GEOMETRY OF TROPICAL MUTATION SURFACES WITH A SINGLE MUTATION

TOMOKI ODA

Abstract. Recently, Escobar, Harada, and Manon introduced the theory of polyptych lattices. This theory gives a general framework for constructing projective varieties from polytopes in a polyptych lattice. When all the mutations of the polyptych lattice are linear isomorphisms, this framework recovers the classical theory of toric varieties. In this article, we study rank two polyptych lattices with a single mutation. We prove that the associated projective surface X is a $\mathbb{G}_m$-surface that admits an equivariant 1-complement $B \in |-K_X|$ such that B supports an effective ample divisor. Conversely, we show that a $\mathbb{G}_m$-surface X that admits an equivariant 1-complement $B \in |-K_X|$ supporting an effective ample divisor comes from a polyptych lattice polytope. Finally, we compute the complexity of the pair (X, B) in terms of the data of the polyptych lattice, we describe the Cox ring of X, and study its toric degenerations.

Contents

1. Introduction

In [9], Escobar, Harada, and Manon introduced the notion of polyptych lattices. A polyptych lattice consists of a collection of lattices called charts, equipped with piecewise linear identifications between them, referred to as mutations in the literature [1, 18]. Polyptych lattices provide a framework for a natural generalization of toric geometry. To a polyptych lattice $\mathcal{M}$, the authors associate a commutative algebra $A_{\mathcal{M}}$; any such algebra is called a detropicalized algebra of $\mathcal{M}$ (cf. Definition 2.1). The algebra $A_{\mathcal{M}}$ plays a role analogous to the Laurent polynomial ring in toric geometry, and its spectrum $U_{A_{\mathcal{M}}} := \mathrm{Spec}(A_{\mathcal{M}})$ is called the *affine tropical mutation variety* (cf. Definition 2.1).

In this article, we focus on the shearing polyptych lattices $\mathcal{M}_s$ for s integer, introduced in [5]. Each $\mathcal{M}_s$ is obtained by gluing two lattices along a shear across two linear regions (cf. Definition 2.11); it is arguably the simplest example of a polyptych lattice with nontrivial mutations. Thus, $\mathcal{M}_s$ provides a natural testing ground for studying tropical mutation varieties of higher rank or additional linear domains. These surfaces

2020 *Mathematics Subject Classification.* Primary: 14M25, 14E30; Secondary: 14E15.

Key words and phrases. complexity, Cox ring, toric degenerations, log Calabi–Yau varieties, polyptych lattice.

The author was partially supported by NSF research grant DMS-2443425.

extend classical toric phenomena to the log Calabi–Yau setting and also exhibit new phenomena absent from toric geometry.

For instance, unlike the case of $\mathbb{G}_m^2$, affine tropical mutation surface $U_{A_{\mathcal{M}_s}}$ admits nontrivial deformations. As a first result, we classify all isomorphism classes of detropicalizations of $\mathcal{M}_s$.

Theorem 1.1 (cf. Theorem 3.8)**.** Let $s \geq 1$. The coarse moduli space of detropicalizations of the polyptych lattice $\mathcal{M}_s$ is given by the quotient

$$\mathbb{A}^{s-1}/D_{2s},$$

where D_{2s} denotes the dihedral group generated by the cyclic subgroup μ_s of s-th roots of unity, generated by ζ and a reflection s.

The group D_{2s} acts on the coordinate space $\mathbb{A}^{s-1} = \operatorname{Spec} \mathbb{K}[b_1, \ldots, b_{s-1}]$ by

$$\zeta \cdot (b_1, \ldots, b_{s-1}) = (\zeta b_1,\ \zeta^2 b_2,\ \ldots,\ \zeta^{s-1} b_{s-1}) \qquad (\zeta \in \mu_s),$$

and

$$s \cdot (b_1, \ldots, b_{s-1}) = (b_{s-1},\ b_{s-2},\ \ldots,\ b_1).$$

We also proved that the detropicalization of $\mathcal{M}_s$ is determined by the choice of degree s monic polynomial $f \in \mathbb{K}[y]$ with constant term 1 and b_i are the coefficients y^i. We denote by $U_f := \operatorname{Spec}(A_f)$ the affine tropical mutation variety corresponding to the detropicalization of $\mathcal{M}_s$ defined by f.

