} \lesssim \|F_0\|_{L^q_\#(64nB')}$. By Proposition 9.5, it suffices to show $\|E_{Q_0}G\|_{L^q_\#(w_{B'})} \lesssim \|E_Q g\|_{L^q_\#(w_B)}$.

Reversing the change of variable as in (9.13), we get

$$\int |E_{Q_0}G(z)|^q w_{B'}(z)dz = \int |E_Q g(u)|^q w_{B'}(Lu)du.$$

But as $L$ is Lipschitz, $|Lu - c_{B'}| \sim |u - L^{-1}(c_{B'})| = |u - c_B|$, hence

$$\int |E_Q g(u)|^q w_{B'}(Lu)du \sim \int |E_Q g(u)|^q w_B(u)du.$$

This shows that $\|E_{Q_0}G\|_{L^q_\#(w_{B'})} \lesssim \|E_Q g\|_{L^q_\#(w_B)}$. □

### 9.3.2 Several further reductions

All that is left is the proof of Proposition 9.5, and this is where Fourier analytic tools will be used. By an abuse of notation we will write $Q_0 = Q = [-\delta/2, \delta/2]^{n-1}$.

Then in the new notations,

$$F_0(x) = \sum_{T_Q \in \mathcal{F}_Q} \sup_{y \in 16T_Q(x)} \|E_Q G\|_{L^q_\#(w_{B(y,\delta^{-1})})} 1_{4T_Q}(x),$$

and we will be proving

$$\|F_0\|_{L^q_\#(64nB)} \lesssim \|E_Q G\|_{L^q_\#(w_B)},$$

for any spatial cube $B$ of length $\delta^{-2}$.

- Given any $B$ as above, consider the subcollection $\mathcal{F}_Q^B$ of tubes $T_Q \in \mathcal{F}_Q$ that intersect

$64nB$. Since $4T_Q$'s have bounded overlap, we have

$$\begin{aligned}
&\|F_0\|^q_{L^q_\#(64nB)} \\
&= \frac{1}{|64nB|}\int_{64nB}\left(\sum_{T_Q\in\mathcal{F}_Q^B}\sup_{y\in 16T_Q(x)}\|E_QG\|_{L^q_\#(w_{B(y,\delta^{-1})})}1_{4T_Q}(x)\right)^q dx \\
&\leq \frac{1}{|64nB|}\int_{64nB}\left(\sum_{T_Q\in\mathcal{F}_Q^B}\sup_{y\in 16T_Q}\|E_QG\|^q_{L^q_\#(w_{B(y,\delta^{-1})})}1_{4T_Q}(x)\right)\left(\sum_{T_Q\in\mathcal{F}_Q^B}1_{4T_Q}(x)\right)^{\frac{q}{q'}} dx \\
&\lesssim \delta^{2n}\sum_{T_Q\in\mathcal{F}_Q^B}\sup_{y\in 16T_Q}\|E_QG\|^q_{L^q_\#(w_{B(y,\delta^{-1})})}\int_{64nB}1_{4T_Q}(x)dx \\
&\lesssim \delta^{n-1}\sum_{T_Q\in\mathcal{F}_Q^B}\sup_{y\in 16T_Q}\|E_QG\|^q_{L^q_\#(w_{B(y,\delta^{-1})})}.
\end{aligned}$$

it suffices to show

$$\sum_{T_Q\in\mathcal{F}_Q^B}\sup_{y\in 16T_Q}\|E_QG\|^q_{L^q_\#(w_{B(y,\delta^{-1})})}\lesssim \delta^{1-n}\|E_QG\|^q_{L^q_\#(w_B)}. \tag{9.15}$$

- We show that it suffices to prove that (9.15) holds for $B$ centred at 0. Given any cube $B$ with centre $c$, we define $G'(\xi) = G(\xi)e(c'\cdot\xi + c_n\cdot|\xi|^2)$ so that $E_QG'(z) = E_QG(z+c)$. Hence for any $y\in 16T_Q$, letting $y = x+c$,

$$\begin{aligned}
\|E_QG\|^q_{L^q(w_{B(y,\delta^{-1})})} &= \int |E_QG(z)|^q w_{B(y,\delta^{-1})}(z)dz \\
(u = z-c) &= \int |E_QG'(u)|^q w_{B(y,\delta^{-1})}(u+c)du \\
&= \int |E_QG'(u)|^q w_{B(x,\delta^{-1})}(u)du.
\end{aligned}$$

Let $T'_Q = T_Q - c$, which is another tube in $\mathcal{F}_Q$ and such that $16T'_Q$ contains $x = y-c$. Since $T_Q$ intersects $64nB$, $T'_Q$ intersects $64nB - c$. By (9.15) applied to the cube $B-c$ centred at 0, we have

$$\sum_{T'_Q\in\mathcal{F}_Q^{B-c}}\sup_{x\in 16T'_Q}\|E_QG'\|^q_{L^q_\#(w_{B(x,\delta^{-1})})}\lesssim \delta^{1-n}\|E_QG'\|^q_{L^q_\#(w_{B-c})}.$$

Next we change variables back to get

$$\sum_{T_Q\in\mathcal{F}_Q^B}\sup_{y\in 16T_Q}\|E_QG\|^q_{L^q_\#(w_{B(y,\delta^{-1})})}\lesssim \delta^{1-n}\|E_QG\|^q_{L^q_\#(w_B)},$$

as required. So this process above is almost trivial.

- For each $T_Q\in\mathcal{F}_Q^B$, denote $c_{T_Q} = (c_1,\dots,c_{n-1},c_n)$. Let $B'$ be a $\sqrt{n}\delta^{-1}$-cube centred at $(c_1,\dots,c_{n-1},0)$. Let $\mathcal{B}$ be the collection of all such cubes $B'$ as $T_Q$ ranges through $\mathcal{F}_Q^B$.

We have the following weight inequality:

$$\sum_{B' \in \mathcal{B}} w_{B'}(x', \delta x_n) \lesssim w_B(x). \tag{9.16}$$

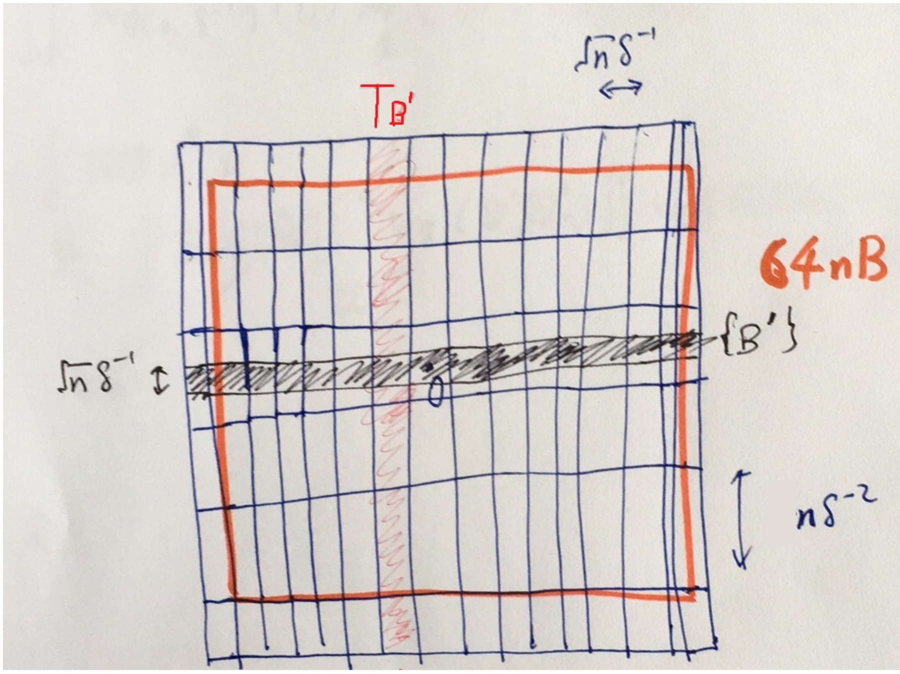


Figure 8: Covering by tiles $T_Q$

Hence by (9.16), the right hand side of (9.15) can be bounded from below:

$$\begin{aligned}\delta^{1-n}\|E_Q G\|^q_{L^q_\#(w_B)} &= \delta^{1-n}\frac{1}{|B|}\int |E_Q G(x)|^q\, w_B(x)dx \\ &\gtrsim \delta^{n+1}\sum_{B' \in \mathcal{B}}\int |E_Q G(x)|^q\, w_{B'}(x', \delta x_n)dx.\end{aligned}$$

For each $B' \in \mathcal{B}$, there are at most $O(1)$ tubes $T_Q \in \mathcal{F}^B_Q$ that have the same entries as $B'$ in the first $(n-1)$-coordinates; let $T_{B'}$ denote the slightly larger tube formed by the union of the aforesaid tubes (see Figure 8 again), so $T_{B'}$ has dimensions $O(\delta^{-1})$ in the first $(n-1)$ coordinates and $O(\delta^{-2})$ in the last coordinate. Note also the $n$-th coordinate of $c(T_{B'})$ is 0. To show (9.15), it thus suffices to show for each $B' \in \mathcal{B}$,

$$\sup_{y \in 16T_{B'}} \|E_Q G\|^q_{L^q_\#(w_{B(y,\delta^{-1})})} \lesssim \delta^{n+1}\int |E_Q G(x)|^q\, w_{B'}(x', \delta x_n)dx. \tag{9.17}$$

- To use the oscillatory feature of the Fourier transform, we will use some distribution theory. Recall how we showed in 4.2 of Section 4 that $\widehat{Eg}$ is a distribution supported on the compact set $\mathbb{P}^{n-1} \subseteq \mathbb{R}^n$. Similarly, $\widehat{E_Q G}$ is also a distribution supported on the paraboloid above $Q$. Thus we can write $E_Q G(z) = \widehat{E_Q G}(e(z\cdot))$.

