THE RIEMANN-HILBERT CORRESPONDENCE AND CLASSICAL ANALYSIS

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ABSTRACT. The Hilbert twenty-first problem asked to find the regular singular differential equations for a given monodromy matrix. Deligne solved the twenty-first problem in any given complex manifold. To We interpret his solution as a prototype of Riemann-Hilbert correspondence. Today,the Riemann-Hilbert correspondence is referred to as categorical equivalence between holonomic De Rham functor(differential equations) and holonomic solution functor(solutions).

CONTENTS

1. INTEGRABLE CONNECTIONS

Let's X be an n dimensional complex manifold and consider the ring $\mathcal{D}_X$ of a partial differential operators with $\mathcal{O}_X$ coefficient.

$$\mathcal{D}_X(U) = \{P = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha(x)\partial_x^\alpha\} | a_\alpha(x) \text{is holomorphic } \}$$

the product for any $P, Q \in \mathcal{D}_X(U)$, is defined by $PQ \in \mathcal{D}_X(U)$ as $(PQ)f = P(Qf)$ for $f \in \mathcal{O}_X(U)$. The order of partial differential operators are defined by

$$ordP := \{|\alpha| | a_\alpha(x) \neq 0\}$$

.
This formulation depend on the coordinate system. We can also construct $\mathcal{D}_X$ without of relying on such a coordinate by considering sheaf of holomorphic vector field Θ.

$$\Theta_X := \{\theta \in \mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X) | \theta(fg) = (\theta(f))g + g\theta(f)\}$$

We can regard $\mathcal{D}_X$ as

$$\mathcal{D}_X := \{\text{Subring of } \mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X) \text{generated by } \mathcal{O}_X \text{and } \Theta_X\}$$

In this thesis,we will reformulate some of the classical theorems from the classical differential equation theory into the language of sheaf on the manifold via $\mathcal{D}_X$-module. Usually $\mathcal{D}_X$-module means left $\mathcal{D}_X$-module if we don't particular say anything. The most important $\mathcal{D}_{\mathcal{X}}$ module for the differential equations are for which we can locally write as $\mathcal{M} = \mathcal{D}_X^{N_0}/\mathcal{D}_X^{N_1}P$ by using locally $\mathcal{D}_X$ coefficient matrix $P = (P_{ij}) \in M(N_1, N_2, \mathcal{D}_X)$. In other words, when there is an exact sequence

$$\mathcal{D}_X^{N_1} \xrightarrow{\times P} \mathcal{D}_X^{N_0} \longrightarrow \mathcal{M} \longrightarrow 0$$

Such a $\mathcal{M}$ are called a coherent $\mathcal{D}_X$ module. The left exact functor $\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(*, \mathcal{O}_X)$ will made the exact sequence to

$$0 \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X^{N_0} \xrightarrow{\times P} \mathcal{O}_X^{N_1}$$

Here we used the isomorphism $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{O}_X) \cong \mathcal{O}_X$ by $\phi \to \phi(1)$.
By observing the exact sequence, we will get the isomorphism $\mathcal{H}om(\mathcal{M}, \mathcal{O}_X) \cong \{u(x) \in \mathcal{O}_X^{N_0} | Pu(x) = 0\} \subset \mathcal{O}_X^{N_0}$. Therefore $\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is nothing but sheaf of solutions $Pu = 0$. Sato began to study a PDE by utilizing local free resolution for coherent ring $\mathcal{D}_X$.

Example 1.1. $\mathcal{D}_X$ is coherent ring in particular Noetherian.

This example allows us to decompose $\mathcal{D}_X$-module $\mathcal{M}$ by locally free module so that

$$\dots \longrightarrow \mathcal{D}_X^{N_k} \xrightarrow[\times P_k]{} \dots \xrightarrow[\times P_2]{} \mathcal{D}_X^{N_1} \xrightarrow[\times P_1]{} \mathcal{D}_X^{N_0} \xrightarrow[\times P_1]{} \mathcal{M} \longrightarrow 0$$

Defining a complex of the sheaf over $\mathbb{C}_X$-module

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$$

$$= [\ 0 \longrightarrow \mathcal{O}_X^{N_0} \xrightarrow[P_1\times]{} \mathcal{O}_X^{N_1} \xrightarrow[P_2\times]{} \mathcal{O}_X^{N_2} \xrightarrow[P_3\times]{} \dots\ ]$$

In particular the isomorphism $H^0\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ exist.
Next lemmas are immediate because $\mathcal{D}_X$ is generated by $\mathcal{O}_X$ and Θ

Lemma 1.2. *Let $\mathcal{M}$ be an $\mathcal{O}_X$ module, Giving left $\mathcal{D}_X$-module to $\mathcal{M}$ is equivalent to giving $\mathbb{C}_X$linear homomorphism*

$$\nabla : \Theta \to \mathcal{E}nd_X(\mathcal{M}) \quad (\theta \to \nabla_\theta)$$

satisfies following conditions
(1)$\nabla_{f\theta}(m) = f\nabla_\theta(m)(f \in \mathcal{O}_X, \theta \in \Theta_X, m \in \mathcal{M})$
(2)$\nabla_\theta(fm) = \theta(f)m + f\nabla_\theta$
(3)$\theta_{[\theta_1,\theta_2]}(m) = [\nabla_{\theta_1}, \nabla_{\theta_2}(m)]$

Example 1.3. Let's Ω_X^j be a sheaf of holomorphic j-form then $(\mathcal{M}, \nabla)$ induces homomorphism

$$\nabla : \mathcal{M} \to \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{M} \quad (m \to \sum_{i=1}^{m} dx_i \otimes \nabla_{\partial_i}(m))$$

can extend the j-form map

$$\nabla_i : \Omega_X^j \otimes \mathcal{M} \to \Omega_X^{j+1} \otimes \mathcal{M} \quad (\omega \otimes m \to d\omega \otimes m + (-1)^j \omega \wedge \nabla m)$$

We can easily compute this as a complex, so we can get the De Rham complex

$$0 \longrightarrow \mathcal{M} \xrightarrow{\nabla} \Omega^1_X \otimes \mathcal{M} \xrightarrow{\nabla_1} \Omega^2_X \otimes \mathcal{M}_X \xrightarrow{\nabla_2} \dots$$

Example 1.4. When $\mathcal{M}$ is a rank N left free $\mathcal{D}_X$-module , we can describe $\nabla_{\partial_i} : \mathcal{M} \to \mathcal{M}$ with the local coordinate $(x_1 \dots x_n)$. The local isomorphism $m \in \mathcal{M} \cong \mathcal{O}_X^N$ can be described by using regular matrix A_i

$$\nabla_{\partial_i}(m) = \nabla_{\partial_i}(m) + A_i m$$

The condition $[\partial_i, \partial_j] = 0$ gives $[\nabla_{\partial_i}, \nabla_{\partial_j}] = 0$ from lemma 1.2, so we have $0 = [\partial_i + A_i, \partial_j + A_j]m = (\partial_i A_j - \partial_j A_i + [A_i, A_j])m$ thus we have

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} = [A_i, A_j] \tag{1.5}$$

for an arbitrary i, j (1.5) is called integrability condition and ∇ is called integrable connection.

Definition 1.6. Locally free finite rank left $\mathcal{D}_X$ module $\mathcal{M}$ is called integrable connection.

Example 1.7. Horizontal section of integrable connection $\mathcal{M}^\nabla$ is a subsheaf of $\mathcal{M}$ such that vanishes under the action of ∇ i.e $\mathcal{M}^\nabla = \{m \in \mathcal{M} | \nabla m = 0\}$

Definition 1.8. Local system $\mathcal{L}$ is a locally constant sheaf of finite-dimensional complex vector space.

Roughly speaking, Riemann-Hilbert correspondence states that an abelian category of integrable connection and the local system of the complex manifold are equivalent. Now we will see the most primitive version of it [Szamuely].
As local systems are vector space, their homomorphism define the abelian category Loc(X). The category of integrable connection is categorically equivalent to abelian category of local system. To prove that fact, we use the Cauchy-Kowalevski theorem from PDE theory.

Lemma 1.9. *Let $X = \mathbb{C}^n$ and $Y = \{x_n = 0\} = \mathbb{C}^n \subset X$ be a complex hypersurface. Consider a regular matrix $A(x) = (A_{ij})_{1 \le i,j \le N}$ such that each component of matrix is holomorphic around the origin of X. Then for an arbitrary vector valued holomorphic functions around the origin $\vec{a}(x_1 \dots x_{n-1}) \in \mathcal{O}_{Y,0}^N$, there is a solution for an initial value problem*

$$(\partial_n + A(x))\vec{u}(x_1 \dots x_n) = 0$$

$$\vec{u}(x_1 \dots x_{n-1}, 0) = \vec{a}(x_1 \dots x_n)$$

Theorem 1.10. *Suppose X is connected, with a rank N integrable connection $(\mathcal{M}, \nabla)$, horizontal connection $\mathcal{M}^\nabla$ is a local system with rank N. We have an isomorphism of $\mathcal{D}_X$-module*

$$\mathcal{O}_X \otimes \mathcal{M}^\nabla \cong \mathcal{M}$$

There is a categorical equivalence of abelian category.

$$\Phi : \mathcal{M} \in Conn(X) \to Loc(X) \ni \mathcal{M}^\nabla$$

Proof. This is local problem so it is sufficient to consider neighborhood around the origin and think X as $X = \mathbb{C}^n$. Under the local isomorphism $\mathcal{M} \cong \mathcal{O}_X^N$ and by example 1.4,there exist connection matrix $A_i(x) \in M_N(\mathcal{O}_{X,0})$ and $\nabla_{\partial_i}(m) = \partial_i m + A_i m$. and $[\partial_i + A_i, \partial_j + A_j] = 0$ from integrable condition. By applying lemma 1.9 iterative, we can show that isomorphism of the stalk by $\mathcal{M}_0^\nabla \cong \mathbb{C}^N \quad (m(x) \to m(0))$. This means that $\mathcal{M}^\nabla$ is a local system, and same rank with $\mathcal{M}$.
$U \to \mathcal{M}^\nabla(U) \otimes \mathcal{O}_X(U)$ defines locally free sheaf $\mathcal{M}$ on X. We define connection map $\nabla_{\mathcal{M}^\nabla}$ as follows. Given an open subset U with $\mathcal{M}^\nabla \cong \mathbb{C}^N$ and fix $\mathbb{C}$ basis $s_1 \dots s_N$, so that each section of $\mathcal{M}$ can be written as $\sum s_i \otimes f_i$ with $f_i \in \mathcal{O}_X$. We can define $\nabla_{\mathcal{M}^\nabla}$ as $\nabla_{\mathcal{M}^\nabla}(\sum s_i \otimes f_i) = \sum s_i df_i$. As two choice of s_i just depend on $\mathbb{C}$ coefficient matrix the difference is annihilate by the action of connection. $\nabla_{\mathcal{M}^\nabla}$ does not depend on choice of s_i.$\nabla_{\mathcal{M}^\nabla}$ is defined over all the patch of X, so this is global. □

We will generalize this correspondence into a context of holonomic systems higher dimensional cases. Although we won't see on here, this concept is now developed more into a non-holonomic system. This generalization was made by Mochizuki, Kedlaya, Kashiwara, and D'Agnolo.
Also there is another important ingredients to discuss Riemann-Hilbert correspondence, which is called a regularity. We will see discussion of regularity in the next section.
Before concluding this section, see a technique of invertible sheaf $\mathcal{M}$ that can be applied for a $\mathcal{D}_X$-module.