Given a polytope $\mathcal{P} \subset \mathcal{M}$, Escobar–Harada–Manon showed in [9, Theorem 7.11] that $U_{A_{\mathcal{M}}}$ admits a compactification by adding a boundary divisor $B(\mathcal{P})$, producing a projective variety $X_{A_{\mathcal{M}}}(\mathcal{P})$. We call $X_{A_{\mathcal{M}}}(\mathcal{P})$ the *tropical mutation variety*, $B(\mathcal{P})$ the *tropical mutation boundary*, and the pair $\big(X_{A_{\mathcal{M}}}(\mathcal{P}),\, B(\mathcal{P})\big)$ a *tropical mutation pair*. When no mutation is present, this construction specializes to the usual notion of a toric pair[1]. For the shearing polyptych lattice $\mathcal{M}_s$, we denote the tropical mutation surface by $X_f(\mathcal{P})$.

The second theorem studies tropical mutation pairs $(X_f(\mathcal{P}), B(\mathcal{P}))$ through the invariant known as the complexity (cf. Definition 5.1). In the toric case, every toric pair $(T(P), B)$ has complexity zero, independent of the choice of polytope P. We extend this to shearing tropical mutation surface pairs, showing that for $(X_f(\mathcal{P}), B(\mathcal{P}))$, the complexity is likewise independent of the choice of $\mathcal{P}$.

Theorem 1.2 (cf. Corollary 5.11)**.** Let $\mathcal{P}$ be a polytope in the rank-two shearing polyptych lattice $\mathcal{M}_s$ for $s \geq 1$, and let $(X_f(\mathcal{P}), B(\mathcal{P}))$ be a tropical mutation pair associated to degree s polynomial $f \in \mathbb{K}[y]$. Then

1. $B(\mathcal{P})$ support an effective ample divisor;
2. $\mathbb{G}_m \leq \operatorname{Aut}(X_f(\mathcal{P}), B(\mathcal{P}))$;
3. $\big(X_f(\mathcal{P}), B(\mathcal{P})\big)$ is a cluster type pair; and
4. the complexity of $(X_f(\mathcal{P}), B(\mathcal{P}))$ is equal to the number of distinct roots of f.

Cluster type varieties were introduced by Enwright–Figueroa-Moraga (cf. [7, Definition 2.26]) as a natural generalization of toric varieties from a birational perspective.

The third theorem provides a simple geometric characterization of tropical mutation surface pairs.

Theorem 1.3 (cf. Theorem 6.6)**.** Let (X, B) be an index one log Calabi–Yau surface pair with B supports an effective ample divisor and $\mathbb{G}_m \leq \operatorname{Aut}(X, B)$. Then (X, B) is a tropical mutation surface pair.

In the proof of Theorem 1.3, we use the toric degeneration $\pi_\alpha \colon \mathcal{X}_\alpha \longrightarrow \mathbb{A}^1$ constructed by Escobar–Harada–Manon [9, Theorem 7.22]. These families are trivial away from the origin: the general fiber is $X_{A_{\mathcal{M}}}(\mathcal{P})$, while the fiber over the origin is the toric variety $T(P_\alpha)$, where P_α denotes the chart image of $\mathcal{P}$.

Our fourth theorem constructs a family degenerating $T(P_1)$ and $T(P_2)$, and provides its description via the divisorial fan $\mathcal{S}_f$.

[1] i.e, the variety X is a projective toric variety and B is the reduced sum of the torus invariant divisors.

Theorem 1.4 (cf. Theorem 2.9, Theorem 7.4)**.** Let U_f be an affine tropical mutation variety associated with the polyptych lattice $\mathcal{M}_s$, and let $\mathcal{P} \subset \mathcal{M}_s$ be a polytope with chart images P_1 and P_2. Let $T(P_i)$ be the projective toric variety associated to P_i and $X_f(\mathcal{P})$ be the compactification of U_f associated to $\mathcal{P}$. Then the following hold:

(1) there exists a projective flat family $\pi_{f,\mathcal{P}}\colon \mathcal{X}_f(\mathcal{P}) \to \mathbb{P}^1$ whose special fibers at $\{0\}$ and $\{\infty\}$ are isomorphic to

$$\mathcal{X}_{f,0}(\mathcal{P}) \cong T(P_1), \qquad \mathcal{X}_{f,\infty}(\mathcal{P}) \cong T(P_2),$$

and general fiber is $X_f(\mathcal{P})$. Away from $\{0,\infty\}$ the family is a trivial.

(2) for each f there exists a divisorial fan $\mathcal{S}_f$ on $\mathbb{P}^1 \times \mathbb{P}^1$ such that

$$\mathcal{X}_f(\mathcal{P}) \cong X(\mathcal{S}_f).$$

and the composition $X(S_f) \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\mathrm{pr}_2} \mathbb{P}^1$ equals $\pi_{f,\mathcal{P}}$. Here pr_2 is the projection to the second coordinate.