Fix a bump function $h \in C_c^\infty(\mathbb{R}^n)$ supported on $B(0,1/10)$ with $\int h = 1$. Define $\phi_m(\xi) = \widehat{E_Q G} * h_m(\xi)$ where $h_m(\xi) = (\delta^{-2}m)^n h(\delta^{-2}m\xi)$. Then we have the following properties:

- $\sup_m \|\phi_m\|_1 < \infty$. (Refer to 4.2 in Section 4.)
- $\mathrm{supp}(\phi_m) \subseteq T = [-3\delta/4, 3\delta/4]^{n-1} \times [-\delta^2/10, \delta^2/2]$ for all $m$.
- For any function $f \in C^\infty(\mathbb{R}^n)$, we have $\int \phi_m f \to \widehat{E_Q G}(f)$.

Taking $f_z(\xi) = e(\xi \cdot z)$ above, we have

$$\phi_m^\vee(z) = \int \phi_m(\xi) e(\xi \cdot z) d\xi \to E_Q G(z), \text{ for any } z \in \mathbb{R}^n.$$

Moreover, by the dominated convergence theorem, we have $\phi_m^\vee \to E_Q G$ in $L^q_\#(w_{B(y,\delta^{-1})})$ and also in $L^q_\#(w_{B'}(x', \delta x_n))$. Thus we may prove (9.17) with $\widehat{E_Q G}$ replaced by $\phi_m$ for each $m$. However, for simplicity of notations, we shall use an abuse of notation and treat $\widehat{E_Q G}$ as if it was a function supported on $T$.

### 9.3.3 The main proof

We now proof (9.17). Let $B' \in \mathcal{B}$ be given. Write $c_{B'} = c$, $y = \varepsilon + c$ and $z = u + y$. Since $y \in 16T_{B'}$, we have $|\varepsilon_j| \lesssim \delta^{-1}$ for $1 \le j \le n-1$ and $|\varepsilon_n| \lesssim \delta^{-2}$. Then we compute, using Taylor expansion,

$$\begin{aligned}
|E_Q G(z)| &= \left| \int \widehat{E_Q G}(\eta) e(\xi \cdot z) d\eta \right| \\
&= \left| \int \widehat{E_Q G}(\eta) e(\eta \cdot (u + c + \varepsilon) d\eta \right| \\
&= \left| \int \widehat{E_Q G}(\eta) e(\xi \cdot (u + c)) \sum_{j_1=0}^\infty \cdots \sum_{j_n=0}^\infty \frac{(2\pi i \eta_1^{j_1} \varepsilon_1^{j_1}) \cdots (2\pi i \eta_n^{j_n} \varepsilon_n^{j_n})}{j_1! \cdots j_n!} d\eta \right| \\
&\le \sum_{j_1=0}^\infty \cdots \sum_{j_n=0}^\infty \frac{C^{j_1+\cdots+j_n}}{j_1!\cdots j_n!} \delta^{-(j_1+\cdots+j_{n-1}+2j_n)} \left| \int \widehat{E_Q G}(\eta) e(\eta \cdot (u+c)) \eta_1^{j_1} \cdots \eta_n^{j_n} d\eta \right|.
\end{aligned}$$

For each $j \ge 0$, let $M_j \in C_c^\infty(\mathbb{R})$ be such that

- $M_j(t) = t^j$ for $|t| \le 3/4$,
- $M_j(t) = 0$ for $|t| > 7/8$, and
- $\sup_{j \ge 0} \|M_j^{(k)}\|_\infty \lesssim_k 1$.

(Refer to (5.9) for existence of such functions.) Then using the assumption that $\widehat{E_Q G}$ is supported on $T$ with dimensions $\delta^{n-1} \times \delta^2$, we have, for any $y \in 16T_{B'}$,

$$\begin{aligned}
|E_Q G(u+y)| &\le \sum_{j_1=0}^\infty \cdots \sum_{j_n=0}^\infty \frac{C^{j_1+\cdots+j_n}}{j_1!\cdots j_n!} \left| \int \widehat{E_Q G}(\eta) e(\eta \cdot (u+c)) m_\alpha(\delta^{-1}\eta', \delta^{-2}\eta_n) d\eta \right| \\
&= \sum_{j_1=0}^\infty \cdots \sum_{j_n=0}^\infty \frac{C^{j_1+\cdots+j_n}}{j_1!\cdots j_n!} \left| E_Q G * (m_\alpha(\delta^{-1}\cdot', \delta^{-2}\cdot_n)^\vee)(u+c) \right|,
\end{aligned}$$

where

$$m_\alpha(\eta) = M_{j_1}(\eta_1)\cdots M_{j_{n-1}}(\eta_{n-1})M_{j_n}(\eta_n), \text{ and where } \alpha = (j_1,\ldots,j_n). \tag{9.18}$$

By the triangle inequality, to show (9.17), it suffices to show that

$$\delta^n \int \left|E_Q G * (m_\alpha(\delta^{-1}\cdot', \delta^{-2}\cdot_n)^\vee)(u+c)\right|^q w_{B(0,\delta^{-1})}(u)du \lesssim \delta^{n+1} \int |E_Q G(x)|^q w_{B'}(x', \delta x_n)dx.$$

Letting $H(u) = E_Q G(u+c)$ and changing $v = x - c$, the above is equivalent to

$$\int \left|H * (m_\alpha(\delta^{-1}\cdot', \delta^{-2}\cdot_n)^\vee)(u)\right|^q w_{B(0,\delta^{-1})}(u)du \lesssim \delta \int |H(v)|^q w_{B(0,\delta^{-1})}(v', \delta v_n)dv. \tag{9.19}$$

Next we compute

$$(m_\alpha(\delta^{-1}\cdot', \delta^{-2}\cdot_n)^\vee)(u) = \delta^{n+1} m_\alpha^\vee(\delta u', \delta^2 u_n), \tag{9.20}$$

so in particular, $\|m_\alpha(\delta^{-1}\cdot', \delta^{-2}\cdot_n)^\vee\|_1 \lesssim 1$. Now we can apply Jensen's inequality with $q \geq 1$ to get

$$\begin{aligned}
&\int \left|H * (m_\alpha(\delta^{-1}\cdot', \delta^{-2}\cdot_n)^\vee)(u)\right|^q w_{B(0,\delta^{-1})}(u)du \\
&= \int \left|\int H(v)(m_\alpha(\delta^{-1}\cdot', \delta^{-2}\cdot_n)^\vee)(u-v)dv\right|^q w_{B(0,\delta^{-1})}(u)du \\
&\leq \int\int |H(v)|^q \left|m_\alpha(\delta^{-1}\cdot', \delta^{-2}\cdot_n)^\vee(u-v)\right| dv\; w_{B(0,\delta^{-1})}(u)du \\
&= \int |H(v)|^q \int \left|m_\alpha(\delta^{-1}\cdot', \delta^{-2}\cdot_n)^\vee(u-v)\right| w_{B(0,\delta^{-1})}(u)dudv.
\end{aligned}$$

Thus (9.19) will be true if we can prove the following weight inequality:

$$\int \left|m_\alpha(\delta^{-1}\cdot', \delta^{-2}\cdot_n)^\vee(u-v)\right| w_{B(0,\delta^{-1})}(u)du \lesssim \delta w_{B(0,\delta^{-1})}(v', \delta v_n). \tag{9.21}$$

By (9.20) and using the symmetry of $w_{B(0,\delta^{-1})}$, the left hand side of (9.21) is equal to

$$\delta^{n+1} \int \left|m_\alpha^\vee(\delta v' - \delta u', \delta^2 v_n - \delta^2 u_n)\right| w_{B(0,\delta^{-1})}(u)du.$$

Now we use the definition of $m_\alpha$ (9.18) and the uniform derivative bound of $M_j^{(k)}$ to get $|m_\alpha^\vee(u)| \lesssim_E w_{B(0,1),E}(u)$ for any $E$, and hence

$$\begin{aligned}
&\delta^{n+1} \int \left|m_\alpha^\vee(\delta v' - \delta u', \delta^2 v_n - \delta^2 u_n)\right| w_{B(0,\delta^{-1})}(u)du \\
&\lesssim \delta^{n+1} \int w_{B(0,1)}(\delta v' - \delta u', \delta^2 v_n - \delta^2 u_n) w_{B(0,\delta^{-1})}(u)du \\
&= \delta^{n+1} \int w_{B(0,\delta^{-1})}(v' - u', \delta v_n - \delta u_n) w_{B(0,\delta^{-1})}(u)du.
\end{aligned}$$

Using Proposition 5.2, we get

$$\delta^{n+1} \int w_{B(0,\delta^{-1})}(v' - u', \delta v_n - \delta u_n) w_{B(0,\delta^{-1})}(u)du \lesssim \delta w_{B(0,\delta^{-1})}(v', \delta v_n),$$

which is the right hand side of (9.21).