Definition 1.11. Invertible sheaf $\mathcal{M}$ is locally free $\mathcal{O}_X$ module of rank 1.

Example 1.12. Canonical sheaf Ω_X, and it's dual $\Omega_X^{\otimes -1} := \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ are examples of invertible $\mathcal{O}_X$ sheaf. Functors

$$\mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_X^{op}) \quad (\mathcal{M} \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M})$$

$$\mathrm{Mod}(\mathcal{D}_X^{op}) \to \mathrm{Mod}(\mathcal{D}_X) \quad (\mathcal{M} \to \Omega_X^{\otimes -1} \otimes_{\mathcal{O}_X} \mathcal{M})$$

gives categorical equivalence each other. This shows we can interchange right $\mathcal{D}_X$-module and left $\mathcal{D}_X$-module in appropriate moment.

2. The Hilbert's Twenty-First Problem

We how mathematician tried to solve Hilbert's twenty-first problem and where it lead to.
Let X be a compact Riemann surface and consider the differential equations on open subset $U \subset X$, such that X/U are finite sets $z_1 \dots z_n$

$$\{(\frac{d}{dz})^n + a_1(z)(\frac{d}{dz})^{n-1} + \dots a_n(z)\}f(z) = 0 \tag{2.1}$$

$a_1(z) \dots a_n(z) \in \mathscr{O}_X(U)$ and possible poles on $z_1, \dots, z_n$.

Definition 2.2. Let $D = x\frac{d}{dx}$, if (2.1) can be rewrite as

$$\{D^n + b_1(z)D^{n-1} + \dots b_1(z)\}f = 0 \tag{2.3}$$

and $b_1 \dots b_n$ are holomorphic on z_i. Then we will call (2.3) has a regular singularity at z_i

Let's consider the solution of (2.3) around regular singularity z_i. By letting polar coordinate $z = exp(2\pi it)$ around punctured disk around z_i. Then solutions of (2.3) on punctured disk $u_1(t) \dots u_n(t)$ lift to solutions of the universal covering space of punctured disk $\tilde{u}_1(t) \dots \tilde{u}_n(t)$. There is a matrix $M \in GL(n, \mathbb{C})$ to satisfy $\tilde{u}(t+1) = M\tilde{u}(t)$. This matrix M is usually called the monodromy. Another way to think monodromy representation is that for rank N local system $\mathcal{L}$, we can think a representation of fundamental group $\pi_1(X, z_i)$ and we can define

$$(\mathcal{L}, z_i) : \pi_1(X, z_i) \to GL(N.\mathbb{C})$$

for a loop $[\gamma] \in \pi_1(X, z_i)$. What interesting property of regular singularity is, the Theorem (2.6) tells that equation is essentially determined by a monodromy.

Definition 2.4. The two systems

$$\frac{d}{dz}U = AU \quad \frac{d}{dz}V = BV \tag{2.5}$$

are equivalent when there is an matrix with meromorphic coefficient M such that

$$B = (\frac{d}{dz}M)M^{-1} + MAM^{-1}$$

Theorem 2.6. *Two differential equations with regular singularities are equivalent iff monodromy is conjugate.*

Hilbert's twenty-first problem asks whether any finite-dimensional complex representation of $\pi_1(U)$ can be obtained from monodromy representation of the differential equations with regular singularities. When $U = \mathbb{P} - \{0, 1, \infty\}$, this problem was solved affirmatively in classical Hypergeometric series theory involved by Gauss, Kummer, and Riemann. Hilbert's twenty-first problem is well post in the following sense. If we didn't impose singularities to a regularity, then there are too many answer for a given monodromy. On the other hand, in case of regular singularities, by the theorem 2.6, there is an essentially unique equation for a given monodromy if there is a solution.
Combine with the theorem 1.10 and next theorem shows category of the monodromy representations and category of connections are equivalent. This means that Hilbert's twenty-first problem is true for a connection.

Theorem 2.7. *Let X be a connected $n-$dimensional complex manifold, and x a point in X. The category of complex local systems on X is equivalent to the category of finite dimensional left representations of $\pi_1(X, x)$.*

Proof. General theory of representation tells that category of representation is equivalent to the category of group rings, so we will prove that local system is categorical equivalent to category of group rings.
$\mathcal{L}$ is $\mathbb{C}_X$-module with $\pi_1(X, x)$ action as a set with some representation . To show that it is an $\mathbb{C}[\pi_1(X, x)]$-module we have to show that action of $\pi_1(X, x)$ is compatible with $\mathbb{C}_X$-module structure.
For this let $\mathcal{L} \times \mathcal{L}$ be the direct product sheaf defined by $(\mathcal{L} \times \mathcal{L})(U) = \mathcal{L}(U) \times \mathcal{L}(U)$ over all open $U \subset X$; its stalk over a point x is just $\mathcal{L}_x \times \mathcal{L}_x$. The addition law on $\mathcal{L}$ is a morphism of sheaves $\mathcal{L} \times \mathcal{L}$ given over an open set U by the formula $(s_1, s_2) \to s_1 + s_2$; the morphism $\mathcal{L}_x \times \mathcal{L}_x \to \mathcal{L}_x$ induced on the stalk at x is the addition law on $\mathcal{L}_x$. But this latter map is a map of $\pi_1(X, x)$-sets, which means precisely that $\sigma(s_1 + s_2) = \sigma s_1 + \sigma s_2$ for all $s_1, s_2 \in \mathcal{L}_x$ and $\sigma \in \pi_1(X, x)$.

We can show the argument of multiplication of $\mathcal{C}$ in a similar way. For the other direction, take the universal covering space $p : \tilde{X} \to X$. The group $\pi_1(X, x)$ acts on $\tilde{X}$ as covering transformation. Given a representation $\rho : \pi_1(X, x) \to GL(n, \mathbb{C})$, we define its associated local system $\mathcal{L}$ via the introduction of the constant sheaf $\mathcal{L}_{\tilde{X}}$ on $\tilde{X}$; we put:

$$\Gamma(U, \mathcal{L}) := s \in \Gamma(p^{-1}(U), \mathcal{L}_{\tilde{X}}) : \forall u \in p^{-1}(U), \forall \in \pi_1(X, x), s(\sigma.u) = \rho(\sigma).s(u)$$

then we can check the functionality of this constructed local constant sheaf □

Deligne's was a one of the contributor for the Hilbert's twenty-first problem and generally, he is regarded the one solved this problem. His solution of Hilbert's twenty-first problem was extended the notion of regular singularities for an arbitrary dimension and showed categorical equivalence between category of regular connection is categorical equivalence to the category of integrable connection. More precisely, Delingne meant equations with a regular singularity at normal crossing divisor D and the equation is defined on X/D.

Definition 2.8. Let X be a non singular variety and D a divisor on X. We say that a divisor D is of normal crossings at a point $x \in X$ if there exists a regular parameter system $x_1 \dots x_n$ at x such that D is defined by $x_1 \cdots x_k = 0 \quad (k \leq n)$.

Let X be a complex manifold and $D \subset X$ a normal crossing divisor. We denote by $\mathscr{O}_X[D]$ the sheaf of meromorphic functions on X that are holomorphic on $Y := X/D$ and have poles along D

Definition 2.9. Assume that a coherent $O_X[D]$-module $\mathcal{M}$ is endowed with a $\mathbb{C}$-linear morphism

$$\nabla : \mathcal{M} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathcal{M}$$

satisfying the conditions

$$\nabla(fs) = df \otimes s + f\nabla s \quad (f \in \mathscr{O}_X[D], \quad s \in \mathcal{M})$$
$$[\nabla_\theta, \nabla'_\theta] = \nabla_{[\theta,\theta']} \quad (\theta, \theta' \in \Theta_X)$$

Then we call the pair $(\mathcal{M}, \nabla)$ a meromorphic connection along the divisor D
We denote by $Conn(X; D)$ the category of meromorphic connections along D.

In a Riemann surface we say that a meromorphic connections (M, ∇) at $x = 0$ is regular if there exists a finitely generated $\mathcal{O}_X$-submodule $\mathcal{N} \subset \mathcal{M}$ which is stable by the action of $D = x\nabla$ (i.e., $DL \subset L$) and generates $\mathcal{M}$ over $\mathscr{O}_X[x]$.
Regular connections for general algebraic variety is defined by a pull back of the unit ball $B = \{x \in \mathbb{C} | |x| < 1\}$. For a morphism $i : B \to X$ such that $i^{-1}D = \{0\}$ the stalk $(i^*\mathcal{M})_0$ at $0 \in B$ is a meromorphic connections of one-variable and if this is regular, then we will say $\mathcal{M}$ is regular
We denote by $Conn(X; D)^{reg}$ the category of meromorphic connections along D.

Theorem 2.10. *Deligne's Riemann-Hilbert Correspondence*
With above hypothesis,then the restriction functor $\mathcal{N} \to \mathcal{N}|Y$ *induces an equivalence*

$$Conn^{reg}(X; D) \to Conn(Y)$$

of categories

The main theme of this note is the vast generalization of Deligne's RiemannHilbert correspondence by Masaki Kashiawara. Kashiawara's Riemann-Hilbert correspondence was formulated in the derived category. Today, this is an essential tool in algebraic geometry, representation theory and number theory etc...

3. Holonomic $\mathcal{D}$-modules

To understand coherent $\mathcal{D}_X$-module $\mathcal{M}$, we use geometrical interpretation of $\mathcal{M}$. This geometric idea naturally lead to the concept of characteristic variety and holonomic $\mathcal{D}_X$-module.

Definition 3.1. For a coherent $\mathcal{D}_X$-module $\mathcal{M}$ if there is an increasing sequence $\{F_i\mathcal{M}\}$ of subsheaf $F_i\mathcal{M} \subset \mathcal{M}$ satisfies following properties
(1)$\mathcal{M} = \cup F_i\mathcal{M}$, $F_i\mathcal{M} = 0$ when $i \leq 0$
(2) $F_i\mathcal{M}$ is coherent $\mathcal{F}\mathcal{M}_0 := \mathcal{O}_X$ module
(3)$(F_i\mathcal{M}) \cdot (F_j\mathcal{M}) \subset F_{i+j}\mathcal{M}$
then we will call $\mathcal{F}$ as filtration of $\mathcal{M}$ over $\mathcal{D}_X$ and the pair $(\mathcal{M}, F)$ filtered $\mathcal{D}_X$ module.

Example 3.2. $\mathcal{D}_X$ naturally equip filtration $\mathcal{F}$ from the order of partial differential operators.