Finally, we provide a complete description of the Cox rings of tropical mutation surfaces associated with the shearing polyptych lattices $\mathcal{M}_s$. As an application, we obtain a concrete combinatorial criterion that characterizes precisely when a tropical mutation surface $X_f(\mathcal{P})$ is toric.

Theorem 1.5 (cf. Theorem 8.5, Corollary 8.6)**.** Let $f(y) = \prod_{i=1}^{\gamma}(y-\alpha_i)^{\beta_i}$ be a degree s polynomial with $\alpha_i \in \mathbb{K}^*$, $\sum_{i=1}^{\gamma}\beta_i = s$, and $\prod \alpha_i = 1$. Let $\mathcal{P} \subset \mathcal{M}_s$ be a polytope defined by tropical points $p_i = (a_i, b_i, c_i), i = 1, \ldots, n$, where $c_1, \ldots, c_j \geq 0$ and $c_{j+1}, \ldots, c_n < 0$.

Let D_i is the irreducible component of the tropical mutation boundary corresponding to tropical point p_i and $C_1, \ldots, C_{2\gamma}$ are interior curves in U_f under compactification to $X_f(\mathcal{P})$. Then the following hold:

(1) the class group of $X_f(\mathcal{P})$ is isomorphic to

$$\mathbb{Z}\langle D_1,\ldots,D_n,C_1,\ldots,C_{2\gamma}\rangle \Big/ \left\langle \sum_{i=1}^{n} c_iD_i,\ \sum_{i=1}^{n} a_iD_i + \sum_{i=1}^{\gamma}\beta_iC_{2i},\ \sum_{i=1}^{j} -c_iD_i + C_{2k-1} + C_{2k} \ \Big|\ k = 1,\ldots,\gamma \right\rangle.$$

(2) the Cox ring of $X_f(\mathcal{P})$ is

$$\mathrm{Cox}(X_f(\mathcal{P})) \cong \mathbb{K}[w_1,\ldots,w_{n+2\gamma}] \Big/ \left\langle w_{n+2i-1}w_{n+2i} + \alpha_i\, w_1^{c_1}\cdots w_j^{c_j} - w_{j+1}^{-c_{j+1}}\cdots w_n^{-c_n} \ \Big|\ i = 1,\ldots,\gamma \right\rangle,$$

where w_i corresponds to $[D_i]$ for $1 \leq i \leq n$ and w_{n+i} corresponds to $[C_i]$ for $1 \leq i \leq 2\gamma$ under the grading of $\mathrm{Cl}(X_f(\mathcal{P}))$.

(3) the surface $X_f(\mathcal{P})$ is toric if and only if f is equivalent to $(y+1)^s$, and either:
   (a) all but one of the coefficients c_i are $+1$ and the others are nonpositive, or
   (b) all but one of the coefficients c_i are -1 and the others are nonnegative.

In earlier work, Haussen-Süß [13] and Altmann-Petersen [2] computed Cox rings of varieties with torus action by describing the corresponding polyhedral divisors. Using the language of $\mathbb{T}$-varieties one can immediately prove Theorem 1.5 from Theorem 1.4. Our approach is different, as we intend to highlight the combinatorial data of the tropical mutation variety and the tropical mutation boundary.

Acknowledgments. The author is grateful to Joshua Enwright, Laura Escobar, Megumi Harada, Nathan Ilten, Christopher Manon, Joaquín Moraga, and Burt Totaro for many helpful comments and suggestions. The author is especially grateful to Joaquín Moraga for his invaluable guidance and supervision throughout this project. The author thanks Nathan Ilten for many comments that helped to improve the content of this paper.

2. Preliminaries

Throughout the article, we assumed that $\mathbb{K}$ is an algebraically closed field of characteristic zero. We use the term polyptych lattice to mean a finite polyptych lattice.

We adopt the framework of polyptych lattices and detropicalized algebras developed in [9], and restate some of their consequences with minor adjustments in notation and conventions. For the complete construction, we refer the reader to [9].

Definition 2.1. (cf. [9, Definition 6.3]) Let $A_{\mathcal{M}}$ be the detropicalized algebra associated with a polyptych lattice $\mathcal{M}$. The *affine tropical mutation variety* is defined as $U_{A_{\mathcal{M}}} := \mathrm{Spec}(A_{\mathcal{M}})$.

Throughout this article, we consider finitely generated $\mathbb{K}$-detropicalized algebras, as opposed to the more general Noetherian $\mathbb{K}$-algebras considered in [9].