# 10 Decoupling in the Range $2 \le p \le \frac{2n}{n-1}$

In this section we show $\mathrm{Dec}_n(\delta, p) \lesssim_\varepsilon \delta^{-\varepsilon}$ for $2 \le p \le \frac{2n}{n-1}$, using induction on scales. We first introduce some general notations which may be used in this section as well as all subsequent sections.

## 10.1 Notations and conventions

Let $p_0 = \frac{2n}{n-1}$. Let $g : [0,1]^{n-1} \to \mathbb{C}$ be given. We will fix $g$ in the rest of the whole section, so we drop the dependence on $g$ of all terms in this section. But keep in mind that all implicit constants will be independent of $g$.

Fix $0 < \nu < 1$ and $n$ $\nu$-transverse cubes $Q_1, \dots, Q_n \subseteq [0,1]^{n-1}$ with the same side length $l(Q_i) := \mu$. All implicit constants will be independent of $\mu$ and the positions of the frequency cubes $Q_i$, $1 \le i \le n$.

Let $E \ge 100n$. Notice that we allow every implicit constant including the decoupling constants to depend on $E$, so we will hide the parameter $E$ from our notations.

For $s \in \mathbb{N}$ and $K \ge 1$, let $B_K^s$ denote a cube in $\mathbb{R}^n$ with side length $K^s$ with arbitrary centre. For simplicity, if $s = 1$, we will usually abbreviate $B_K^1$ as $B_K$.

We will only be dealing with $K$'s with $\mu \ge K^{-1}$, i.e. $K \ge \mu^{-1}$.

Let $1 \le t, p < \infty$ be exponents and $q \le s \le r$ be positive integers. As $\mu \ge K^{-1} \ge K^{-q}$, we can define

$$D_t(q, B_K^r) = \prod_{i=1}^{n} \left( \sum_{Q_{i,q} \in \mathcal{P}_{K^{-q}}(Q_i)} \left\| E_{Q_{i,q}} g \right\|^2_{L^t_\#(w_{B_K^r})} \right)^{\frac{1}{2n}}.$$

Then $q = \frac{p_0(n-1)}{n} = 2$ in (9.2). For $\varepsilon > 0$ and $\nu$ fixed above, let $C_{\varepsilon,\nu}$ be the implicit constant in the inequality (9.1) $\le C_{\varepsilon,\nu}$(9.2). For $K \ge 1$, taking the $\delta$ in (9.2) to be $K^{-1}$, we have (9.2) $= K^\varepsilon D_2(1, B_K^2)$.

We write $\mathcal{B}^s(B_K^r)$ to be the (unique) partition of $B_K^r$ with cubes $B_K^s$ of side length $K^s$. Define

$$A_p(q, B_K^r, s) = \left( \frac{1}{\#\mathcal{B}^s(B_K^r)} \sum_{B_K^s \in \mathcal{B}^s(B_K^r)} D_2(q, B_K^s)^p \right)^{\frac{1}{p}}.$$

Hence if $p_0 = \frac{2n}{n-1}$, taking $\delta = K^{-1}$ in (9.1), we have (9.1) is equal to $A_{p_0}(1, B_K^2, 1)$. In particular, Theorem 9.2 says

$$A_{p_0}(1, B_K^2, 1) \le C_{\varepsilon,\nu} D_2(1, B_K^2).$$

Thus if $p \le p_0$, then by definition of $A_p$ and Jensen's inequality, we trivially have

$$A_p(1, B_K^2, 1) \le A_{p_0}(1, B_K^2, 1) \le C_{\varepsilon,\nu} D_2(1, B_K^2). \tag{10.1}$$

### 10.1.1 General properties

With the notations above, we state and prove some general properties. Note all propositions in this subsection are true as long as $2 \le p < \infty$.

**Proposition 10.1.** *Let $1 \le t, p < \infty$ be exponents and let $q \le s \le r$ be positive integers. For any $K \ge \mu^{-1}$ and $m \ge 1$, we have the following equalities:*

$$D_t(q, B^r_{K^m}) = D_t(mq, B^{mr}_K), \tag{10.2}$$
$$A_p(q, B^r_{K^m}, s) = A_p(mq, B^{mr}_K, ms), \tag{10.3}$$
$$A_p(q, B^r_K, r) = D_2(q, B^r_K). \tag{10.4}$$

They are immediate by definition.

**Proposition 10.2.** *If $K \ge \mu^{-1}$, then we have*

$$D_2(1, B_K) \gtrsim \prod_{i=1}^n \|E_{Q_i} g\|^{\frac{1}{n}}_{L^2_\#(w_{B_K})}.$$

*Proof.* By the $L^2$-decoupling Theorem 6.1 with $R = K$ and $l(Q_i) = \mu \ge K^{-1}$,

$$D_2(1, B_K) = \prod_{i=1}^n \left( \sum_{Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)} \|E_{Q_{i,1}} g\|^2_{L^2_\#(w_{B_K})} \right)^{\frac{1}{2n}} \gtrsim \prod_{i=1}^n \|E_{Q_i} g\|^{\frac{1}{n}}_{L^2_\#(w_{B_K})}.$$

□

**Proposition 10.3.** *If $K \ge \mu^{-1}$, $m \ge 1$, and $2 \le p < \infty$, then we have*

$$A_p(1, B^m_K, 1) \lesssim D_2(1, B^m_K) \le D_p(1, B^m_K).$$

*Proof.* We only prove the first inequality, as the second follows immediately from Jensen's inequality.

$$\begin{aligned} A_p(1, B^m_K, 1)^p &= \frac{1}{\#\mathcal{B}^1(B^m_K)} \sum_{B_K \in \mathcal{B}^1(B^m_K)} D_2(1, B_K)^p \\ &= \frac{1}{\#\mathcal{B}^1(B^m_K)} \sum_{B_K \in \mathcal{B}^1(B^m_K)} \prod_{i=1}^n \left( \sum_{Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)} \|E_{Q_{i,1}} g\|^2_{L^2_\#(w_{B_K})} \right)^{\frac{p}{2n}} \\ &\le \frac{1}{\#\mathcal{B}^1(B^m_K)} \prod_{i=1}^n \left( \sum_{B_K \in \mathcal{B}^1(B^m_K)} \left( \sum_{Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)} \|E_{Q_{i,1}} g\|^2_{L^2_\#(w_{B_K})} \right)^{\frac{p}{2}} \right)^{\frac{1}{n}}, \end{aligned}$$

where the last inequality follows from the following Hölder's inequality for $n$-terms:

$$\sum_j \prod_{i=1}^n |a_{i,j}|^{\frac{1}{n}} \le \prod_{i=1}^n \left( \sum_j |a_{i,j}| \right)^{\frac{1}{n}}.$$

Since $p \geq 2$, using Hölder's inequality we have

$$\begin{aligned}
&\sum_{B_K \in \mathcal{B}^1(B_K^m)} \left( \sum_{Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)} \left\| E_{Q_{i,1}} g \right\|^2_{L^2_{\#}(w_{B_K})} \right)^{\frac{p}{2}} \\
&\leq \left( \sum_{B_K \in \mathcal{B}^1(B_K^m)} \sum_{Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)} \left\| E_{Q_{i,1}} g \right\|^2_{L^2_{\#}(w_{B_K})} \right)^{\frac{p}{2}} \left( \sum_{B_K \in \mathcal{B}^1(B_K^m)} 1 \right)^{1-\frac{p}{2}} \\
&\lesssim K^{-\frac{np}{2}} K^{n(m-1)\left(1-\frac{p}{2}\right)} \left( \sum_{Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)} \left\| E_{Q_{i,1}} g \right\|^2_{L^2(w_{B_K^m})} \right)^{\frac{p}{2}},
\end{aligned}$$

where we have used the following weight inequality as a consequence of Proposition 3.1:

$$\sum_{B_K \in \mathcal{B}^1(B_K^m)} w_{B_K} \lesssim w_{B_K^m}.$$

Hence

$$\begin{aligned}
&\frac{1}{\#\mathcal{B}^1(B_K^m)} \prod_{i=1}^{n} \left( \sum_{B_K \in \mathcal{B}^1(B_K^m)} \left( \sum_{Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)} \left\| E_{Q_{i,1}} g \right\|^2_{L^2_{\#}(w_{B_K})} \right)^{\frac{p}{2}} \right)^{\frac{1}{n}} \\
&\lesssim K^{-(m-1)n} K^{-\frac{np}{2}} K^{n(m-1)\left(1-\frac{p}{2}\right)} \prod_{i=1}^{n} \left( \sum_{Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)} \left\| E_{Q_{i,1}} g \right\|^2_{L^2(w_{B_K^m})} \right)^{\frac{p}{2n}} \\
&= \prod_{i=1}^{n} \left( \sum_{Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)} \left\| E_{Q_{i,1}} g \right\|^2_{L^2_{\#}(w_{B_K^m})} \right)^{\frac{p}{2n}}.
\end{aligned}$$

Hence

$$A_p(1, B_K^m, 1) \lesssim \prod_{i=1}^{n} \left( \sum_{Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)} \left\| E_{Q_{i,1}} g \right\|^2_{L^2_{\#}(w_{B_K^m})} \right)^{\frac{1}{2n}} = D_2(1, B_K^m).$$

□

## 10.2 Intermediate steps

The intermediate steps consist of two parts: an induction on scales argument, followed by an appliaction of Corollary 4.3, which a Bernstein-type inequality (or reverse-Hölder's inequality or locally constant property).