Now we prove that coherent $\mathcal{D}_X$-module$\mathcal{M}$ always has a filtration. $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module so there exist a sheaf $\mathcal{K}$ that satisfies an exact sequence

$$0 \to \mathcal{K} \to \mathcal{D}_X^{\otimes m} \xrightarrow{\Phi} \mathcal{M} \to 0$$

Take a filter for $\mathcal{D}_X$ from example 3.2. Next lemmas will show that $F_i\mathcal{M} := \Phi(F_i\mathcal{D}_X^{\otimes m})$ is indeed coherent $\mathcal{O}_X$ module so that it is a filtration.

Lemma 3.3. *$F_i\mathcal{K} := \mathcal{K} \cap (F_i\mathcal{D}_X)^{\oplus m}$ is a coherent $F_0\mathcal{D}_X := \mathcal{O}_X$ sheaf*

Proof. (Sketch of the proof)
Each $(F_i\mathcal{D}_X)^{\oplus m}$ are coherent $F_0\mathcal{D}_X := \mathcal{O}_X$-module. Since $\mathcal{D}_X$ is a union of filtration, $\mathcal{K}$ is generated by a some coherent $F_0\mathcal{D}_X$ submodule $\mathcal{K}_0 \subset \mathcal{K} \subset \mathcal{D}_X^m$: $\mathcal{K} = \mathcal{D}_X \cdot \mathcal{K}_0$. Thus if we put $\mathcal{K}_j = (F_j\mathcal{D}_X) \cdot \mathcal{K}_0 \subset \mathcal{K}$, $\mathcal{K} = \sum_{j\in\mathbb{Z}} \mathcal{K}_j$. Each $\mathcal{K}_j$ are coherent $\mathcal{O}_X$-module. $(F_i\mathcal{D})^m$are notherian as $\mathcal{O}_X$-module, so

$$\mathcal{K} \cap (F_i\mathcal{D}_X)^{\oplus m} = \sum \{\mathcal{K}_j \cap (F_i\mathcal{D}_X)^{\oplus m}\}$$

is coherent $\mathcal{O}_X$-module. □

Utilizing the lemma 3.3, we can prove the existence of a filtration.

Lemma 3.4. *Coherent $\mathcal{D}_X$-module $\mathcal{M}$ which is also finitely generated $\mathcal{O}_X$-module is coherent $\mathcal{O}_X$ module.*

Proof. There is an exact sequence

$$0 \to F_i\mathcal{K} \to (F_i\mathcal{D}_X)^{\oplus m} \to F_i\mathcal{M} \to 0$$

$F_i\mathcal{K}$ is coherent $F_0\mathcal{D}_X = \mathcal{O}_X$ sheaf and $F_i\mathcal{D}_X^{\oplus m}$ is also coherent $\mathcal{O}_X$-module. Thus $F_i\mathcal{M}$ is also coherent$\mathcal{O}_X$ module. $\mathcal{M} = \cup_{i\in\mathbb{Z}} F_i\mathcal{M}$ so $\mathcal{M}$ is also coherent $\mathcal{O}_X$-module □

To visualize information of coherent $\mathcal{M}$ module, we need to choose a specific filter.

Definition 3.5. A graded ring $gr^F\mathcal{M}$ is defined by $gr_i^F\mathcal{M} := F_i\mathcal{M}/F_{i-1}\mathcal{M}$ and $gr^F\mathcal{M} = \oplus gr_i^F\mathcal{M}$. If $gr^F\mathcal{M}$ is coherent over $gr^F\mathcal{D}_X-$module then filter for $\mathcal{M}$ is called good filtration.

By utilizing similar argument to the lemma 3.3 and 3.4 , we can show that coherent $\mathcal{D}_X$-module $\mathcal{M}$ always has a local good filter. Although in global case, Deligne provided a counterexample for the existence in the category of analytic $\mathcal{D}_X$-module. Now we can begin "geometrical interpretation" of coherent $\mathcal{D}_X$-module $\mathcal{M}$. Our setting is complex manifold X of dimension n with a local coordinate $x = (x_1 \ldots x_n)$. $\partial_i \in F_1\mathcal{D}_X$ define a section $[\partial_i] \in gr_1^F\mathcal{D}_X = F_1\mathcal{D}_X/F_0\mathcal{D}_X$ then there is a isomorphism of sheaf of ring.

$$\mathcal{O}_X[T_1 \ldots T_n] \cong gr^F\mathcal{D}_X = \oplus gr_i^F\mathcal{D}_X \quad (T_i \to [\partial_i])$$

This fact can be stated without using the local coordinates. Consider a canonical projection cotangent bundle, $\pi : T^*X \to X$ and consider sheaf of a holomorphic function $\mathcal{O}_{T^*X}$ over T^*X, and their direct image of the $\pi_*\mathcal{O}_{T^*X}$. For each section $[P] \in gr_i^F\mathcal{D}_X$, we can define an injection $gr^F\mathcal{D}_X \to \pi_*\mathcal{O}_{T^*X}$ by taking a principal symbol $\sigma(P)(x,\xi) \in \pi_*\mathcal{O}_{T^*X}$. The section of subsheaf $gr^F\mathcal{D}_X \subset \pi_*\mathcal{O}_{T^*X}$ is a polynomial of each fiber of $\pi : T^*X \to X$, $\pi^{-1}(x) = T_x^*X \cong \mathbb{C}^n$. This gives sequence of injective homomorphism

$$\pi^{-1}gr^F\mathcal{D}_X \to \pi^{-1}\pi_*\mathcal{O}_{T^*X} \to O_{T^*X}$$

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ module and $(\mathcal{M}, F)$ be a local good filter. Then $gr^F\mathcal{M}$ is coherent $gr^F\mathcal{D}_X$ module so there is a local free resolution

$$(gr^F\mathcal{D}_X)^{\oplus m_1} \to (gr\mathcal{D}_X)^{\oplus m_0} \to gr^F(\mathcal{M}) \to 0$$

Apply a right exact functor $\widetilde{*} := \mathcal{O}_{T^*X} \otimes_{\pi^{-1}(gr^F\mathcal{D}_X)} \pi^{-1}(*)$ so that $\mathcal{O}_{T^*X}$ module $\widetilde{gr^F\mathcal{M}} = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}gr^F\mathcal{M}} \pi^{-1}gr^F\mathcal{M}$ has a free resolution

$$(\mathcal{O}_{T^*M})^{m_1} \to (\mathcal{O}_{T^*M})^{m_0} \to \widetilde{gr^F\mathcal{M}} \to 0$$

so $gr^F\mathcal{M}$ is coherent $\mathcal{O}_{T^*X}$module. support of the $\widetilde{gr^F(\mathcal{M})}$ reflects analytic and algebraic property of the $\mathcal{M}$. We will call it a characteristic variety.

Definition 3.6. The characteristic variety of $\mathcal{M}$ is $ch\mathcal{M} := \mathrm{supp}(\widetilde{gr^F\mathcal{M}})$

The definition of a $ch(\mathcal{M})$ looks like involving filtration F, but in fact, one can prove that $ch(\mathcal{M})$ is not dependent on the way of filtration.

Proposition 3.7. *A supp(ch($\mathcal{M}$)) is not depend on the choice of good filter.*

Proof. See [Hotta-Takeuchi-Tanizaki] □

Let's see why this is called the characteristic variety.

Example 3.8. The coherent $\mathcal{D}_X$-module $M = \mathcal{D}_X/\mathcal{D}_XP$ defined by a partial differential operators $P \in \Gamma(X; \mathcal{D}_X)$ has a filter that induced from the filter from $F_i\mathcal{D}_X$

$$F_i\mathcal{M} = (F_i\mathcal{D}_X + \mathcal{D}_XP)/\mathcal{D}_XP \quad gr^F\mathcal{M} \cong gr^F\mathcal{D}_X/gr^F\mathcal{D}_X\sigma(P)$$

with the principal symbol $\sigma(P)$. In this case, characteristic variety can be describe as

$$ch\mathcal{M} = \{(x,\xi) \in T^*X | \sigma(P)(x,\xi) = 0\}$$

Namely characteristic variety of $\mathcal{M}$ is zero points of principal symbol $\sigma(P)$. This is the motivative example of characteristic variety and also compared with classical Cauchy-Kowalevski theorem, we can understand why this is called characteristic variety.

Example 3.9. However, when it comes to a coherent $\mathcal{D}_X$ module $\mathcal{M} := \mathcal{D}_X / \sum D_X P_j$ defined by several partial differential operators $P_1 \dots P_n$ things is getting more complicated. In order to find appropriate results, we must restrict which module can be appropriate. For an ideal $\mathcal{I} = \sum \mathcal{D}_X P_j \subset \mathcal{D}_X$ has a good filter

$$F_i\mathcal{M} = (F_i\mathcal{D}_X + \mathcal{I})/\mathcal{I} \subset \mathcal{M} \quad F_i\mathcal{I} = \mathcal{I} \cap F_i\mathcal{D}_X \subset \mathcal{I}$$

$$gr^F\mathcal{M} \cong gr^F\mathcal{D}_X/gr^F\mathcal{I}$$

Here, $gr^F\mathcal{I} \subset gr^F\mathcal{D}_X$ is an ideal generated by principal symbols. The characteristic variety of $\mathcal{M}$ is

$$ch\mathcal{M} = \{(x,\xi) \in T^*X | \sigma(Q)(x,\xi) = 0 \quad (Q \in \mathcal{I})\}$$

Although there is a problem. If we try to interpret characteristic variety as a zero sets of principal symbol on cotangent bundle, then generally speaking $\sum gr^F(\mathcal{D}_X)\sigma(P_i) \subset gr^F(\mathcal{I})$ is not equality. Means that $ch\mathcal{M}$ could be smaller than common zero points of $\sigma(P_i)$s. We want to focused on modules such that $gr^F(\mathcal{D}_X)\sigma(P_i) = gr^F\mathcal{I}$ holds. This problem has several ways to discuss. Our point is dimension.According to [P.Schapira], when dim$ch\mathcal{M}$ =dim$X$ then we have desired equality $gr^F I = \sum gr^F(\mathcal{D}_X)\sigma(P_i)$.

Dimension of characteristic variety is bounded below by the dimension of X i.e. dim $ch\mathcal{M}$ $\geq$dim$X$. This inequality is called Bernstein's inequality. If equality holds, then we will call the module as a holonomic $\mathcal{D}_X$-module.

Theorem 3.10. *If coherent $\mathcal{D}_X$-module $\mathcal{M}$is non-zero, then $2dimX \geq dimch\mathcal{M} \geq dimX$. In particular if $ch\mathcal{M} = dimX$, then we call $\mathcal{M}$ is holonomic $\mathcal{D}_X$-module.*

Example 3.11. $\mathcal{O}_X$ is a holonomic $\mathcal{D}_X$-module.

Intuitively speaking, the least possible dimension for ch$\mathcal{M}$ means that the ideal of the system of equation is the largest possible. We expect that solution space to be finite-dimension. We can safely call it as appropriate module to discuss differential equation.