A detropicalized algebra associated with strictly dualizable polyptych lattice (cf. [9, Definition 4.1]) admits a natural $\mathbb{K}$-vector space basis, called an adapted basis (cf. [9, Definition 6.5]). In this article, we denote elements of the adapted basis by θ, using a notation that differs slightly from that of the original.

We use the term polytope in place of what is referred to as a PL polytope (cf. [9, Definition 5.1]) in the original literature. Unless explicitly stated otherwise, all polytopes will be assumed to be convex and integral.

Definition 2.2. (cf. [9, Theorem 7.6]) The *polytope algebra* is the graded algebra

$$A_{\mathcal{M}}^{\mathcal{P}} := \bigoplus_{k=0}^{\infty} \Gamma(U_{A_{\mathcal{M}}}, k\mathcal{P}) \cdot t^k.$$

The associated projective variety

$$X_{A_{\mathcal{M}}}(\mathcal{P}) := \mathrm{Proj}\, A_{\mathcal{M}}^{\mathcal{P}}$$

is called the *tropical mutation variety associated with* $\mathcal{P}$.

Definition 2.3. (cf. [9, Theorem 7.11]) Let $\mathcal{M}$ be a strictly dualizable polyptych lattice and $(\mathcal{M}, \mathcal{N}, \mathrm{v}, \mathrm{w})$ a strictly dualizable pair. $\mathcal{M}$ is *degenerable* if it is detropicalizable and there exists a strictly dual pair such that, for every $n_i \in \mathcal{N}$, there is a chart $\alpha \in I$ with $\pi_\alpha(\mathrm{v}(n_i))$ linear on M_α.

For the clarity of exposition, we use the term *tropical point* in place of what is referred to as a "point" in the original literature (cf. [9, Definition 3.1]). Now we establish a correspondence between polytopes and the boundary divisors of tropical mutation varieties.

Definition 2.4. Let $\mathcal{M}$ be a degenerable polyptych lattice, and let $\mathcal{P} \subset \mathcal{M}$ be a polytope bounded by inequalities of the form $p_i - \alpha_i$ for tropical points $p_i \in \mathrm{Sp}(\mathcal{M})$ and integers $\alpha_i \in \mathbb{Z}_{<0}$. A *Facet* is the codimension one subset $\mathcal{F}_i := \{m \in \mathcal{M}_{\mathbb{R}} \mid p_i(m) - \alpha_i = 0\}$.

Theorem 2.5. (cf. [9, Theorem 7.11]) Let $X_{A_{\mathcal{M}}}(\mathcal{P})$ be the tropical mutation variety associated with $\mathcal{P}$, and let $U_{A_{\mathcal{M}}}$ denote the corresponding affine tropical mutation variety. Then:

(1) $U_{A_{\mathcal{M}}}$ is a dense open subvariety of $X_{A_{\mathcal{M}}}(\mathcal{P})$;
(2) the complement $B(\mathcal{P}) := X_{A_{\mathcal{M}}}(\mathcal{P}) \backslash U_{A_{\mathcal{M}}}$ is a divisor;
(3) for each irreducible component $D_i \in \mathrm{Supp}(B(\mathcal{P}))$, there exists a facet $\mathcal{F}_i$ of $\mathcal{P}$ such that D_i is the vanishing locus of the adapted basis away from the facet $\mathcal{F}_i$; and
(4) $B(\mathcal{P})$ supports an effective ample divisor.

Definition 2.6. Let $\mathcal{P} \subset \mathcal{M}$ be a polytope. We define the divisor $B(\mathcal{P}) := X_{A_{\mathcal{M}}}(\mathcal{P}) \setminus U_{A_{\mathcal{M}}}$ and refer to it as the *tropical mutation boundary*. The pair $\big(X_{A_{\mathcal{M}}}(\mathcal{P}), B(\mathcal{P})\big)$ is called the *tropical mutation pair*. For each facet $\mathcal{F}_i$ of $\mathcal{P}$, we denote by D_i the irreducible component of $B(\mathcal{P})$ corresponding to $\mathcal{F}_i$, and refer to D_i as the *boundary component associated with* $\mathcal{F}_i$.

When the choice of $\mathcal{P}$ is clear, we will simply denote tropical mutation boundary as B. From now on, we will also assume all polyptych lattices are degenerable. Furthermore every facet is defined by an equation of the form $p_i - \alpha_i = 0$ for a tropical point $p_i \in \mathrm{Sp}(\mathcal{M})$ and $\alpha_i \in \mathbb{Z}_{<0}$.