### 10.2.1 The induction argument

We start with a lemma which facilitates our induction argument.

**Lemma 10.4.** *Let $2 \le p \le p_0$, $K \ge \mu^{-1}$, and $M \ge 2$. Then*

$$A_p(1, B_K^M, 1) \lesssim_{\varepsilon,\nu} K^\varepsilon A_p(2, B_K^M, 2).$$

*Proof.* We first prove the case $M = 2$:

$$A_p(1, B_K^2, 1) \lesssim_{\varepsilon,\nu} K^\varepsilon A_p(2, B_K^2, 2) = K^\varepsilon D_2(2, B_K^2), \tag{10.5}$$

where the last equality follows from (10.4). Recall (10.1) which says

$$A_p(1, B_K^2, 1) \le C_{\varepsilon,\nu} D_2(1, B_K^2).$$

Hence it suffices to show $D_2(1, B_K^2) \lesssim K^\varepsilon D_2(2, B_K^2)$, which raised to the power $2n$ is equivalent to

$$\prod_{i=1}^{n} \sum_{Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)} \left\| E_{Q_{i,1}} g \right\|^2_{L^2_\#(w_{B_K^2})} \lesssim \prod_{i=1}^{n} \sum_{Q_{i,2} \in \mathcal{P}_{K^{-2}}(Q_i)} \left\| E_{Q_{i,2}} g \right\|^2_{L^2_\#(w_{B_K^2})}. \tag{10.6}$$

The right hand side of (10.6) is equal to

$$\prod_{i=1}^{n} \sum_{Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)} \sum_{Q_{i,2} \in \mathcal{P}_{K^{-2}}(Q_{i,1})} \left\| E_{Q_{i,2}} g \right\|^2_{L^2_\#(w_{B_K^2})}.$$

Thus (10.6) is true if for every $Q_{i,1} \in \mathcal{P}_{K^{-1}}(Q_i)$, we have

$$\left\| E_{Q_{i,1}} g \right\|_{L^2_\#(w_{B_K^2})} \lesssim \left( \sum_{Q_{i,2} \in \mathcal{P}_{K^{-2}}(Q_{i,1})} \left\| E_{Q_{i,2}} g \right\|^2_{L^2_\#(w_{B_K^2})} \right)^{\frac{1}{2}}.$$

But the above just follows from the $L^2$-decoupling inequality (6.1) with $R = K^2$ and $l(Q) = l(Q_{i,1}) = K^{-1} \ge K^{-2} = R$. This proves (10.5).

Next we prove Lemma 10.4 for a general $M \ge 2$, which raised to the power $p$ is equivalent to

$$\frac{1}{\#\mathcal{B}_1(B_K^M)} \sum_{B^1 \in \mathcal{B}_1(B_K^M)} D_2(1, B_K^1)^p \lesssim_{\varepsilon,\nu} \frac{1}{\#\mathcal{B}_2(B_K^M)} \sum_{B_K^2 \in \mathcal{B}_2(B_K^M)} D_2(2, B_K^2)^p.$$

The left hand side of the above inequality is equal to

$$\frac{1}{\#\mathcal{B}_2(B_K^M)} \sum_{B^2 \in \mathcal{B}_2(B_K^M)} \frac{1}{\#\mathcal{B}_1(B_K^2)} \sum_{B_K^1 \in \mathcal{B}_1(B_K^2)} D_2(1, B_K^1)^p.$$

Hence it suffices to show that for each $B_K^2 \in \mathcal{B}_2(B_K^M)$, we have

$$\frac{1}{\#\mathcal{B}_1(B_K^2)} \sum_{B_K^1 \in \mathcal{B}_1(B_K^2)} D_2(1, B_K^1)^p \lesssim_{\varepsilon,\nu} K^\varepsilon D_2(2, B_K^2)^p,$$

Raising both sides to the power $1/p$, we see the above is exactly (10.5) which we proved to be true. □

Now we come to the main induction step.

**Proposition 10.5.** *Let $2 \le p \le p_0$, $R \ge \mu^{-1}$ and $m \ge 1$. Then for each $\varepsilon > 0$, there is $C_{\varepsilon,\nu}$ such that*

$$A_p(1, B_R^{2^m}, 1) \le C_{\varepsilon,\nu}^{m-1} R^{2^{m-1}\varepsilon} A_p(1, B^2_{R^{2^{m-1}}}, 1).$$

*Proof.* Applying Lemma 10.4 with $M = 2^m$ and $K = R$, we have

$$A_p(1, B_R^{2^m}, 1) \le C_{\varepsilon,\nu} R^{\varepsilon} A_p(2, B_R^{2^m}, 2) = C_{\varepsilon,\nu} R^{\varepsilon} A_p(1, B_{R^2}^{2^{m-1}}, 1).$$

Applying Lemma 10.4 with $M = 2^{m-1}$ and $K = R^2$, we have

$$A_p(1, B_{R^2}^{2^{m-1}}, 1) \le C_{\varepsilon,\nu} R^{2\varepsilon} A_p(2, B_{R^2}^{2^{m-1}}, 2) = C_{\varepsilon,\nu} R^{2\varepsilon} A_p(1, B_{R^4}^{2^{m-2}}, 1).$$

We can perform the above process $(m-1)$ times until we get

$$A_p(1, B^4_{R^{2^{m-2}}}, 1) \le C_{\varepsilon,\nu} R^{2^{m-2}\varepsilon} A_p(2, B^4_{R^{2^{m-2}}}, 2) C_{\varepsilon,\nu} = A_p(1, B^2_{2^{m-1}}, 1).$$

(The reason we stop at the $(m-1)$-th step is that we like to obtain a term of the form $A_p(1, B_K^2, 1)$ which facilitates a decoupling that matches the right hand side of Definition 8.4.) Combining the above inequalities gives

$$A_p(1, B_R^{2^m}, 1) \le C_{\varepsilon,\nu}^{m-1} R^{2^{m-1}\varepsilon} A_p(1, B^2_{R^{2^{m-1}}}, 1).$$

□

### 10.2.2 Applying a Bernstein-type inequality

We first use Proposition 4.3 to get to an inequality close enough to the form in Definition 8.4.

**Proposition 10.6.** *Let $0 < \nu \le 1$ and $Q_1, \dots, Q_n$ be an $n$-tuple of $\nu$-transverse cubes with the same side length $\mu$. If $2 \le p \le p_0$, then for any $m \ge 1$ and any $R \ge \mu^{-2^m}$, we have*

$$\left( \sum_{B' \in \mathcal{P}_{R^{2^{-m}}}(B_R)} \left( \prod_{i=1}^n \|E_{Q_i} g\|^p_{L^p(B')} \right)^{\frac{1}{n}} \right)^{\frac{1}{p}}$$
$$\le C^m_{\varepsilon,\nu} R^{\frac{\varepsilon}{2}} R^{\frac{n/2-n/p}{2^m}} \left( \prod_{i=1}^n \sum_{Q_{i,1} \in \mathcal{P}_{R^{-1/2}}(Q_i)} \left\| E_{Q_{i,1}} g \right\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2n}},$$

*where $C^m_{\varepsilon,\nu}$ is independent of $\mu$, $R$ and the positions of $B_R$ and $Q_i, 1 \le i \le n$.*

*Proof.* Applying Propositions 10.3 and 10.5 with $2 \le p \le p_0$, $m \ge 1$ and $K = R^{2^{m-1}} \ge \mu^{-1}$, we have, for some absolute constant $C > 0$,

$$A_p(1, B_R^{2^m}, 1) \le C C_{\varepsilon,\nu}^{m-1} R^{2^{m-1}\varepsilon} D_2(1, B^2_{R^{2^{m-1}}}).$$

(Assuming $C \le C_{\varepsilon,\nu}$, we may bound $CC_{\varepsilon,\mu}^{m-1} \le C_{\varepsilon,\nu}^m$.) We compute

$$A_p(1, B_R^{2^m}, 1) = \left( \frac{1}{\#\mathcal{B}^1(B_R^{2^m})} \sum_{B_R \in \mathcal{B}^1(B_R^{2^m})} D_2(1, B_R)^p \right)^{\frac{1}{p}}$$

$$\text{(by Prop 10.2)} \gtrsim \left( \frac{1}{\#\mathcal{B}^1(B_R^{2^m})} \sum_{B_R \in \mathcal{B}^1(B_R^{2^m})} \left( \prod_{i=1}^n \|E_{Q_i} g\|_{L^2_\#(w_{B_R})}^p \right)^{\frac{1}{n}} \right)^{\frac{1}{p}}. \tag{10.7}$$

We now invoke Proposition 4.3. As $R \ge \mu^{-1} = l(Q_i)^{-1}$, using (4.3) we have

$$\|E_{Q_i} g\|_{L^2_\#(w_{B_R})} \gtrsim (\mu R)^{\frac{n}{p} - \frac{n}{2}} \|E_{Q_i} g\|_{L^p_\#(B_R)} \ge R^{\frac{n}{p} - \frac{n}{2}} \|E_{Q_i} g\|_{L^p_\#(B_R)}.$$

We shall see that we can afford such a loss of the power on $R$.