Theorem 3.12. *If $\mathcal{M}$ and $\mathcal{N}$ are holonomic $\mathcal{D}_X$-module, then for any point $x \in X$,$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N})_x$ is finite dimension. [Kashiwara]*

From now on, we will occasionally use conormal bundle $T^*_Y X$

Definition 3.13. We assume Y can be described by a local coordinate $Y = \{x_1 = \dots x_d = 0\}$ and X can be written as $(x_1 \dots x_n)$. Conormal bundle of smooth submanifold $Y \subset X$ be a $T^*_Y X \subset T^*X$ such that

$$T^*_Y X = \{(x,\xi) \in T^*X | x_1 = \dots = x_d = \xi_{d+1} = \dots = \xi_n = 0\}$$

Now, we would like to focus again for the connection. One of the nice consequences of introducing the idea of characteristic variety is that we can now geometrically understand the connection. If $\mathcal{M}$ is a connection, namely a locally free sheaf of finite rank, then the example 3.14 will shows that $ch\mathcal{M} = T^*_X X$. The characteristic variety can be completely described in terms of conormal bundle. In the proposition 3.15, we will show that this is actually an equivalent statement.

Example 3.14. Let $\mathcal{M}$ be a connection of rank N. By theorem 1.10 we have a local isomorphism $\mathcal{M} \cong \mathcal{O}_X^{\oplus N}$. Then we can define a good filter for $\mathcal{M}$ by $F_i\mathcal{M} = \mathcal{M}$ when $i \geq 0$ and $F_i\mathcal{M} = 0$ if $i < 0$. We have a local isomorphism

$$gr^F\mathcal{M} \cong (\mathcal{O}_X[T_1 \dots T_n] / \sum_{i=1}^{n} \mathcal{O}_X[T_1 \dots T_n]T_i)^{\oplus N} \quad (n = dimX)$$

Then for a zero section of T^*X, i.e.$j : X \cong T^*_X X \to T^*X$. we have a local isomorphism of $\mathcal{O}_X$-module

$$\widetilde{gr^F\mathcal{M}} = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}gr^F\mathcal{D}_X} \pi^{-1}gr^F\mathcal{M} \cong j_*(\mathcal{O}_X)^{\oplus N}$$

This implies ch$\mathcal{M} = T^*_X X$. $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module.

Let's show the full statement in regard relationship between connection and holonomic $\mathcal{D}_X$-module.

Proposition 3.15. *For a coherent $\mathcal{D}_X$-module $\mathcal{M} \ncong 0$ these conditions are equivalent*
(1)$\mathcal{M}$ is locally finitely generated over $\mathcal{O}_X$
(2) $\mathcal{M}$ is a connection
(3) ch$\mathcal{M} = T^*_X X$

Proof. (1)→(2) By lemma 3.4 we saw $\mathcal{M}$ is coherent $\mathcal{O}_X$-module. We will prove this is a locally free so that $\mathcal{M}$ is a connection. For proof, we will use Nakayama's lemma.

Lemma 3.16. *Let A be a commutative ring, $J = J(A)$ be a Jacobson radical. If we have $M = \mathcal{I}M + N$ for a finitely generated A-module M with their submodule $N \subset M$ and ideal $\mathcal{I} \subset J$, then $M = N$*

The local ring $\mathcal{O}_{X,x}$, $x \in X$ has a unique maximal ideal $\mathfrak{m}_x$ so that we have $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong \mathbb{C}$. Apply the right exact functor $\mathcal{O}_{X,x}/\mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}} (*)$ to the surjective homomorphism

$$\mathcal{O}_{X,x}^{\oplus N} \to \mathcal{M} \to 0$$

so that

$$\mathbb{C}^N \to \overline{\mathcal{M}} := \mathbb{C} \otimes_{\mathcal{O}_{X,x}} \mathcal{M}_x \to 0$$

thus $\overline{\mathcal{M}}_x$ is finite dimension over $\mathbb{C}$. Let's $\overline{s}_1 \dots \overline{s}_N$ be a basis of $\overline{\mathcal{M}_x}$ over $\mathbb{C}$. Then Nakayama's lemma implies $\mathcal{M}_x = \sum_{i=1}^N \mathcal{O}_{X,x}s_i$ We will prove s_i is a basis of $\mathcal{M}_x$ as a $\mathcal{O}_{X,x}$ free module.
Suppose there is a nontrivial relation

$$\sum_{i=1}^{N} f_i s_i = 0 \quad (f_i \in \mathcal{O}_{X,x}). \tag{3.17}$$

For each $(1 \leq i \leq N)$, we will define

$$ord(f_i) = max\{k | f_i \in \mathfrak{m}_x^k\}$$

. If apply a partial differential operator ∂_i to (2.17), then we have

$$0 = \sum_{i=1}^{N} \{((\partial_j f_i)s_i + f_i(\partial_j s_i))\} = \sum_{i=1}^{N} g_i s_i$$

for some $g_i \in \mathcal{O}_{X,x}$ and we will have

$$min_{1\leq i\leq N}\{ordf_i\} > min_{1\leq i\leq N}\{ord(g_i)\}$$

Repeating this process iterative, we will have a nontrivial relationship for $\mathcal{M}_x \cong \mathbb{C}^N$ such that

$$\sum_{i=1}^{N} \overline{h}_i\overline{s}_i = 0 \quad (\overline{h}_i \in \mathbb{C}, (\overline{h}_1, \ldots, \overline{h}_N) \neq (0, \ldots, 0))$$

but this contradict the hypothesis of $\overline{s}_1 \ldots \overline{s} \in \overline{\mathcal{N}}$ being a $\mathbb{C}$-basis. Thus we have the isomorphism

$$\mathcal{M}_x \cong \mathcal{O}_{X,x}s_1 \oplus \cdots \oplus \mathcal{O}_{X,x}s_N$$

For each point $x \in X$ there is a homomorphism of coherent $\mathcal{O}_X$-module

$$\Phi : \mathcal{O}_X^{\oplus N} \to \mathcal{M} \quad ((f_1, \ldots, f_N) \to \sum_{i=1}^{N} f_is_i)$$

this induces an isomorphism of stalk $\Phi_x : \mathcal{O}_{X,x}^{\oplus N} \cong \mathcal{M}_x$. This is isomorphism around the x.
We already saw (2)→(3) in example 3.14.
(3)→(1) we can show the statement by introducing the other approach to constructing a characteristic variety. Let F be a good filtration for $\mathcal{M}$, and annihilating ideal for coherent $gr^F\mathcal{D}_X$-module $gr^F\mathcal{M}$,

$$Ann_{gr^F\mathcal{D}_X}(gr^F\mathcal{M}) = \{s \in gr^F\mathcal{D}_X | sm = 0 \quad (m \in gr^F\mathcal{M})\}$$

and then it's vanishing set

$$V(Ann_{gr^F\mathcal{D}_X}(gr^F(\mathcal{M}))) := \{(s,\xi) \in T^*X | s(x,\xi) = 0 \quad (s \in Ann_{gr^F\mathcal{D}_X}(gr^F(\mathcal{M}))) \in T^*X\}$$

is indeed coincide with the characteristic variety.
$gr^F(\mathcal{D}_X) \cong \mathcal{O}_X[T_1 \ldots T_n] \quad (n = dimX, T_i = [\partial_i] \in gr_1^F(\mathcal{D}_X))$. (3) told that we have an equality of the ideal

$$\sqrt{Ann_{gr^F_{\mathcal{D}_X}}\mathcal{M}} = \sum_{i=1}^{n} \mathcal{O}_X[T_1, \ldots, T_n]T_i$$

means that for an ideal $\mathcal{J} := \sum_{i=1}^{n} \mathcal{O}_X[T_1, \ldots, T_n]T_i \subset gr^F(\mathcal{D}_X)$ there is some k such that

$$\mathcal{J}^k \subset Ann_{gr^F\mathcal{D}_X}(gr^F\mathcal{M})$$

$T^\alpha = T_1^{\alpha_1} \ldots T_n^{\alpha_n}$ with $|\alpha| = k$ are elements of $\mathcal{J}^k$ so we have

$$\partial_x^\alpha F_i\mathcal{M}_x \subset F_{i+k-1}\mathcal{M}$$

Since F is a good filter, there is j_0 such that

$$F_i\mathcal{D}_X \cdot F_j\mathcal{M} = F_{i+j}\mathcal{M}(j \leq j_0)$$

So for an arbitrary $j \geq j_0$

$$F_{j+k}\mathcal{M} = F_k\mathcal{D}_XF_j\mathcal{M} \subset F_{j+k-1}\mathcal{M}$$

thus the increasing sequence

$$\cdots \subset F_j\mathcal{M} \subset F_{j+1}\mathcal{M} \subset \ldots$$

terminate as $F_j\mathcal{M} = F_{j+1}\mathcal{M}$ so $\mathcal{M} = F_j\mathcal{M}$ is coherent. □

4. The Cauchy-Kowalevski-Kashiwara Theorem

For a morphism between complex manifold $f : Y \to X$ and consider a sheaf of $\mathcal{O}_Y$ module described as

$$\mathcal{D}_{Y\to X} := f^*\mathcal{D}_X = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X$$

$\mathcal{D}_{Y\to X}$ is a right $f^{-1}\mathcal{D}_X$ module. Also the homeomorphism of $\mathcal{O}_Y$ induced by a differential by f^*

$$\Theta_Y \to f^*\Theta_X = \mathcal{O}_Y \otimes f^{-1}\Theta_X$$

shows $\mathcal{D}_{Y\to X}$ is also a left $\mathcal{D}_Y$-module. This construction is general. Means that we can define a inverse image for any $\mathcal{D}_X$-module $\mathcal{M}$, so that

$$f^*\mathcal{M} := \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{M} = \mathcal{D}_{Y\to X} \otimes_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M}$$

is a left $\mathcal{D}_Y$-module and right $f^{-1}\mathcal{D}_X$-module. This shows we construct a functor

$$f^* : \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_Y)$$

Example 4.1. If Y is a submanifold of complex X and $f : Y \to X$ is an embedding. By utilizing a description of local coordinate , $(x_1 \dots x_n)$ for the X and $Y = \{x_1 = \dots x_d\}(d = \mathrm{codim} Y)$. Then we have a local isomorphism

$$D_{Y\to X} \cong \mathcal{D}_X/(x_1\mathcal{D}_X + \dots x_d\mathcal{D}_X)|_Y$$

Example 4.2. If $f : Y \to X$ is submersion, we can locally write $Y = S \times X$ so $y = (y_1 \dots y_d, y_{d+1} \dots y_n)$. f can be described as

$$y = (y_1 \dots y_d, y_{d+1} \dots y_n) \to (y_{d+1} \dots y_n)$$

Therefore we have local isomorphism

$$\mathcal{D}_{Y\to X} \cong \mathcal{D}_Y/\mathcal{D}_Y\partial_1 + \dots \mathcal{D}_Y\partial_d$$

The functor f^* can be extend to the functor between a derived category.

$$Lf^* : D^+(\mathrm{Mod}(\mathcal{D}_X)) \to D^+(\mathrm{Mod}(\mathcal{D}_Y))$$

$$\mathcal{M}_\bullet \to \mathcal{D}_{Y\to X} \otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M}_\bullet$$

by associative property of tensor product, we have

$$Lf^*(\mathcal{M}_\bullet) \cong (\mathcal{O}_Y \otimes^L_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X) \otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M}_\bullet$$

$$\cong \mathcal{O}_Y \otimes^L_{f^{-1}\mathcal{O}_Y} (f^{-1}\mathcal{D}_X \otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M}_\bullet)$$

$$\cong \mathcal{O}_Y \otimes^L_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{M}_\bullet$$

as $\mathcal{D}_X$ is flat $\mathcal{O}_X$ module. We call $Lf^*\mathcal{M}_\bullet$ as an inverse image of f^*. We can also show that $Lg^*(Lf^*(\mathcal{M}_\bullet)) \cong L(f \circ g)^*(\mathcal{M}_\bullet)$

This inverse image is one of the basic operations of a derived category. In this chapter, we want to generalize Cauchy-Kowalevski theorem(lemma 1.9) in the context of the derived category. Now we will claim that f is a closed embedding throughout this chapter. Actually, many theorems we will introduce here would also hold in case of f is a submersion. However for the sake of Riemann-Hilbert correspondence and to shorten proofs , it is enough to consider the case f is a closed embedding.