Let $X_{A_\mathcal{M}}(\mathcal{P})$ be a tropical mutation variety whose affine tropical mutation variety $U_{A_\mathcal{M}}$ is a normal variety, and $\mathcal{P}$ is a normal polytope (cf. [9, Definition 7.13])[2] , then $X_{A_\mathcal{M}}(\mathcal{P})$ is embedded to the projective spaces by adapted basis.

Proposition 2.7. (cf. [9, Lemma 7.14 and Proposition 7.16]) Let $\mathcal{M}$ be a polyptych lattice and $\mathcal{N}$ its strict dual $\mathrm{w}\colon \mathcal{N} \to \mathrm{Sp}(\mathcal{M})$. Let $\mathcal{P} \subset \mathcal{M}$ be a normal polytope defined by the tropical points $\mathrm{w}(n_1) = \alpha_1, \ldots, \mathrm{w}(n_k) = \alpha_k$. Let D_i be the divisor associated to the tropical point $\mathrm{w}(n_i)$. If $A_\mathcal{M}$ is normal, then the following statements hold:

(1) $X_{A_\mathcal{M}}(\mathcal{P})$ is normal;
(2) the valuation $\mathrm{ord}_{D_i}\colon A_\mathcal{M} \backslash \{0\} \to \mathbb{Z}$ is the composition $\mathfrak{v} \circ n_i$; and
(3) for any integral polytope $\mathcal{P}$, the algebra $A^{\mathcal{P}}_{\mathcal{M}}$ is generated in degree 1.

Tropical mutation varieties admit several toric degenerations over $\mathbb{A}^1$. In Theorem 2.9 below, we combine these degenerations into a single family defined over projective space.

Theorem 2.8. (cf. [9, Theorem 7.21]) Let $\mathcal{M} = (\{M_\alpha\}_{\alpha\in I}, \{\mu_{\alpha,\beta}\}_{\alpha,\beta\in I})$ be a polyptych lattice, and let $\mathcal{P} \subset \mathcal{M}$ be a polytope. Write $X_{A_\mathcal{M}}(\mathcal{P})$ for the associated tropical mutation variety. For each chart M_α, let P_α denote the chart image of $\mathcal{P}$ and let $T(P_\alpha)$ be the toric variety associated with P_α. Then there exists a projective flat family

$$\mathcal{X}_\alpha \longrightarrow \mathbb{A}^1$$

which is trivial away from the origin, whose general fiber is isomorphic to $X_{A_\mathcal{M}}(\mathcal{P})$ and whose special fiber is the toric variety $\mathcal{X}_{\alpha,0} \cong T(P_\alpha)$.

Theorem 2.9. Let $\mathcal{M}$ be a polyptych lattice with lattice charts indexed by I with $|I| = m$. Let $\mathcal{P}$ be a polytope in $\mathcal{M}$. Let P_α be the polytope induced by $\mathcal{P}$ in the chart M_α with $\alpha \in I$. Then, there exists a flat projective family $\mathcal{X}_{A_\mathcal{M}}(\mathcal{P}) \to \mathbb{P}^{m-1}$ satisfying the following conditions:

(1) the fiber over the algebraic torus of $\mathbb{P}^{m-1}$ is isomorphic to $X_{A_\mathcal{M}}(\mathcal{P})$; and
(2) the fiber over a general point of the hyperplane H_α in $\mathbb{P}^{m-1}$ is isomorphic to $T(P_\alpha)$.

Proof. Let

$$A_{\mathcal{M},\tau_1,\ldots,\tau_m} := A_\mathcal{M}[\tau_1, \cdots, \tau_m]$$

be a multigraded detropicalized algebra obtained by adding m variables. For each chart $\alpha \in I$, define a valuation $\overline{\mathfrak{v}_\alpha}$ by extending the valuation $\tilde{\mathfrak{v}}_\alpha\colon A_\mathcal{M}[\tau_\alpha] \to \overline{\mathbb{Z}}$ which is defined in [9, lemma 7.8].

$$\overline{\mathfrak{v}_\alpha}\colon A_{\mathcal{M},\tau_1,\ldots,\tau_m} \longrightarrow \overline{\mathbb{Z}}.$$

$$f\tau_1^{k_1}\ldots\tau_m^{k_m} \in A_{\mathcal{M},\tau_1,\ldots,\tau_m}, \quad \overline{\mathfrak{v}_\alpha}(f\tau_1^{k_1}\cdots\tau_m^{k_m}) := \tilde{\mathfrak{v}}_\alpha(f\tau_\alpha^{k_\alpha}),$$