Thus we have

$$A_p(1, B_R^{2^m}, 1) \gtrsim R^{\frac{n}{p} - \frac{n}{2}} \left( \frac{1}{\#\mathcal{B}^1(B_R^{2^m})} \sum_{B_R \in \mathcal{B}^1(B_R^{2^m})} \left( \prod_{i=1}^n \|E_{Q_i} g\|_{L^p_\#(w_{B_R})}^p \right)^{\frac{1}{n}} \right)^{\frac{1}{p}}$$

$$= R^{-\frac{2^m n}{p}} R^{\frac{n}{p} - \frac{n}{2}} \left( \frac{1}{\#\mathcal{B}^1(B_R^{2^m})} \sum_{B_R \in \mathcal{B}^1(B_R^{2^m})} \left( \prod_{i=1}^n \|E_{Q_i} g\|_{L^p(w_{B_R})}^p \right)^{\frac{1}{n}} \right)^{\frac{1}{p}}.$$

Also by (10.3),

$$D_2(1, B^2_{R^{2^{m-1}}}) \le R^{-\frac{2^m n}{p}} \left( \prod_{i=1}^n \sum_{Q_{i,1} \in \mathcal{P}_{R^{-2^{m-1}}}(Q_i)} \left\| E_{Q_{i,1}} g \right\|^2_{L^p(w_{B_{R^{2^m}}})} \right)^{\frac{1}{2n}}.$$

Thus (hiding any absolute constant $C$ inside $C_{\varepsilon,\nu}$ as before),

$$\left( \sum_{B_R \in \mathcal{B}^1(B_R^{2^m})} \left( \prod_{i=1}^n \|E_{Q_i} g\|_{L^p(w_{B_R})}^p \right)^{\frac{1}{n}} \right)^{\frac{1}{p}}$$

$$\le C_{\varepsilon,\nu}^m R^{\frac{n}{2} - \frac{n}{p}} R^{2^{m-1}\varepsilon} \left( \prod_{i=1}^n \sum_{Q_{i,1} \in \mathcal{P}_{R^{-2^{m-1}}}(Q_i)} \left\| E_{Q_{i,1}} g \right\|^2_{L^p(w_{B_{R^{2^m}}})} \right)^{\frac{1}{2n}}.$$

Changing variables $R^{2^m} \mapsto R$, we have for any $m \ge 1$ and any $R \ge \mu^{-2^m}$,

$$\left( \sum_{B' \in \mathcal{P}_{R^{2^{-m}}}(B_R)} \left( \prod_{i=1}^n \|E_{Q_i} g\|_{L^p(w_{B'})}^p \right)^{\frac{1}{n}} \right)^{\frac{1}{p}}$$

$$\le C_{\varepsilon,\nu}^m R^{\frac{\varepsilon}{2}} R^{\frac{n/2 - n/p}{2^m}} \left( \prod_{i=1}^n \sum_{Q_{i,1} \in \mathcal{P}_{R^{-1/2}}(Q_i)} \left\| E_{Q_{i,1}} g \right\|^2_{L^p(w_{B_R})} \right)^{\frac{1}{2n}}.$$

$\square$

## 10.3 The final argument

### 10.3.1 Bounding the multilinear decoupling constant

The main result is the following:

**Proposition 10.7.** *Let $0 < \nu \le 1$. If $2 \le p \le p_0$, then for all $m \ge 1$ and $0 < \delta \le 1$,*

$$\mathrm{MDec}_n(\delta, p, \nu, m, E) \le C^m_{\varepsilon,\nu} R^{\frac{\varepsilon}{2}} R^{\frac{n/2-n/p}{2^m}} = C^m_{\varepsilon,\nu} \delta^{-\frac{\varepsilon}{2}} \delta^{-\frac{n/2-n/p}{2^m}}.$$

*Proof.* Refer to Definition 8.4. Let $g : [0,1]^{n-1} \to \mathbb{C}$. Let $m' \ge 1$, $0 < \delta \le 1$. Let $B$ with $l(B) = \delta^{-1}$. Let $0 < \nu \le 1$ and $\{Q_i : 1 \le i \le n\}$ be an $n$-tuple of $\nu$-transverse cubes in $[0,1]^{n-1}$ such that $l(Q_i) = \mu \ge \delta^{2^{-m'}}$.

If we can show

$$\left[\sum_{\Delta \in \mathcal{P}_{\mu^{-1}}(B)} \prod_{i=1}^{n} \|E_{Q_i} g\|^{p/n}_{L^p(\Delta)}\right]^{\frac{1}{p}} \le A \prod_{i=1}^{n} \left(\sum_{q_i \in \mathcal{P}_{\delta^{1/2}}(Q_i)} \|E_{q_i} g\|^2_{L^p(w_B)}\right)^{\frac{1}{2n}}, \qquad (10.8)$$

for some multiplicative factor $A$ independent of $g$, $\mu$ and the positions of $\{Q_i, 1 \le i \le n\}$, then we have $\mathrm{MDec}_n(\delta, p, \nu, m', E) \le A$.

To apply Proposition 10.6, we let $R = \delta^{-1}$ and let $m = m'$. Thus $\mu \ge R^{-2^{-m}}$.

We then compute

$$\begin{aligned}
&\left[\sum_{\Delta \in \mathcal{P}_{\mu^{-1}}(B)} \prod_{i=1}^{n} \|E_{Q_i} g\|^{p/n}_{L^p(w_\Delta)}\right]^{\frac{1}{p}} \\
&= \left[\sum_{B' \in \mathcal{P}_{R^{2^{-m}}}(B)} \sum_{\Delta \in \mathcal{P}_{\mu^{-1}}(B')} \prod_{i=1}^{n} \|E_{Q_i} g\|^{p/n}_{L^p(w_\Delta)}\right]^{\frac{1}{p}} \\
(n\text{-Hölder}) &\le \left[\sum_{B' \in \mathcal{P}_{R^{2^{-m}}}(B)} \left(\prod_{i=1}^{n} \sum_{\Delta \in \mathcal{P}_{\mu^{-1}}(B')} \|E_{Q_i} g\|^{p}_{L^p(w_\Delta)}\right)^{\frac{1}{n}}\right]^{\frac{1}{p}} \\
&\lesssim \left[\sum_{B' \in \mathcal{P}_{R^{2^{-m}}}(B)} \left(\prod_{i=1}^{n} \|E_{Q_i} g\|^{p}_{L^p(w_{B'})}\right)^{\frac{1}{n}}\right]^{\frac{1}{p}},
\end{aligned}$$

where the last line follows from $\sum_{\Delta \in \mathcal{P}_{\mu^{-1}}(B')} w_\Delta \lesssim w_{B'}$. Hence we can use Proposition

10.6 to get

$$\left[\sum_{B'\in\mathcal{P}_{R^{2^{-m}}}(B)}\left(\prod_{i=1}^{n}\|E_{Q_i}g\|^p_{L^p(w_{B'})}\right)^{\frac{1}{n}}\right]^{\frac{1}{p}}$$

$$\lesssim C^m_{\varepsilon,\nu}R^{\frac{\varepsilon}{2}}R^{\frac{n/2-n/p}{2^m}}\left(\prod_{i=1}^{n}\sum_{Q_{i,1}\in\mathcal{P}_{R^{-1/2}}(Q_i)}\left\|E_{Q_{i,1}}g\right\|^2_{L^p(w_{B_R})}\right)^{\frac{1}{2n}}$$

$$= C^m_{\varepsilon,\nu}R^{\frac{\varepsilon}{2}}R^{\frac{n/2-n/p}{2^m}}\prod_{i=1}^{n}\left(\sum_{q_i\in\mathcal{P}_{\delta^{1/2}}(Q_i)}\|E_{q_i}g\|^2_{L^p(w_B)}\right)^{\frac{1}{2n}}.$$

Hence we see our $A$ in (10.8) can be taken to be $C^m_{\varepsilon,\nu}R^{\frac{\varepsilon}{2}}R^{\frac{n/2-n/p}{2^m}}$. □

### 10.3.2 Proof of decoupling inequality

**Theorem 10.8.** *Let $n\geq 2$ and $E\geq 100n$. If $2\leq p\leq p_0$, then we have*

$$\mathrm{Dec}_n(R^{-1},p,E)\lesssim_{\varepsilon,n,p,E}R^{\varepsilon},$$

*for all $R\geq 1$.*

*Proof.* We first prove the case $n=2$. Recall (8.16) of Theorem 8.9, which says that for each $0<\nu\leq 1$ and $m\geq 1$, there is $C_{\nu,m}=C_{\nu,p,m,E}$ and $\eta(\nu)=\eta(\nu,p,E)$ such that $\lim_{\nu\to 0^+}\eta(\nu)=0$ and for each $R\geq\nu^{-2^m}$, we have

$$\mathrm{Dec}_2(R^{-1},p,E)\leq C_{\nu,m,p}R^{\eta(\nu)}\sup_{1\leq R'\leq R}\mathrm{MDec}_2(R'^{-1},p,\nu,m,E). \tag{10.9}$$

Let $\varepsilon>0$. Since $\lim_{\nu\to 0^+}\eta(\nu)=0$, take $0<\nu\leq 1$ with $\eta(\nu)<\varepsilon$.