Consider a natural morphism induced by $f : Y \to X$

$$T^*Y \xleftarrow{\rho_f} Y \times_X T^*X \xrightarrow{\omega_f} T^*X$$

Where ρ_f is a homomorphism of holomorphic vector bundle. If f is a closed embedding, then ρ_f is a surjective. It's kernel $\rho_f^{-1}(T_Y^*Y) \subset Y \times_X T^*X$ is a conormal bundle T_Y^*X

Definition 4.3. $\mathcal{M}$ is non-characteristic if we have

$$\omega_f^{-1}(ch(\mathcal{M})) \cap \rho_f^{-1}(T^*Y) \subset Y \times_X T_X^*X$$

Example 4.4. Let $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P \quad (P \in \mathcal{D}_X)$, $f : Y = \{x_n = 0\} \to X$ is a closed embedding. Then principal symbol $\sigma(P)(x,\xi)$ of P is a homogeneous over ξ and $ch(\mathcal{M}) = \{(x,\xi) \in T^*X | \sigma(P) = 0\}$. f is non-characteristic for $\mathcal{M}$ $ch(\mathcal{M}) \cap T_X^*Y \subset Y \times_X T_X^*X$ if and only if

$$\sigma(P)((x_1 \dots x_{n-1}, 0)(0 \dots, 0, 1)) \neq 0$$

By multipling some holomorphic function to P from left, we can replace a generator of $\mathcal{D}_X P \subset \mathcal{D}_X$ as

$$P' = \partial^m + \sum_{0 \geq j \geq m-1} Q_j(x, \partial_1 \dots \partial_{n-1})\partial_n^j$$

with $m = ordP, ordQ_j \leq ordP - j$.The condition of a f being non-characteristic is equivalent to Y is a non-characteristic for a partial differential operator P in classical PDE. We may treat as if P' is a degree m polynomial and divide $\mathcal{D}_X$ to get a local isomorphism

$$f^*\mathcal{M} = (\mathcal{D}_X/x_n\mathcal{D}_X + \mathcal{D}_X P) \cong \mathcal{D}_Y^{\otimes m}$$

Moreover, by a free resolution of right $f^{-1}\mathcal{D}_X$-module $\mathcal{D}_{Y\to X}$

$$0 \to f^{-1}\mathcal{D}_X \xrightarrow{x_n \times} f^{-1}\mathcal{D}_X \to \mathcal{D}_{Y \to X} \to 0$$

We have

$$Lf^*(\mathcal{M}) = [0 \to (\mathcal{D}_X/\mathcal{D}_X P)|_Y \xrightarrow{x_n \times} (\mathcal{D}_X/\mathcal{D}_\mathcal{X}\mathcal{P})|_Y \to 0)]$$

We have $H^j Lf^*(\mathcal{M}) = 0$ and $H^0 Lf^*(\mathcal{M}) \cong \mathcal{D}_Y^{\otimes m}$. This example will be generalized by theorem 3.8. Before proving statements, let's see lemma for proving statement.

Lemma 4.5. *If $f : Y \to X$ is closed embedding, then f is non-characteristic for $\mathcal{M}$ if and only if ρ_f is finite over $\omega_f^{-1}(ch\mathcal{M})$*

Proof. See [Takeuchi] □

Definition 4.6. A subset S of T^*X, $S \subset T^*X$ is called conic when S is closed under the action of $\mathbb{C}^*$ on T^*X, as these characteristic variety

Lemma 4.7. *A characteristic variety for the coherent $\mathcal{D}_X$-module $\mathcal{M}$ is conic.*

Proof. Chose a local generators of $\mathcal{M}$, $u_1 \dots u_m \in \mathcal{M}$ then put

$$\mathcal{I}_j := \{P \in \mathcal{D}_X | Pu_j = 0\} \subset \mathcal{D}_X \quad (1 \leq j \leq m)$$

so there is an injection homomorphism of coherent $\mathcal{D}_X$-module

$$\mathcal{D}_X/\mathcal{I}_j \to \mathcal{M}$$

and surjective homomorphism

$$\oplus_{j=1}^m (\mathcal{D}_X/\mathcal{I}_j) \to \mathcal{M}$$

then by the formula following lemma,(proof can be found on [Hotta-Takeuchi-Tanizaki]) we have a $ch\mathcal{M} = \cup_{j=1}^{m}(ch(\mathcal{D}_X/\mathcal{I}))$ and actually, $\cup_{j=1}^{m}(ch(\mathcal{D}_X/\mathcal{I}))$ is conic because of homogenity of principal symbol.

Lemma 4.8. *If there is a short exact sequence of coherent $\mathcal{D}_X$-module*

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

then there is an equality

$$ch\mathcal{M} = ch\mathcal{M}' \cup ch\mathcal{M}''$$

□

Theorem 4.9. *If a closed embedding $f : Y \to X$ of coherent $\mathcal{D}_X$-module $\mathcal{M}$ is a non-characteristic, then following statements hold.*
(1) $H^j Lf^*(\mathcal{M}) = 0$ $(j \neq 0)$
(2) $H^0 Lf^*(\mathcal{M}) \cong f^*\mathcal{M}$ is coherent $\mathcal{D}_X$-module.
(3) $chf^*\mathcal{M} \subset \rho_f \omega_f^{-1}(ch\mathcal{M})$

Proof. (Step 1) Let's attempt (1). If f is closed embedding of codimension 1 surface $f : Y \to X$ then

$$Lf^* = [0 \to f^{-1}\mathcal{M} \xleftarrow{x_n \times} f^{-1}\mathcal{M} \to 0]$$

Let $u \in \mathcal{M}$ be a $f^{-1}u \in f^{-1}\mathcal{M}$ with $x_n(f^{-1}u) = 0$. Then ideal $\mathcal{I} := \{P \in \mathcal{D}_X | Pu = 0\}$ defines injective homomorphism

$$0 \to \mathcal{D}_X/I \to \mathcal{M}$$

gives an inclusion $ch(\mathcal{D}_X/\mathcal{I}) \subset ch\mathcal{M}$. This shows that a $f : Y \to X$ is non-characteristic for a coherent $\mathcal{D}_X$-module $\mathcal{D}_X/I$. Since T_Y^*X is complex line bundle over Y and $ch(\mathcal{D}/\mathcal{I})$ is a conic, we see that, locally exist a $P \in \mathcal{I}$ and we have

$$\{(x,\xi) \in T^*X | \sigma(P)(x,\xi) = 0\} \cap T_Y^*X \subset Y \times_X T_X^*X$$

holds.This Implies $f : Y \to X$ is also non-characteristic for $\mathcal{D}_X/\mathcal{D}_X P$. Furthermore, without loss of generality, we can assume $P = \partial_n^m + \sum Q_j(x,_1 \ldots, \partial_{n-1})\partial_n^j \quad (m = ordP, ordQ_j \leq ordP - j)$. For this $P \in \mathcal{I}$, we put $ad_{x_n}(P) := [x_n, P] = x_n \circ P - P \circ x_n \in \mathcal{D}_X$. In this case, $x_n(f^{-1}u) = P(f^{-1}u) = 0$. For arbitrary $k > 0$, $ad_{x_n}^m P(f^{f^{-1}u}) = 0$. Especially when $ad_{x_n}^m(P)$ is holomorphic function. Thus $f^{-1}u = 0$. This shows the first statement.
Let's show the second statement. By the argument of (1), for a local generators of $u_1 \ldots u_l \in \mathcal{M}$, there is some $P_i \in \mathcal{D}_X$ with $P_i u_i = 0$. In addition to that, $f : Y \to X$ is a non-characteristic for $\mathcal{D}_X/\mathcal{D}_X P_j$. This induce a short exact sequence

$$0 \to \mathcal{N} \to \oplus_{j=1}^{l} \mathcal{D}_X/\mathcal{D}_X P_j \to \mathcal{M} \to 0$$

Here f is non-characteristic for $\mathcal{N}$. From computation of example 3.4, we have

$$0 \to f^*\mathcal{N} \to \mathcal{D}_Y^{\oplus N} \to f^*\mathcal{M} \to 0$$

with $N := \sum_{i=1}^{l} ordP_i$. This shows that $f^*\mathcal{M}$ is locally finitely generated $\mathcal{D}_Y$ module. Since $f^*\mathcal{N}$ also satisfies same hypothesis, we can claim $f^*\mathcal{M}$ is coherent $\mathcal{D}_Y$ module.
(3)Let $P_j \in \mathcal{D}_X$ be what we used in(2) with $m_j := ordP_j$. We have an isomorphism of $\mathcal{D}_Y$

$$\Phi_j : \mathcal{D}_Y^{\oplus m_j} \cong f^*(\mathcal{D}_X/\mathcal{D}_X P_j) \cong \mathcal{D}_X/x_n\mathcal{D}_X + \mathcal{D}_X P_j$$

$((R_0, R_1 \dots R_{m_j-1})) \to [\sum_{k=0}^{m_j-1} R_k \circ \partial_n^k]$ satisfies following property.
(4.10)

$$\Phi_j(F_i\mathcal{D}_Y \oplus F_{i-1}\mathcal{D}_Y \oplus \cdots \oplus F_{i-m_j+1}\mathcal{D}_Y) = \frac{F_i\mathcal{D}_X + (x_n\mathcal{D}_X + \mathcal{D}_X P_j)}{x_n\mathcal{D}_X + \mathcal{D}_X P_j} \subset f^*(\mathcal{D}_X/\mathcal{D}_X P_j)$$

Here, define the filtration of $f^*(\mathcal{D}_X/\mathcal{D}_X P_j) \cong \mathcal{D}_Y^{\oplus m_j}$ by

$$F_i(\mathcal{D}_Y^{\oplus m_j}) := F_i\mathcal{D}_Y \oplus \dots F_{i-m_j+1}\mathcal{D}_Y \subset f^*(\mathcal{D}_X/\mathcal{D}_X P_j)$$