One can define the multi-index filtration of $\mathbf{k} := \{(k_1, \ldots, k_m) \mid k_i \in \mathbb{Z}_{\geq 0}\}$ with respect to the valuation $\overline{\mathfrak{v}_\alpha}$ by setting

$$F_{\leq\mathbf{k}} := \{f\tau_1\cdots\tau_m \in A_{\mathcal{M},\tau_1,\ldots,\tau_m} \mid \overline{\mathfrak{v}_\alpha}(f\tau_1\cdots\tau_m) \leq k_\alpha \text{ for all } \alpha \in I\}.$$

Using this filtration, we obtain an $\mathbb{K}[\tau_1, \ldots, \tau_m]$-algebra $\mathcal{R}^{\mathcal{P}}_{A_\mathcal{M}}$ such that

$$\mathcal{R}^{\mathcal{P}}_{A_\mathcal{M}} := \bigoplus_{\mathbf{k}\in\mathbb{Z}^n} F_{\leq\mathbf{k}} A^{\mathcal{P}}_{\mathcal{M},\tau_1,\ldots\tau_m}.$$

---

[2]Throughout this article we assume polytopes are normal. Note, every two-dimensional integral polytope is normal [6, Corollary 2.2.13], and by Proposition 3.1 shows that $A_{\mathcal{M}_s}$ is normal under any detropicalization.

By [3, Proposition 5.1] and [9, Lemma 7.9], we have

$$\mathcal{R}^{\mathcal{P}}_{A_{\mathcal{M}}}[\tau_1^{-1},\dots \hat{\tau_\alpha},\dots \tau_m^{-1}]/(\tau_\alpha) \cong S_\alpha[\tau_1^{-1},\dots \tau_m^{-1}],$$

for $T(P_\alpha) = \mathrm{Proj}(S_\alpha)$ and

$$\mathcal{R}^{\mathcal{P}}_{A_{\mathcal{M}}}[\tau_1^{-1},\dots,\hat{\tau}_\alpha,\dots,\tau_m^{-1}] \cong A^{\mathcal{P}}_{\mathcal{M}}[\tau_1^{-1},\dots,\tau_m^{-1}].$$

Let define $\mathcal{X}_{A_{\mathcal{M}}}(\mathcal{P}) := \mathrm{Proj}_{\mathbb{P}^{m-1}}(\mathcal{R}^{\mathcal{P}}{}_{A_{\mathcal{M}}})$ as $\mathbb{K}[\tau_1,\dots,\tau_m]$ algebra. The above argument shows that for each fiber over $\mathbb{G}_m^{m-1} := D(\tau_1\cdots\tau_m)$ on $\mathcal{X}_{A_{\mathcal{M}}}(\mathcal{P})$ is trivial and isomorphic to $X_{A_{\mathcal{M}}}(\mathcal{P})$. Moreover, for the fiber over $\mathbb{G}_{m,\alpha}^{m-2} := D(\tau_1\cdots\hat{\tau}_\alpha\cdots\tau_m)$ is the toric variety $T(P_\alpha)$. □

Based on the theorem above, we define the *global tropical mutation variety*.

Definition 2.10. Let $\mathcal{M}$ be a polyptych lattice with m charts, and let $\mathcal{P}\subset\mathcal{M}$ be a polytope. Define the Rees algebra $\mathcal{R}^{P}_{A_{\mathcal{M}}} := \bigoplus_{\mathbf{k}\in\mathbb{Z}^n_{\geq 0}} F_{\leq \mathbf{k}} A^{P}_{\mathcal{M},\tau_1,\dots,\tau_m}$, associated to the multi–index filtration $F_{\leq\mathbf{k}}$ constructed above. The *global tropical mutation variety associated with* $\mathcal{P}$ is the relative Proj

$$\mathcal{X}_{A_{\mathcal{M}}}(\mathcal{P}) := \mathrm{Proj}_{\mathbb{P}^{m-1}}(\mathcal{R}^{P}_{A_{\mathcal{M}}}).$$

In this article, we mainly study tropical mutation surfaces arising from the polyptych lattice with a single shear, introduced by Cook–Escobar–Harada–Manon [5]. Let's call tropical mutation surface associated to this polyptych lattices as *shearing tropical mutation surface.*

Definition 2.11. (cf. [5, Section 3]) For $s\in\mathbb{Z}_{>0}$[3] the *shearing polyptych lattice* $\mathcal{M}_s$ is defined by:

$$\mathcal{M}_s := (M_1, M_2, \mu_{1,2}), \qquad \mu_{1,2}(x,y) = \begin{cases}(-x,\,y), & y\geq 0,\\ (sy-x,\,y), & y\leq 0.\end{cases}$$