By Corollary 7.4, it suffices to assume $R\in 2^{\mathbb{N}}$.

Let $1\leq R'\leq R$. Take $m\geq 1$ large enough such that $(\frac{n}{2}-\frac{n}{p})2^{-m-1}\leq\varepsilon$ (so $m$ depends on $n$ and $\varepsilon$ only). Then by Proposition 10.7 with $\delta=R'^{-1}$, we have

$$\mathrm{MDec}_2(R'^{-1},p,\nu,m,E)\leq C^m_{\varepsilon,\nu}R'^{\frac{\varepsilon}{2}}R'^{\frac{n/2-n/p}{2^m}}\leq C^m_{\varepsilon,\nu}R^{\varepsilon},$$

which holds for all $1\leq R'\leq R$. Taking $C'(\varepsilon)=C'(\varepsilon,p,E)=C_{\nu,m}C^m_{\varepsilon,\nu}$ which depends in turn on $\varepsilon$, $p$ and $E$ only, we have, by (10.9), for all $R\geq\nu^{-2^m}$,

$$\mathrm{Dec}_2(R^{-1},p,E)\leq C'(\varepsilon)R^{2\varepsilon}.$$

Denote $K=\nu^{-2^m}$ which depends in turn on $\varepsilon,n,E$ only. It remains to prove that for $1\leq R\leq K$ we also have decoupling. Write $I=[1,K]\cap 2^{\mathbb{N}}$, which has finite cardinality $\#I:=N=N(\varepsilon,p,E)$. Write $I=\{R_1,\ldots,R_N\}$. For each $1\leq i\leq N$, we use trivial decoupling 7.2 to get $\mathrm{Dec}_2(R_i^{-1},p,E)\leq C_i=C_{i,\varepsilon,p,E}$. Take $C''(\varepsilon)=\max_{1\leq i\leq N}C_i$, and then take $C(\varepsilon)=\max\{C'(\varepsilon),C''(\varepsilon)\}=C(\varepsilon,p,E)$. Thus for all $R\geq 1$, we have

$$\mathrm{Dec}_2(R^{-1},p,E)\leq C(\varepsilon)R^{2\varepsilon}.$$

Now assume for some $n \geq 2$, we have $\mathrm{Dec}_n(R^{-1}, p, E) \lesssim_{\varepsilon,n,p,E} R^{\varepsilon}$. Then by Theorem 5.1, we have

$$\mathrm{Dec}_n(R^{-1}, p, \Gamma_{n-1}(E)) \lesssim_{\varepsilon,n,p,E} R^{\varepsilon}.$$

Using (8.17) of Theorem 8.9, we have for each $\varepsilon > 0$, $0 < \nu \leq 1$ and $m \geq 1$, there is $C_{\nu,m} = C_{n,\nu,p,m,E}$ and $\eta_\varepsilon(\nu) = \eta(\nu, \varepsilon, p, n, E)$ such that $\lim_{\nu\to 0^+} \eta_\varepsilon(\nu) = 0$ and for each $R \geq \nu^{-2^m}$, we have

$$\mathrm{Dec}_n(R^{-1}, p, E) \leq C_{\nu,m} R^{\eta_\varepsilon(\nu)+\varepsilon} \sup_{1\leq R'\leq R} \mathrm{MDec}_n(R'^{-1}, p, \nu, m, E).$$

Now argue similarly as in the case $n = 2$. □

# 11 Decoupling in the Range $\frac{2n}{n-1} < p \leq \frac{2(n+1)}{n-1}$

In this section we show $\mathrm{Dec}_n(\delta, p) \lesssim_\varepsilon \delta^{-\varepsilon}$ for $\frac{2n}{n-1} < p \leq \frac{2(n+1)}{n-1}$, using an interpolation argument in addition to the arguments in the previous section. We will still use the notations and conventions as in the last section.

Let $q = \frac{p(n-1)}{n}$, so $q < p$. For simplicity, let $p_0 = \frac{2n}{n-1}$ and $p_1 = \frac{2(n+1)}{n-1}$. Since $p \geq p_0$, we have $q \geq 2$. Let $\alpha = \alpha_p$ such that

$$\frac{1}{q} = \frac{1-\alpha}{2} + \frac{\alpha}{p}. \tag{11.1}$$

We first prove the non-endpoint case $p < p_1$. In the end, we will prove the endpoint case $p = p_1$.

## 11.1 The induction argument

The corresponding lemma to Lemma 10.4 is the following:

**Lemma 11.1.** *Let $p_0 < p < \infty$, $K \geq \mu^{-1}$, and $M \geq 2$. Then*

$$A_p(1, B_K^M, 1) \lesssim_{\varepsilon,\nu} K^{\varepsilon} A_p(2, B_K^M, 2)^{1-\alpha} D_p(1, B_K^M, 1)^{\alpha}.$$

*Proof.* Imitating the proof of Lemma 10.4, it suffices to prove the case $M = 2$:

$$A_p(1, B_K^2, 1) \lesssim_{\varepsilon,\nu} K^{\varepsilon} D_2(2, B_K^2)^{1-\alpha} D_p(1, B_K^2)^{\alpha}. \tag{11.2}$$

Recall Theorem 9.2 says that for some $C_{\varepsilon,\nu} = C_{\varepsilon,\nu,n,p,E}$, we have

$$A_p(1, B_K^2, 1) \leq C_{\varepsilon,\nu} D_q(1, B_K^2).$$

Using Hölder's inequality twice (first by (11.1) and then with respect to $l^2$), we have

$$D_q(1, B_K^2) \leq D_2(1, B_K^2)^{1-\alpha} D_p(1, B_K^2)^{\alpha},$$

which completes the proof of (11.2). □

The main proposition which follows by induction is the following.

**Proposition 11.2.** *Let $p_0 < p < p_1$, $R \geq \mu^{-1}$ and $m \geq 1$. Then for each $\varepsilon > 0$, there is $C_{\varepsilon,\nu} = C_{\varepsilon,\nu,\alpha(p)} = C_{\varepsilon,\nu,p,n,E}$ such that*

$$A_p(1, B_R^{2^m}, 1) \lesssim_{\varepsilon,\nu,\alpha} R^{\beta_m} A_p(1, B^2_{R^{2^{m-1}}}, 1)^{(1-\alpha)^{m-1}} \prod_{l=0}^{m-2} D_p(1, B^{2^{m-l}}_{R^{2^l}})^{\alpha(1-\alpha)^l},$$

*for some $\beta_m = \beta_{m,p}$.*

*Proof.* Applying Lemma 11.1 with $M = 2^m$ and $K = R$, we have

$$\begin{aligned} A_p(1, B_R^{2^m}, 1) &\leq C_{\varepsilon,\nu} R^\varepsilon A_p(2, B_R^{2^m}, 2)^{1-\alpha} D_p(1, B_R^{2^m})^\alpha \\ &= C_{\varepsilon,\nu} R^\varepsilon A_p(1, B_{R^2}^{2^{m-1}}, 1)^{1-\alpha} D_p(1, B_R^{2^m})^\alpha. \end{aligned}$$

Applying Lemma 11.1 again with $M = 2^{m-1}$ and $K = R^2$, we have

$$\begin{aligned} A_p(1, B_{R^2}^{2^{m-1}}, 1) &\leq C_{\varepsilon,\nu} R^{2\varepsilon} A_p(2, B_{R^2}^{2^{m-1}}, 2)^{1-\alpha} D_p(1, B_{R^2}^{2^{m-1}})^\alpha \\ &= C_{\varepsilon,\nu} R^{2\varepsilon} A_p(1, B_{R^4}^{2^{m-2}}, 1)^{1-\alpha} D_p(1, B_{R^2}^{2^{m-1}})^\alpha. \end{aligned}$$

We can perform the above process $(m-1)$ times until we get

$$\begin{aligned} A_p(1, B^4_{R^{2^{m-2}}}, 1) &\leq C_{\varepsilon,\nu} R^{2^{m-2}\varepsilon} A_p(2, B^4_{R^{2^{m-2}}}, 2)^{1-\alpha} D_p(1, B^4_{R^{2^{m-2}}})^\alpha \\ &= C_{\varepsilon,\nu} A_p(1, B^2_{2^{m-1}}, 1) D_p(1, B^4_{R^{2^{m-2}}})^\alpha. \end{aligned}$$

Combining the above inequalities gives

$$\begin{aligned} A_p(1, B_R^{2^m}, 1) \leq &\left[ C_{\varepsilon,\nu}^{1+(1-\alpha)+\cdots+(1-\alpha)^{m-2}} (R^\varepsilon)^{1+2(1-\alpha)+\cdots+[2(1-\alpha)]^{m-2}} \right] \\ &\cdot \left[ A_p(1, B^2_{R^{2^{m-1}}}, 1)^{(1-\alpha)^{m-1}} \prod_{l=0}^{m-2} D_p(1, B^{2^{m-l}}_{R^{2^l}})^{\alpha(1-\alpha)^l} \right]. \end{aligned}$$

Since we have $p > p_0$, that is, $\alpha < 1$, we have

$$1 + (1-\alpha) + \cdots + (1-\alpha)^{m-2} \leq \alpha^{-1}.$$

Similarly, we have

$$1 + 2(1-\alpha) + \cdots + [2(1-\alpha)]^{m-2} \leq \max\left\{ \frac{(2-2\alpha)^m}{1-2\alpha}, m, \frac{1}{2\alpha-1}, \right\} := \beta_m. \qquad (11.3)$$

By a slight abuse of notation, this finishes the proof of the proposition. □

**Remark:** The first bound works for the case $\alpha < 1/2$, the second bound works for the case $\alpha = 1/2$ and the third bound works for the case $\alpha > 1/2$. Since we assume in the assumption that $p < p_1$, we have

$$\alpha = \frac{np - p - 2n}{(n-1)(p-2)} = 1 - \frac{2}{(n-1)(p-2)} < \frac{1}{2}.$$

Hence we use the first bound $(2-2\alpha)^m(1-2\alpha)^{-1}$. However, the exact value of $\beta_m$ is not important, since it will contribute to a power that finally goes to 0 as $m \to \infty$.