Then surjective homeomorphism $\Psi : \oplus_{j=1}^l f^*(\mathcal{D}_X/\mathcal{D}_X P_j) \to f^*\mathcal{M}$ from (2) induces the good filtration $F_i(f^*\mathcal{M}) := \Psi(\oplus_{j=1}^l \Phi_j(F_i(\mathcal{D}_Y^{m_j})))$ also by the above argument, we have $F_i(f^*\mathcal{M}) = (F_i\mathcal{M} + x_n\mathcal{M})/x_n\mathcal{M} \subset f^*\mathcal{M}$. This gives a surjection

$$f^*gr^F\mathcal{M} := (gr^F\mathcal{M}/x_n gr^F\mathcal{M}) \to gr^F(f^*\mathcal{M})$$

. From the projection, $\pi_Y : T^*Y \to Y$ and $\pi : Y \times_X T^*X \to Y$ we can] construct the $\mathcal{O}_{T^*Y}$ module

$$\widetilde{f^*gr^F}\mathcal{M} := \mathcal{O}_{T^*} \otimes_{\pi^{-1}gr^F\mathcal{D}_Y} \pi^{-1}(f^*gr^F\mathcal{M})$$

and $\mathcal{O}_{Y\times_X T^*X}$-module

$$\omega_f^*(\widetilde{gr^F\mathcal{M}}) := \mathcal{O}_{Y\times_X T^*X} \otimes_{\pi^{-1}(f^*gr^F\mathcal{D}_X)} \pi^{-1}(f^*gr^F\mathcal{M})$$

Since this is embedding, ρ_f is finite over compact support of $\omega_f^*(\widetilde{gr^F\mathcal{M}})$ and we have an isomorphism

,

$$f^*gr^F\mathcal{M} \cong \rho_f\omega_f^*(\widetilde{f^*gr^F}\mathcal{M})$$

We can get inclusion $supp(\widetilde{f^*gr^F}\mathcal{M}) \subset \rho_f\omega_f^{-1}(ch\mathcal{M})$. By combining surjection of $\mathcal{O}_{T^*Y}$ module $\widetilde{f^*gr^F}\mathcal{M} \to gr^F(\widetilde{f^*\mathcal{M}})$ we will get what we want.

(Step 2) In case of $f : Y = \{x_{n-d+1} = x_{n-d+2} \cdots = x_n = 0\} \to X$ defines closed embedding. We can prove by induction from (Step1). Take a complex hyperplane H such that $Y \subset H := x_n = 0 \subset X$. Let $g : H \to X$ be an embedding then $\rho_f : Y \times_X T^*X \to T^*Y$ decomposed into

$$Y \times_X T^*X = Y \times_H (H \times_X T^*X) \xrightarrow{id_Y \times_H \rho_g} Y \times_H T^*H \to T^*Y$$

By the hypothesis, ρ_f is finite over $\omega_f^{-1}(ch\mathcal{M})$, so the neighborhood of Y in X, $\rho_g : H \times_X T^*X \to T^*H$ is also a finite on $\omega_g^{-1}(ch\mathcal{M})$. Thus $g : H \to X$ is non-characteristic for $\mathcal{M}$ around Y. From the (step1), $H^jLg^*(\mathcal{M}) = 0 \quad (j \neq 0)$ and we have $H^0Lg^*(\mathcal{M}) \cong g^*\mathcal{M}$ with $ch(g^*\mathcal{M}) \subset \rho_g\omega^{-1}(ch\mathcal{M})$.
By the induction hypothesis, $H^jLh^*(g^*\mathcal{M}) = 0 \quad (j \neq 0)$ and $H^0Lh^*(g^*\mathcal{M}) \cong h^*g^*\mathcal{M} \cong f^*\mathcal{M}$ is coherent. We also have $ch(f^*\mathcal{M}) \cong \rho_h\omega^{-1}\rho_g\omega^{-1}(ch\mathcal{M}) \subset \rho_f\omega_f^{-1}(ch\mathcal{M})$. □

We will write $H^0Lf^*\mathcal{M} \cong f^*\mathcal{M}$ as $\mathcal{M}_Y$. We will call $\mathcal{M}_Y$ as an induced system over Y. Since $f^*\mathcal{O}_X \cong \mathcal{O}_Y$, we have a homomorphism of sheaf

$$\begin{aligned} &f^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \\ &\to \mathcal{H}om_{f^{-1}\mathcal{D}_X}(f^{-1}\mathcal{M}, f^{-1}\mathcal{O}_X) \\ &\to \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X} \otimes_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M}, \mathcal{D}_{Y\to X} \otimes_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{O}_X) \end{aligned}$$

$$\cong \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y) \tag{4.11}$$

From this fact, we can consider $\mathcal{M}_Y$ as a restriction of solution of PDE of X over Y. In particular, if Y is a hyperplane $\{x_n = 0\}$ with closed embedding $f : Y \to X$ is non-characteristic over $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$, then we have a local isomorhism

$$\mathcal{D}_Y^{\oplus m} \cong \mathcal{D}_X/x_n\mathcal{D}_X + \mathcal{D}_X P \cong \mathcal{M}_Y$$

In fact, by considering the map $((R_0, R_1 \dots R_{m-1})) \to [\sum_{k=0}^{m-1} R_k \circ \partial_n^k]$, we can describe a morphism (3.11) in the following way

$$\{u \in \mathcal{O}_X|_Y | Pu = 0\} \to \mathcal{O}_Y^{\oplus m}$$

$$u \to (u|_Y, \partial_n u|_Y, \dots \partial_n^{m-1} u|_Y)$$

The classical Cauchy-Kowalevski theorem claims that this map is an isomorphism. This map can be extend to a derived category by the following way.

$$f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(Lf^*\mathcal{M}, \mathcal{O}_Y)$$

In the case of a hyperplane, we can calculate as $H^0 f^{-1}\mathcal{R}(\mathcal{M}, \mathcal{O}_X) = \{u \in \mathcal{O}_X|_Y | Pu = 0\}, H^1 f^{-1}\mathcal{R}(\mathcal{M}, \mathcal{O}_X) = (\mathcal{O}_X/P\mathcal{O}_X)|_Y$ and when $j \neq 0, 1$, cohomology vanishes.
$H^0\mathcal{R}(\mathcal{M}_Y, \mathcal{O}_Y) = \mathcal{O}_Y^{\oplus m}$ and otherwise all cohomoloies vanish.
The Cauchy-Kowalevski theorem claims that 1st cohomology vanishes due to the solvability of PDE so we can confirm an isomorphism $f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(Lf^*\mathcal{M}, \mathcal{O}_Y)$. The Cauchy-Kowalevski-Kashiwara's theorem generalize this fact.

Theorem 4.12. *An embedding $f : Y \to X$ is non-characteristic for coherent $\mathcal{D}_X$ module $\mathcal{M}$. Then we have an isomorphism of derived category of*

$$f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(Lf^*\mathcal{M}, \mathcal{O}_Y)$$

Proof. (1)Consider the case $f : Y := \{x_n = 0\} \to X$ is a closed embedding. Utilize the exact sequence of coherent $\mathcal{D}_X$-module in a proof of theorem 3.9

$$0 \to \mathcal{N} \to \oplus_{j=1}^{l} \mathcal{D}_X/\mathcal{D}_X P_j \to \mathcal{M} \to 0$$

The classical Cauchy-Kowalevski theorem told that there is an isomorphism for $\mathcal{L} := \oplus_{j=1}^{l} \mathcal{D}_X/\mathcal{D}_X P_j$ with

$$f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{O}_X) \cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{L}_Y, \mathcal{O}_Y)$$

so we will have the commutative diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
f^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X) & \xrightarrow[A]{} & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_Y,\mathcal{O}_Y) \\
\downarrow & & \downarrow \\
f^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L},\mathcal{O}_X) & \xrightarrow[\cong]{} & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}_Y,\mathcal{O}_Y) \\
\downarrow & & \downarrow \\
f^{-1}\mathcal{H}om(\mathcal{N},\mathcal{O}_X) & \xrightarrow[B]{} & \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}_Y,\mathcal{O}_Y) \\
\downarrow & & \downarrow \\
f^{-1}\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X) & \xrightarrow[C]{} & \mathcal{E}xt^1_{\mathcal{D}_Y}(\mathcal{M}_Y,\mathcal{O}_Y) \\
\downarrow & & \downarrow \\
f^{-1}\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{L},\mathcal{O}_X) & \xrightarrow[\cong]{} & \mathcal{E}xt_{\mathcal{D}_Y}(\mathcal{L}_Y,\mathcal{O}_Y) \\
\downarrow & & \downarrow \\
f^{-1}\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{N},\mathcal{O}_X) & \longrightarrow & \mathcal{E}xt^1_{\mathcal{D}_Y}(\mathcal{N},\mathcal{O}_Y) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
$$

Both of vertical sequences are exact. A is injective, and since $\mathcal{N}$ is also non-characteristic with same hypothesis, so B is also injective. Thus, by 5-lemma, A isan isomorphism. Apply the 5-lemma once more, we will have injection of C. By doing this argument iterative, we will have the isomorphism

$$f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X) \cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y,\mathcal{O}_Y)$$

(2) When $f : Y = \{x_{n-d+1} = x_{n-d+2} = \cdots = x_n = 0\} \to X$ is closed embedding of submanifold. With the same method of proof on theorem 3.11 we can prove by the induction of hyperplane H such that

$$Y \xrightarrow{h} H = \{x_n = 0\} \xrightarrow{g} X \quad (f = g \circ h)$$

so by induction we will have the following statement.

$$f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X) \cong h^{-1}g^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)$$

$$\cong h^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_H}(Lg^*\mathcal{M},\mathcal{O}_H)$$

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(Lh^*Lg^*\mathcal{M},\mathcal{O}_Y)$$

$$\cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(Lf^*\mathcal{M},\mathcal{O}_Y)$$

□

5. The De Rham Functor and The Solution Functor

Let's denote the dimension of complex manifold X as d_X, and derived category of bounded complex of $\mathcal{D}_X$-module as $D^b(\mathcal{D}_X)$. Also

$$D^b_{coh}(\mathcal{D}_X) = \{M_\bullet \in D^b(\mathcal{D}_X) | H^j(\mathcal{M}_\bullet)\text{is coherent}\mathcal{D}_X\text{module}\})$$

$$D^b_h(\mathcal{D}_X) = \{M_\bullet \in D^b(\mathcal{D}_X) | H^j(\mathcal{M}_\bullet)\text{is holonomic}\mathcal{D}_X\text{module}\})$$

we can define the solution complex $Sol_X(\mathcal{M}_\bullet)$ and the De Rham complex $DR(\mathcal{M}_\bullet)$ in the following way.

$$Sol_X(\mathcal{M}_\bullet) = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\bullet, \mathcal{O}_X)$$

$$DR(\mathcal{M}_\bullet) = \Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{M}_\bullet$$

The solution functor is solutions of partial differential equations as we saw in chapter 1. In that sense, solution functor generalize a solution of PDE in a derived category. Analogously, De Rham complex is higher dimensional analogous of differential equations.
Next lemma is just a calculation.