, We write an element of $\mathcal{M}_s$ as $((x,y)(x',y))\in M_1\times M_2$ with $x+x' = \min\{0, sy\}$. Since $\mathcal{M}_s$ is self-dual, every tropical point $p\in\mathrm{Sp}(\mathcal{M}_s)$ is represented by itself. Therefore we define a tropical point $\mathrm{w}((a_i,c_i),(b_i,c_i)) : \mathcal{M}_s\to\mathbb{Z}$. Let $((x,y)(x',y))\in\mathcal{M}_s$

$$\mathrm{w}((a_i,c_i),(b_i,c_i))(x,y)(x',y) = \begin{cases} c_i x - b_i y, & y\leq 0,\\ c_i x + a_i y, & y\geq 0,\end{cases} \qquad a_i+b_i = \min\{sc_i, 0\}.$$

3. Classification of affine tropical mutation varieties associated with $\mathcal{M}_s$

In this section, we study the automorphisms of tropical mutation surfaces and describe the moduli of affine tropical mutation varieties.

Proposition 3.1. For $\mathcal{M}_s$, detropicalized algebra $A_{\mathcal{M}_s}$ is isomorphic to

$$A_{\mathcal{M}_s} \cong A_f := \mathbb{K}[x_1,x_2,y^{\pm 1}]/\langle x_1x_2 - f(y)\rangle,$$

for some f with

$$f(y) = \prod_{i=1}^{\gamma}(y-\alpha_i)^{\beta_i}, \qquad \alpha_i\in\mathbb{K}^*, \quad \sum_{i=1}^{\gamma}\beta_i = s.$$

The affine variety $U_f = \mathrm{Spec}\, A_f$ has Du Val $A_{\beta_i - 1}$-singularities at the $(0,0,\alpha_i)$.

[3] The polyptych lattice $\mathcal{M}_s$ is defined for any integer s. However, $\mathcal{M}_s$ and $\mathcal{M}_{-s}$ are strongly isomorphic. Hence for simplicity of notation, we restrict to the case $s>0$. $\mathcal{M}_0$ is strongly isomorphic to the lattice.

Proof. By [5, Theorem 6.20], in the special case $f(y) = y^s + 1$ one obtains the detropicalization

$$A_{\mathcal{M}_s} \cong \mathbb{K}[x_1, x_2, y^{\pm 1}]/\langle x_1x_2 - f(y)\rangle.$$

The same argument applies verbatim if one replaces $y^s + 1$ by any polynomial $f(y) \in \mathbb{K}[y]$ of degree s with nonzero constant term. Indeed, since $\mathbb{K}[x_1, x_2, y^{\pm 1}]$ is a UFD of Krull dimension 3, while any detropicalization $A_{\mathcal{M}_s}$ has dimension 2. The kernel of Φ must be a height one prime ideal, generated by $x_1x_2 - f(y)$.

The type of singularities is clear from the local expression. □

Each lattice point $(a, b) \in M_i$ corresponds to an element of the adapted basis of A_f, represented in terms of three variables x_1, x_2, y as follows:

	M_1	M_2
$b \geq 0$	$x_1^b y^{-a}$	$x_1^b y^{a}$
$b \leq 0$	$x_2^{-b} y^{sb-a}$	$x_2^{-b} y^{a}$

The tropical mutation surfaces U_f admits two torus embeddings:

$$j_1 \colon \mathbb{G}_m^2 \longrightarrow U_f, \qquad (x, y) \longmapsto \left(x, \tfrac{f(y)}{x}, y\right),$$
$$j_2 \colon \mathbb{G}_m^2 \longrightarrow U_f, \qquad (x, y) \longmapsto \left(\tfrac{f(y)}{x}, x, y\right).$$

Proposition 3.2. Let $f, g \in \mathbb{K}[y]$ with $\deg(f) = \mathrm{def}(g) = s$ with nonzero constant term. Then $A_f \cong A_g$ if and only if there exist $\lambda, c \in \mathbb{K}^*$ such that $\lambda f(cy) = g(y)$ or $\lambda y^s f(cy) = g(y)$.