## 11.2 Applying a Bernstein-type inequality

The analogue to Proposition 10.6 is given by

**Proposition 11.3.** *Let $0 < \nu \le 1$ and $Q_1, \dots, Q_n$ be an $n$-tuple of $\nu$-transverse cubes with the same side length $\mu$. If $p_0 < p < p_1$, then for any $m \ge 1$ and any $R \ge \mu^{-2^m}$, we have*

$$\left( \sum_{B' \in \mathcal{P}_{R^{2-m}}(B_R)} \left( \prod_{i=1}^n \|E_{Q_i} g\|_{L^p(B')}^p \right)^{\frac{1}{n}} \right)^{\frac{1}{p}}$$
$$\lesssim_{\varepsilon,\nu,p} R^{\beta_m \varepsilon + \frac{n}{2} - \frac{n}{p}} P(m) \left( \prod_{i=1}^n \sum_{q_i \in \mathcal{P}_{R^{-1/2}}(Q_i)} \|E_{q_i} g\|_{L^p(w_{B_R,E})}^2 \right)^{\frac{1}{2n}}.$$

*where*

$$P(m) := \prod_{l=0}^{m-2} \mathrm{Dec}_n(R^{-1+2^{l+1-m}}, p)^{\alpha(1-\alpha)^l}. \tag{11.4}$$

*Proof.* Applying Propositions 10.3 and 11.2 with $2 < p_0 < p < p_1$, $m \ge 1$ and $K = R^{2^{m-1}} \ge \mu^{-1}$, we have

$$A_p(1, B_R^{2^m}, 1) \lesssim_{\varepsilon,\nu,p} R^{\beta_m \varepsilon} D_p(1, B^2_{R^{2^{m-1}}})^{(1-\alpha)^{m-1}} \prod_{l=0}^{m-2} D_p(1, B^{2^{m-l}}_{R^{2^l}})^{\alpha(1-\alpha)^l}. \tag{11.5}$$

The argument for the lower bound for $A_p(1, B_R^{2^m}, 1)$ is the same as in Proposition 10.6:

$$A_p(1, B_R^{2^m}, 1) \gtrsim R^{\frac{n}{p} - \frac{n}{2} - \frac{2^m n}{p}} \left( \sum_{B_R \in \mathcal{B}^1(B_R^{2^m})} \left( \prod_{i=1}^n \|E_{Q_i} g\|_{L^p(w_{B_R})}^p \right)^{\frac{1}{n}} \right)^{\frac{1}{p}}.$$

For the upper bound of the right hand side of (11.5), fix $0 \le l \le m-2$ and $Q \in \mathcal{P}_{R^{-2^l}}(Q_i)$. We may use parabolic rescaling (Proposition 7.1) with $\delta^{1/2} = R^{-2^{m-1}}$, $l(Q) = \sigma^{1/2} = R^{-2^l}$ (so $\delta \le \sigma$) and $B = B_{R^{2^m}}$ with $l(B) = R^{2^m} = \delta^{-1}$ to bound:

$$\|E_Q g\|_{L^p(w_B)} \lesssim \mathrm{Dec}_n(R^{-2^m + 2^{l+1}}, p) \left( \sum_{q \in \mathcal{P}_{R^{-2^{m-1}}}(Q)} \|E_q g\|_{L^p(w_B)}^2 \right)^{\frac{1}{2}}.$$

This shows that for all $0 \le l \le m-2$,

$$D_p(1, B^{2^{m-l}}_{R^{2^l}}) = \prod_{i=1}^n \left( \sum_{Q \in \mathcal{P}_{R^{-2^l}}(Q_i)} \|E_q g\|_{L^p_\#(w_B)}^2 \right)^{\frac{1}{2n}}$$
$$\lesssim \mathrm{Dec}_n(R^{-2^m + 2^{l+1}}, p) \prod_{i=1}^n \left( \sum_{Q \in \mathcal{P}_{R^{-2^l}}(Q_i)} \sum_{q \in \mathcal{P}_{R^{-2^{m-1}}}(Q)} \|E_q g\|_{L^p_\#(w_B)}^2 \right)^{\frac{1}{2n}}$$
$$= \mathrm{Dec}_n(R^{-2^m + 2^{l+1}}, p) D_p(1, B^2_{R^{2^{m-1}}}).$$

Hence the right hand side of (11.5) is bounded above by

$$R^{\beta_m\varepsilon}\left(\prod_{l=0}^{m-2}\mathrm{Dec}_n(R^{-2^m+2^{l+1}},p)^{\alpha(1-\alpha)^l}\right)D_p(1,B^2_{R^{2^{m-1}}})^{(1-\alpha)^{m-1}+\alpha+\alpha(1-\alpha)+\cdots+\alpha(1-\alpha)^{m-2}}$$
$$=R^{\beta_m\varepsilon}\left(\prod_{l=0}^{m-2}\mathrm{Dec}_n(R^{-2^m+2^{l+1}},p)^{\alpha(1-\alpha)^l}\right)D_p(1,B^2_{R^{2^{m-1}}}).$$

Therefore similar to the proof of Propsition 10.6 we have

$$\left(\sum_{B_R\in\mathcal{B}^1(B_R^{2^m})}\left(\prod_{i=1}^n\|E_{Q_i}g\|^p_{L^p(w_{B_R})}\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}$$
$$\lesssim_{\varepsilon,\nu,p} R^{\beta_m\varepsilon+\frac{n}{2}-\frac{n}{p}}\prod_{l=0}^{m-2}\mathrm{Dec}_n(R^{-2^m+2^{l+1}},p)^{\alpha(1-\alpha)^l}\left(\prod_{i=1}^n\sum_{Q_{i,1}\in\mathcal{P}_{R^{-2^{m-1}}}(Q_i)}\|E_{Q_{i,1}}g\|^2_{L^p(w_{B_{R^{2^m}}})}\right)^{\frac{1}{2n}}.$$

Changing $R^{2^m}\mapsto R$, we have

$$\left(\sum_{B'\in\mathcal{P}_{R^{2^{-m}}}(B_R)}\left(\prod_{i=1}^n\|E_{Q_i}g\|^p_{L^p(w_{B'})}\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}$$
$$\lesssim_{\varepsilon,\nu,p} R^{\frac{\beta_m\varepsilon+n/2-n/p}{2^m}}\prod_{l=0}^{m-2}\mathrm{Dec}_n(R^{-1+2^{l+1-m}},p)^{\alpha(1-\alpha)^l}\left(\prod_{i=1}^n\sum_{q_i\in\mathcal{P}_{R^{-1/2}}(Q_i)}\|E_{q_i}g\|^2_{L^p(w_{B_R})}\right)^{\frac{1}{2n}}.$$

□

### 11.3 Estimating the decoupling constants

The key difference of this argument from the case $2\le p\le p_0$ is that we do not bound the multilinear decoupling constant directly by $\delta^{-\varepsilon}$. Instead, we bound the multilinear decoupling constant by a product of decoupling constants, and then use Theorem 8.9 to get the bound in a reverse direction. Combining the two directions gives the result.