Lemma 5.1. *Left $\mathcal{D}_X$ module $\mathcal{O}_X$ has a free resolution*

$$0 \to \mathcal{D} \otimes_{\mathcal{O}_X} \wedge^{d_X}\Theta_X \to \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^0\Theta \cong \mathcal{D}_X \to \mathcal{O}_X \to 0$$

Morphisms $d_k : \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^k\Theta_X \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^{k-1}\Theta_X$ $(k \neq 0)$ are given by

$$d_k(P \otimes \theta_1 \wedge \dots \theta_k) = \sum_i (-1)^{i+1} P\theta_i \otimes \theta_1 \wedge \dots \breve{\theta_i} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_k$$

$$+ \sum (-1)^{i+j} P \otimes [\theta_j, \theta_j] \wedge \theta_1 \wedge \cdots \wedge \breve{\theta_i} \wedge \cdots \wedge \breve{\theta_j} \wedge \dots \theta_k$$

When $k = 0$ it is $P \to P(1)$

Let By multiplying tensor product $\wedge^{d_X}\Omega_X \otimes_{\mathcal{O}_X} (*)$ to the exact sequence from lemma 5.1, then we also got a free resolution of right $\mathcal{D}_X$-module Ω_X

$$0 \to \wedge^0\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \cong \mathcal{D}_X \to \wedge^1\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \cdots \to \wedge^{d_X}\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \wedge^{d_X}\Omega_X \to 0$$

Here $\Omega^{d_X}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \Omega_X$ and $\delta_k : \Omega^k \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \Omega^{k+1}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$ are given by $\omega \otimes P \to \omega P$ and $\delta_k(\omega \otimes P) = d\omega \otimes P + \sum_{i=1}^{d_X} dx_i \wedge \omega \otimes \partial_i P$.
Combine with the above observations, we can see the De Rham complex $DR_X(\mathcal{M})$ for the coherent $\mathcal{D}_X$-module $\mathcal{M}$ are

$$DR_X(\mathcal{M}) = [0 \xrightarrow{\nabla_0} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\nabla_1} \dots \nabla_{d_X - 1}\Omega^{d_X} \otimes_{\mathcal{O}_X} \mathcal{M} \to 0]$$

with $\nabla_k : \Omega^k \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega^{k+1}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is defined by $\omega \otimes P \to d\omega \otimes P + \sum_{i=1}^{d_x} dx_i \wedge \omega \otimes \partial P_i$

The solution complex and the De Rham complex are actually, in some sense they are dual in a derived category from the following calculation. $DR_X\mathcal{M}_\bullet \cong \Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{M}_\bullet \cong (\Omega^\bullet_X \otimes \mathcal{D}_X[d_X]) \otimes_{\mathcal{D}_X} \mathcal{M}_\bullet \cong \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}_\bullet[d_X]$ so that opposite version of higher direct image functor $\mathbb{R}\mathcal{H}om$ is $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}_\bullet) \cong \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^\bullet\Theta_X, \mathcal{M}_\bullet) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X \wedge^\bullet \Theta_X, \mathcal{M}_\bullet) \cong \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \cong DR_X\mathcal{M}[d_X]$
This means that if we multiple $\otimes_{\mathcal{O}_X}\Omega^{-1}$, we have

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega^{\otimes -1}_X[d_X] \cong \mathcal{O}_X$$

This example motivates to define a dualize functor in the following way.

Definition 5.2. Contravariant functor $\mathbb{D}_X : D^b(\mathcal{D}_X^{op}) \to D^b(\mathcal{D}_X)$ is defined by

$$\mathbb{D}_X(\mathcal{M}_\bullet) = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\bullet, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{-1}[d_X]$$

For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we will call $\mathbb{D}_X(\mathcal{M})$ is a dualizing $\mathcal{D}_X$-module.

Indeed, for a coherent bounded complex $\mathcal{M}_\bullet$, we have an isomorphism $\mathbb{D}_X(\mathbb{D}_X(\mathcal{M})) \cong \mathcal{M}$ and if $\mathcal{M}$ is holonomic, $H^j\mathbb{D}_X(\mathcal{M}) \cong 0$ if $(j \neq 0)$ and $H^0(\mathbb{D}_X(\mathcal{M})) \cong \mathbb{D}_X(\mathcal{M})$ is a holonomic $\mathcal{D}_X$ module. In this sense, $\mathbb{D}_X$ is a dual functor. These fact lead to an important relation between solution functor and De Rham functor.

Theorem 5.3. *For a $\mathcal{M}_\bullet \in D^b_{coh}(\mathcal{D}_X)$, we have an isomorphism*

$$Sol_X(\mathbb{D}_X(\mathcal{M}_\bullet))[d_X] \cong DR_X(\mathcal{M}_\bullet)$$

Proof. We have an isomorphism for $\mathcal{M}_\bullet$

$$Sol(\mathbb{D}_X(\mathcal{M}_\bullet))[d_X] \cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}_X(\mathcal{M}_\bullet), \mathcal{O}_X)[d_X]$$

$$\cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}_X(\mathcal{M}_\bullet), \mathcal{D}_X)[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{O}_X$$

$$\cong \Omega_X \otimes^L_{\mathcal{D}_X} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}_X(\mathcal{M}_\bullet), \mathcal{D}_X) \otimes^L_{\mathcal{O}_X} \Omega_X^{-1}[d_X]$$

$$\Omega_X \otimes^L_{\mathcal{D}_X} \mathbb{D}_X(\mathbb{D}_X(\mathcal{M}_\bullet)) \cong \Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{M}_\bullet$$

Here we used the technique in the example 1.12. □

In the derived category, there is the well-established theory of duality. We want to utilize this in the analytic theory of PDE. Proofs of 4.5 and 4.7 can be found on [Kashiwara-Schapira]

Definition 5.4. Let $f : X \to Y$ be a continuous function between topological space X, Y. The direct image functor $f_*\mathcal{F} \in Sh(Y)$ is defined by

$$f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V)) \quad (V \subset Y)$$

Then subsheaf $f_! \subset f_*(\mathcal{F})$ is defined by

$$f_!\mathcal{F}(V) := \{s \in \mathcal{F}(f^{-1}(V)) | f|_{\text{supp s}} : \text{supp s} \to V \text{is a proper map}\} \quad (V \subset Y)$$

$f_!\mathcal{F}$ is called proper direct image sheaf.

If we define a map of a one point space $a_X : X \to \text{pt}$. The sheaf of abelian group on one point space Sh(pt) is equivalent to ablian category of abelian group $\mathcal{A}$b. So there is an isomorphism

$$(a_X)_* \cong \Gamma(X, *) \quad (a_X)_! \cong \Gamma_c(X, *)$$

If in addition to that assigning hypothesis that there exist $d > 0$ such that for $k > d$ we have $H^k\mathbb{R}f_!\mathcal{F} \cong 0$

Theorem 5.5. *$f : Y \to X$ is a continuous function between a locally compact Hausdorff space with commutative ring A_X and A_X-module $\mathcal{M}$, and A_Y $\mathcal{N}$ of finite global dimension. Poincare-Verdier duality states that there exist a functor of triangle category $f^!$ such that*

$$\mathbb{R}f_*\mathbb{R}\mathcal{H}om_{A_X}(\mathcal{M}_\bullet, f^!\mathcal{N}_\bullet) \cong \mathbb{R}\mathcal{H}om_{A_Y}(\mathbb{R}f_!\mathcal{M}_\bullet, \mathcal{N}_\bullet)$$

We will call $f^!$ as the twisted inverse image functor.

Definition 5.6. THe dualizing complex $\omega_X \in D^b(X)$ for analytic space is defined as $\omega_X = a^! \mathbb{C}_{pt}$. The Verdier duality functor is defined by

$$\mathsf{D}_X(\mathcal{F}_\bullet) := \mathbb{R}\mathcal{H}om(\mathcal{F}_\bullet, \omega_X)$$

These operation commute to $\mathbb{D}_X$ in a following sense

Theorem 5.7. *For a Holonomic and bounded $\mathcal{D}_X$-module $\mathcal{M}$, we have (1)*

$$\mathsf{D}_X(Sol_X(\mathcal{M}[d_X])) \cong DR_X(\mathcal{M})$$

(2) For a category of bounded holonomic $\mathcal{D}_X$-module $D^b_h(\mathcal{D}_X)$, we have an isomorphism

$$DR_X \circ \mathbb{D}_X \cong \mathsf{D}_X \circ DR_X$$

6. The Riemann-Hilbert Correspondence

Definition 6.1. The locally closed analytic subset $X_\alpha \subset X$ $(\alpha \in A)$ define a stratification $X = \sqcup_{\alpha \subset A} X_\alpha$ if X_α satisfy following conditions
(1) For an arbitrary $\alpha \in A$, X_α is a smooth and $\overline{X_\alpha}$ and ∂X_α is an analytic.
(2) For an arbitrary $\alpha \in A$, there is a subset $B \subset A$ such that

$$\overline{X}_\alpha = \sqcup_{\beta \in B} X_\beta$$

We call X_α as an stratum.
The Whitney stratification is a stratification with additional conditions (a),(b)
(a) Assume that a sequence $x_i \in X_\alpha$ of points converges to a point $y \in X_\beta$ $(\alpha \neq \beta)$ and the limit T of the tangent spaces $T_{x_i} X_\alpha$ exists. Then we have $T_y X_\beta \subset T$.
(b) Let $x_i \in X_\alpha$ and $y_i \in X_\beta$ be two sequences of points which converge to the same point $y \in X_\beta (\alpha \neq \beta)$. Assume further that the limit l (resp. T) of the lines l_i jointing x_i and y_i (resp. of the tangent spaces $T_{x_i} X_\alpha$) exists. Then we have $l \subset T$.

Every analytic space has a stratification. Moreover if $\mathcal{M}$ is holonomic $\mathcal{D}_X$-module, then X has a some Whitney stratification $X = \sqcup_{\alpha \in A} X_\alpha$ and we have $\mathrm{ch}\mathcal{M} \subset T^*_{X_\alpha} X$.[Hotta-Takeuchi-Tanizaki]

Definition 6.2. Let X be an analytic space, and $\mathcal{F}$ as sheaf of $\mathbb{C}_X$ module. $\mathcal{F}$ is constructible sheaf if there is a stratification $X = \sqcup_{\alpha \in A} X_\alpha$ with restriction $\mathcal{F}|_{X_\alpha}$ is a local system.

Example 6.3. Consider a $\mathcal{D}_X$ module over complex plane $X = \mathbb{C}$, $\mathcal{M} := \mathcal{D}_X / \mathcal{D}_X(x\partial - \lambda)$ $(\lambda \in \mathbb{C})$, and chose a sheaf $\mathcal{F} = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. $\mathcal{F}|_{X/\{0\}} \cong \mathbb{C}_{X/\{0\}} x^\lambda$ is an local system over $\mathbb{C}/\{0\}$. We have an isomorphism of stalk $\mathcal{F}_0 \cong \mathbb{C}$(if $\lambda = 0, 1, 2 \dots$) and $\mathcal{F} = 0$(otherwise). If we chose stratification as $X = X/0 \sqcup 0$ then $\mathcal{F}$ is an constructible sheaf.