Using the same proof, we get an analogue to Proposition 10.7:

**Proposition 11.4.** *Let $0<\nu\le 1$. If $p_0<p<p_1$, then for all $m\ge 1$ and $0<\delta\le 1$,*

$$\mathrm{MDec}_n(\delta,p,\nu,m,E)\lesssim_{\varepsilon,\nu,p}\delta^{-\frac{\beta_m\varepsilon+n/2-n/p}{2^m}}P(m),$$

*where $P(m)$ was defined in (11.4).*

Thus we are left with estimating $P(m)$. Recall that

$$P(m)=\prod_{l=0}^{m-2}\mathrm{Dec}_n(R^{-1+2^{l+1-m}},p)^{\alpha(1-\alpha)^l}.$$

**Proposition 11.5.** *Let $0 < \nu \le 1$. If $p_0 < p < p_1$, then for any $\varepsilon > 0$, there is $0 < \delta_0 = \delta_0(\varepsilon) \le 1$ such that for all $m \ge 1$ and $0 < \delta < \delta_0$,*

$$\mathrm{MDec}_n(\delta, p, \nu, m, E) \lesssim_{\varepsilon,\nu,p} \delta^{-\frac{\beta_m\varepsilon + n/2 - n/p}{2^m} - \left(1 + \frac{2^{1-m}\alpha}{1-2\alpha} - \frac{(1-\alpha)^m}{1-2\alpha}\right)(\tau+\varepsilon)}.$$

*Proof.* Write $\mathrm{Dec}(\delta) = \mathrm{Dec}_n(\delta, p, E)$. Our first claim is the following:

**Lemma 11.6.** *There is a constant $\tau = \tau_{p,n,E} \in [0, \infty)$ such that*

$$\tau = \sup\left\{s \in \mathbb{R} : \limsup_{\delta\to 0} \mathrm{Dec}(\delta)\delta^s = \infty\right\} = \inf\left\{s \in \mathbb{R} : \lim_{\delta\to 0} \mathrm{Dec}(\delta)\delta^s = 0\right\}.$$

*Proof of lemma.* The existence of a $\tau \in [-\infty, \infty]$ is a general property satisfied by all nonnegative sequences. To show $\tau \in [0, \infty)$, we recall that by trivial decoupling (triangle inequality and Cauchy-Schwarz), $1 \le \mathrm{Dec}(\delta) \lesssim \delta^{-(n-1)/4}$. Thus

$$\sup\left\{s \in \mathbb{R} : \limsup_{\delta\to 0} \mathrm{Dec}(\delta)\delta^s = \infty\right\} \le \frac{n-1}{4}$$

and

$$\inf\left\{s \in \mathbb{R} : \lim_{\delta\to 0} \mathrm{Dec}(\delta)\delta^s = 0\right\} \ge 0.$$

□

Now let $\varepsilon > 0$. Then for $\delta$ small enough, we have $\mathrm{Dec}(\delta)\delta^{\tau+\varepsilon} \le 1$. Using $\alpha \ne 1/2$, we have

$$\begin{aligned}\log_{\delta^{-1}} P(m) &\le \sum_{l=0}^{m-2} \left(1 - 2^{l+1-m}\right)(\tau+\varepsilon)\alpha(1-\alpha)^l \\ &= \sum_{l=0}^{m-2}(\tau+\varepsilon)\alpha(1-\alpha)^l - \sum_{l=0}^{m-2}(\tau+\varepsilon)\alpha 2^{1-m}(2-2\alpha)^l \\ &= \left(1 + \frac{2^{1-m}\alpha}{1-2\alpha} - \frac{(1-\alpha)^m}{1-2\alpha}\right)(\tau+\varepsilon).\end{aligned}$$

Combining with Proposition 11.4, we are done. □

## 11.4 Proof of decoupling inequality

**Theorem 11.7.** *Let $n \ge 2$ and $E \ge 100n$. If $p_0 < p < p_1$, then we have for all $R \ge 1$,*

$$\mathrm{Dec}_n(R^{-1}, p, E) \lesssim_{\varepsilon,n,p,E} R^{\varepsilon}.$$

*Proof.* We give a detailed proof of the case $n = 2$ only. The proof in higher dimensions follows from induction as in the proof of Theorem 10.8.

Recall (8.16) of Theorem 8.9, which says that for each $0 < \nu \le 1$ and $m \ge 1$, there is $C_{\nu,m} = C_{\nu,p,m,E}$ and $\eta(\nu) = \eta(\nu, p, E)$ such that $\lim_{\nu\to 0^+} \eta(\nu) = 0$ and for each $R \ge \nu^{-2^m}$, we have

$$\mathrm{Dec}_2(R^{-1}, p, E) \le C_{\nu,m,p} R^{\eta(\nu)} \sup_{1\le R'\le R} \mathrm{MDec}_2(R'^{-1}, p, \nu, m, E).$$

Since $\eta(\nu) \to 0$, take $0 < \nu \le 1$ small enough such that $\eta(\nu) < \varepsilon$. Then by Proposition 11.5, we have for $R$ large enough,

$$\mathrm{Dec}_2(R^{-1}, p, E) \lesssim_{\varepsilon,m,p} R^{\beta_m \varepsilon + \frac{n/2 - n/p}{2^m} + \left(1 + \frac{2^{1-m}\alpha}{1-2\alpha} - \frac{(1-\alpha)^m}{1-2\alpha}\right)(\tau+\varepsilon)}.$$

Using the first equality of Lemma 11.6, we can find a sequence $R_k \to \infty$ such that

$$\mathrm{Dec}_2(R_k^{-1}, p, E) \ge R_k^{\tau-\varepsilon}.$$

Examining the power of $R$, we are forced to have

$$\tau - \varepsilon \le \beta_m \varepsilon + \frac{n/2 - n/p}{2^m} + \left(1 + \frac{2^{1-m}\alpha}{1-2\alpha} - \frac{(1-\alpha)^m}{1-2\alpha}\right)(\tau + \varepsilon).$$

But this holds for all $m \ge 1$ and all $\varepsilon > 0$. Letting $\varepsilon \to 0$ and multiplying both sides by $2^m$, we have

$$\tau \left( \frac{(2-2\alpha)^m}{1-2\alpha} - \frac{2\alpha}{1-2\alpha} \right) \le \frac{n}{2} - \frac{n}{p}, \tag{11.6}$$

for all $m \ge 1$. Since $\alpha < 1/2$ and $\tau \ge 0$, this forces to $\tau = 0$.

Hence by definition of $\tau$, for any $\varepsilon > 0$,

$$\lim_{\delta \to 0} \mathrm{Dec}_n(R^{-1}, p, E)\delta^{\varepsilon} = 0.$$

Finally, using similar argument as in the proof of Theorem 10.8, the decoupling inequality for $p_0 < p < p_1$ holds for all $R \ge 1$. □

## 11.5 The endpoint case

We prove

**Theorem 11.8.** *Let $n \ge 2$ and $E \ge 100n$. If $p = p_1$, then we have for all $R \ge 1$,*

$$\mathrm{Dec}_n(R^{-1}, p_1, E) \lesssim_{\varepsilon,n,p,E} R^{\varepsilon}.$$

*Proof.* As before, it suffices to show

$$\|Eg\|_{L^{p_1}(B_R)} \lesssim_{\varepsilon} R^{\varepsilon} \left( \sum_{Q \in \mathcal{P}_{R^{-1/2}}([0,1]^{n-1})} \|E_Q g\|^2_{L^{p_1}(w_{B_R,E})} \right)^{\frac{1}{2}}.$$

Let $\varepsilon > 0$. Take $p < p_1$ such that

$$\frac{n}{p} - \frac{n}{p_1} < \frac{\varepsilon}{4}.$$

By (4.3) with $Q = [0,1]^{n-1}$, $q = p_1$ and $p = p$, we have

$$\|Eg\|_{L^{p_1}_{\#}(B_R)} \lesssim R^{\frac{n}{p} - \frac{n}{p_1}} \|Eg\|_{L^{p}_{\#}(w_{B_R,E})} \le R^{\frac{\varepsilon}{4}} \|Eg\|_{L^{p}_{\#}(w_{B_R,E})}. \tag{11.7}$$

Applying the decoupling inequality to $\|Eg\|_{L^p_\#(w_{B,E})}$, we have, for some $C_\varepsilon = C_{\varepsilon,p}$

$$\begin{aligned}\|Eg\|_{L^p_\#(B_R)} &\leq C_\varepsilon R^{\frac{\varepsilon}{2}} \left( \sum_{Q\in\mathcal{P}_{R^{-1/2}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p_\#(w_{B_R},E)} \right)^{\frac{1}{2}} \\ &\leq C_\varepsilon R^{\frac{\varepsilon}{2}} \left( \sum_{Q\in\mathcal{P}_{R^{-1/2}}([0,1]^{n-1})} \|E_Q g\|^2_{L^{p_1}_\#(w_{B_R},E)} \right)^{\frac{1}{2}},\end{aligned}$$

by Jensen's inequality. Thus

$$\begin{aligned}\|Eg\|_{L^p(B_R)} &\leq C_\varepsilon R^{\frac{\varepsilon}{2}+\frac{n}{p}-\frac{n}{p_1}} \left( \sum_{Q\in\mathcal{P}_{R^{-1/2}}([0,1]^{n-1})} \|E_Q g\|^2_{L^{p_1}_\#(w_{B_R},E)} \right)^{\frac{1}{2}} \\ &\leq C_\varepsilon R^{\frac{3\varepsilon}{4}} \left( \sum_{Q\in\mathcal{P}_{R^{-1/2}}([0,1]^{n-1})} \|E_Q g\|^2_{L^{p_1}_\#(w_{B_R},E)} \right)^{\frac{1}{2}}.\end{aligned}$$

Combining with (11.7), we have

$$\|Eg\|_{L^{p_1}_\#(B_R)} \lesssim R^\varepsilon \left( \sum_{Q\in\mathcal{P}_{R^{-1/2}}([0,1]^{n-1})} \|E_Q g\|^2_{L^{p_1}_\#(w_{B_R},E)} \right)^{\frac{1}{2}}.$$

□

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