Definition 6.4. Let X be an analytic space and $\mathcal{F}_\bullet \in D^b(X)$. We call complex $\mathcal{F}_\bullet$ is an constructible if for an arbitrary $j \in \mathbb{Z}$, $H^j(\mathcal{F}_\bullet)$ is constrctible sheaf. We denote full subcategory of constructible complex in $D^b(X)$ as $D^b_c(X) \subset D^b(X)$

The partial claim of Kashiwara's version of Riemann-Hilbert correspondence is about Grothandieck's six operations and constructible sheaf.

Theorem 6.5. *(1)If X is an analytic space then dualizing complex is a constructible $\omega_X \in D^b_c(X)$. The Verdier duality functor preserve category $D^b(X)$ and involutive $\mathsf{D}_X \circ \mathsf{D}_X = id$.*
Let f be a morphism between analytic space then
(2)The functor f^{-1} and $f^! : D^b(Y) \to D^b(X)$ preserves category of constructible sheaf. Moreover we have

$$f^! = \mathsf{D}_X \circ f^{-1} \circ \mathsf{D}_Y, \quad f^{-1}\mathsf{D}_X \circ f^! \circ \mathsf{D}_Y$$

(3) When $f : X \to Y$ is a proper morphism, the functor $\mathbb{R}f_*, \mathbb{R}f_! : D^b(X) \to D^b(Y)$ preserves category of constructible sheaf and

$$\mathbb{R}f_! = \mathcal{D}_Y \circ \mathbb{R}f_* \circ \mathcal{D}_X, \quad \mathbb{R}f_* = \mathcal{D}_Y \circ \mathbb{R}f_! \circ \mathcal{D}_X$$

Theorem 6.6. *The solution complex and the De Rham complex for holonomic $\mathcal{D}_X$-module $\mathcal{M}$ $Sol_X(\mathcal{M})$ $DR_X(\mathcal{M})$ are constructible sheaf.*

Proof. (outline) By the isomorphism $DR(\mathcal{M}) \cong Sol_X(\mathbb{D}_X(\mathcal{M}))$, we only need to prove that $\mathcal{F}_\bullet = Sol_X(\mathcal{M})$ is constructible. Let take a Whiteny straitification $X = \sqcup_{\alpha \in A} X_\alpha$ with ch$\mathcal{M} \subset \sqcup_{\alpha \in A} T^*_{X_\alpha} X$(detailed explanation of the construction is on [Hotta-Takeuchi-Tanizaki]). We want to prove that for each $\alpha \in A$ and $j \in \mathbb{Z}$, $H^j(\mathcal{F}_\bullet)$ is a locally constant sheaf. In order so, it is sufficient to prove for each point $x \in X_\alpha$, there exist an open neighborhood $U \subset X$ of x such that $a_U^{-1}\mathbb{R}(a_U)_*(\mathcal{F}_\bullet|_U) \to \mathcal{F}|_U$ ($a_U : U \to pt$) is an isomorphism. We can prove this by using theorem 6.7 [Kashiwara-Schapira].

Theorem 6.7. *Let X be a complex manifold and $X_\mathbb{R}$ be the real analytic manifold underlying X. Then $\{\Omega_t\}_{t \in \mathbb{R}}$ be a family of relative compact stein open set. Each boundary $\partial\Omega_t \subset X_\mathbb{R}$ is C^∞ hypersurface for $X_\mathbb{R}$. Assume following hypothesis*
(1) $\Omega_s \subset \Omega_t$ $(s < t)$
(2)$\Omega_t = \cup_{s<t}\Omega_s$
(3) For an any $t \in \mathbb{R}$, $\cap_{s>t}(\Omega_s/\Omega_t)\partial\Omega_t$ and $ch\mathcal{M} \cap T^*_{\partial\Omega_t} X_\mathbb{R} \subset T_{X_\mathbb{R}} X_\mathbb{R}$ Then we have a fallowing isomorphism

$$\mathbb{R}\Gamma(\cup\Omega_s; Sol_X(\mathcal{M})) \cong \mathbb{R}\Gamma(\Omega_t; Sol_X(\mathcal{M}))$$

By assumption of a Whitney stratification there is a family of open set $\{\Omega_t \subset X\}_{t \in (0,1]}$ with $\Omega_1 \in U$, $\cap_{t \in (0,1]}\Omega_t = \{x\}$ and $\partial\Omega_t$ is a real C^∞hyperplane in X and $T^*_{\partial\Omega_t}{}^*(X) \cap ch(\mathcal{M}) \subset T^*_X X$ so by above theorem we have a desired isomorphism. Remain is to show $H^j(\mathcal{F})$ is finite dimension, but this requires functional analysis argument, so skip it. □

We can interpret the above theorem as a generalization of the naive form of Riemann-Hilbert correspondence in one dimension as De Rham complex is higher-dimensional analogous of integrable connection and constructible complex is a generalization of local systems. Kashiwara proved that, furthermore, these sheaves are other important sheaves.

Definition 6.8. Let X be an analytic space and $\mathcal{F}_\bullet \in D^b_c(X)$. $\mathcal{F}_\bullet$ is called the perverse sheaf when for arbitrary $j \in \mathbb{Z}$

$$dimsupp(H^j(\mathcal{F}_\bullet)) \leq -j \quad dimsupp(H^j\mathsf{D}_X(\mathcal{F}_\bullet)) \leq -j$$

we will denote full subcategory of perverse sheaf as $Perv(\mathbb{C}_X) \subset D^b_c(X)$

Theorem 6.9. *For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, both $Sol_X(\mathcal{M})[d_X]$ and $DR(\mathcal{M})$ is perverse sheaf.*

Proof. (outline of 6.9) By the isomorphism $DR_X(\mathcal{M}) \cong Sol(\mathbb{D}_X(\mathcal{M}))[d_X]$, we only need to prove $Sol_X(\mathcal{M})[d_X]$ is perverse sheaf. Also the Verdier duality clams that we have

$$\mathsf{D}_X(Sol_X(\mathcal{M})) \cong Sol_X(\mathbb{D}_X(\mathcal{M}))[d_X]$$

so we just only need to prove the first condition of $\mathcal{F}_\bullet := Sol_X(\mathcal{M})$

$$dimsupp(H^j(\mathcal{F}_\bullet)) \leq -j$$

We fix the integer $j \in \mathbb{Z}$ and put $S := suppH^j(\mathcal{F}_\bullet) \subset X$. We can take a Whitney stratification$X = \sqcup_{\alpha \in A} X_\alpha$ and we have $ch\mathcal{M} \subset \sqcup_{\alpha \in A} T^*_{X_\alpha} X$ and also fix the subset $B \subset A$ such that $S = \sqcup_{\alpha \in B} X_\beta$. By the Kashiwara's conductibility theorem, for an arbitarly $\alpha \in A$ and $j \in \mathbb{Z}$, we have $H^j(\mathcal{F}_\bullet)|_{X_\alpha}$ as a local system on X_α. Also there is $\alpha \in B$ such that $dimX_\alpha = dimS$. With that strata, we take normal slice Y on $x \in X_\alpha$. Means that Y is submanifold of X with following properties.
(1) $Y \cap X_\alpha = \{x\}$
(2) $d_Y + d_{X_\alpha} = d_X$
(3) $T_x Y \cap T_x X_\alpha = \{0\}$ Namely Y intersect with X at only one point $\{x\}$ then taking $i_Y : Y \to X$ as a inclusion map, and by the condition $ch\mathcal{M} \subset \sqcup_{\alpha \in A} T^*_{X_\alpha} X$, i_Y is a non-characteristic. By Cauchy-Kowalevski-Kashiwara's theorem, we have a following isomorphism.

$$\mathcal{F}_\bullet|_Y = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y[d_X] \cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)[d_X]$$

Here $\mathcal{M}_Y$ is a coherent $\mathcal{D}_Y$-module. So by our assumption, $x \in X_\alpha \subset S = supp(H^j(\mathcal{F}_\bullet))$ we have $0 \neq H^j(\mathcal{F}_\bullet)_x \cong \mathcal{E}xt^{j+d_X}_{\mathcal{D}_X}(\mathcal{M}_Y, \mathcal{O}_Y)_x$ on the other hand, we have a following lemma

Lemma 6.10. *For a coherent $\mathcal{D}_X$-module $\mathcal{M}$*

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0 \quad (j > d_X)$$

Proof. See [Takeuchi] □

and combine with the fact

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_Y, \mathcal{O}_Y) \cong \mathbb{R}\mathcal{H}om(\mathcal{M}_Y, \mathcal{D}_Y) \otimes^L_{\mathcal{D}_Y} \mathcal{O}_Y$$

we have $\mathcal{E}xt^i_{\mathcal{D}_Y}(\mathcal{M}_\mathcal{Y}, \mathcal{O}_Y) = 0$ $(i > d_Y)$. So the inequality holds for $j + d_X \leq d_Y \leftrightarrow d_X - d_Y \leq -j$ □

Kashiwara reached the ultimate form of Riemann-Hilbert correspondence. This is a categorical equivalence between perverse sheaf and regular singular $\mathcal{D}_X$-module.

Definition 6.11. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and $ch\mathcal{M}$ be a characteristic variety. Let $\mathcal{I}$ be an ideal of $Gr^F(\mathcal{D}_X)$ vanishing on $ch\mathcal{M}$. $\mathcal{M}$ is regular singularity if there is a locally coherent filtration $F(\mathcal{M})$ and $\mathcal{I}Gr^F(\mathcal{M}) = 0$. We will wrote $Mod_{rh}(\mathcal{D}_X)$ as category of regular singular $\mathcal{D}_X$-module.

Theorem 6.12. *Kashiwara's Riemann-Hilbert Correspondence*

$$DR(*) : Mod_{rh}(\mathcal{D}_X) \cong Perv(\mathbb{C}_X)$$

$$Sol_X(*) : Mod_{rh}(\mathcal{D}_X)^{op} \cong Perv(\mathbb{C}_X)$$

Acknowledgments

I want to thank Professor Kiran Kedlaya, who gave me great insights for a Riemann-Hilbert correspondence. Also, his advice on miscellaneous things, mathematical writing, his view of professional mathematician, life and number theory etc.... Our discussions will be a precious memory for life. I also want to credit Jacob Keller for checking the text.

References

[1] R.Hotta, K.Takeuchi, T.Tanisaki"$\mathcal{D}_X$-module Perverse Sheaves, and Representation Theory" Birkhaeuse 2008
[2] M.Kashiwara "Introduction to algebraic analysis" Iwanami Shoten 2000(Japanese)
[3] M.Kashiwara, P.Schapira "Sheaves on Manifold" Springer 2002
[4] N.Katz "An Overview of Deligne's Work on Hilbert's Twenty-First Problem"(Proc. Symp. Pure Math., Vol. 28, Amer. Math. Soc., Providence, R. I., 1976, pp. 537-557)
[5] P.Schapira, "Microdifferential systems in the complex domain" Grundlehren der Mathematischen Wissenschften
[6] T.Szamuely "Galois Groups and Fundamental Groups" Cambridge Studies in Advanced Mathematics,2009
[7] K.Takeuchi"$\mathcal{D}$-module"Kyouritu Shupan 2017(Japanese)
[8] D.Yamakawa "Riemann-Hilbert Correspondence for regular singular connection"https://www.rs.tus.ac.jp/yamakawa/proc-biwako2.pdf