Proof. Let $\Phi : A_f \to A_g$ be the ring isomorphism. Any ring isomorphism sends a unit to a unit, and since g is degree s with having nonconstant term, the difference between the lowest degree term and highest degree term of $\deg\Phi(f(y))$ has to be exactly s. Hence

$$\Phi(y) = cy^{\pm 1} \quad \text{of} \quad c \in \mathbb{K}^*.$$

For x_1 and x_2, the isomorphism permutes generators and preserves relations, so

$$\Phi(x_1) = u_1 x_i \quad , \quad \Phi(x_2) = u_2 x_j, \quad u_1, u_2 \in (\mathbb{K}[y^{\pm}])^* \quad \text{and} \quad \{x_i, x_j\} = \{x_1, x_2\}.$$

Then Φ is an isomorphism of the ring if

$$u_1u_2x_1x_2 - f(cy^{\pm 1}) = u_3(x_1x_2 - g(y)) \quad \text{for } u_3 \in (\mathbb{K}[y])^*.$$

Hence $A_f \cong A_g$ then $\lambda f(cy) = g(y)$ or $\lambda y^s f(cy) = g(y)$ for some $\lambda, c \in \mathbb{K}^*$.

The converse follows by the same construction. □

Definition 3.3. Let $f(y) \in \mathbb{K}[y]$ be a polynomial of degree s, and let μ_s denote the group of s th roots of unity. The *automorphism group of f* is defined by

$$\mathrm{Aut}(f) := \{\, (\epsilon, c) \in \{0, 1\} \times \mu_s \;\mid\; f(y) = y^{s\epsilon} f(cy^{(-1)^\epsilon}) \,\}.$$

Note that $\mathrm{Aut}(f)$ is a subgroup of the dihedral group D_{2s}.

Corollary 3.4. Let U_f be the affine tropical mutation variety associated with f. Then

$$\mathrm{Aut}(U_f) \cong (\mathbb{G}_m \times \mathbb{Z}) \rtimes (\mu_2 \times \mathrm{Aut}(f)).$$

Proof. Any automorphism Φ maps x_1 to $\lambda y^k x_i$ for some $\lambda \in \mathbb{G}_m$, $k \in \mathbb{Z}$, and $x_i \in \{x_1, x_2\}$. Here μ_2 denotes the involution swapping x_1 and x_2, while preserving f. The group $\mu_2 \times \mathrm{Aut}(f)$ acts on $(\mathbb{G}_m \times \mathbb{Z})$ by conjugation. Since an automorphism is determined by the images of x_1, x_2, and y, the automorphism group has the stated structure. □

We write $X_f(\mathcal{P})$ for the projective tropical mutation variety arising from the affine tropical mutation variety U_f and a polytope $\mathcal{P} \in \mathcal{M}_s$. We study the automorphism group of projective tropical mutation surfaces in the following lemma.

Lemma 3.5. Let U_f be the affine tropical mutation variety associated to f. Let $\mathcal{P} \subset \mathcal{M}_s$ be a polytope. Denote by $X_f(\mathcal{P})$ the tropical mutation variety associated to $\mathcal{P}$. Then

$$\mathbb{G}_m \times \mathrm{Aut}(f) \ \leqslant\ \mathrm{Aut}(X_f(\mathcal{P})).$$

Proof. Let $G := \mathbb{G}_m \times \mathrm{Aut}(f)$, and consider the action of $(\lambda, \epsilon, c) \in G$ defined by

$$x_1 \mapsto \lambda x_1, \qquad x_2 \mapsto \lambda^{-1} x_2, \qquad y \mapsto c y^{\epsilon},$$

which preserves the defining relation $x_1 x_2 - f(y) = 0$ so does the torus embedding $j_i \colon \mathbb{G}_m^2 \hookrightarrow U_f$.

We claim that the G-action extends to every irreducible component D_i of the tropical mutation boundary B.

Now consider the projective embedding

$$\Phi_{\mathcal{P},i} \colon U_f \longrightarrow D_i \subset \mathbb{P}^m, \qquad t \longmapsto [0 : \cdots : \theta_j(t) : \cdots : \theta_k(t) : \cdots : 0],$$

for an adapted basis $\{\theta_j, \ldots, \theta_k\}$ corresponding to the facet $\mathcal{F}_i$. G acts on the adapted basis $\theta \in \mathbb{B}$ by $(g \cdot \theta)(t) = \theta(g \cdot t)$. As the G-action preserves torus embedding, $\theta(t) \neq 0$ implies $\theta(g \cdot t) \neq 0$, we have

$$g \cdot t \ \longmapsto\ [0 : \cdots : \theta_j(g \cdot t) : \cdots : \theta_k(g \cdot t) : \cdots : 0] \ \in\ D_i.$$

Therefore, the G-action extends to each boundary component D_i. □

Corollary 3.6. Let $\mathcal{M}_s$ be a shearing polyptych lattice and $\mathcal{P}$ be a polytope. Let $X_f(\mathcal{P})$ be the tropical mutation variety associated to $\mathcal{P}$. If $X_f(\mathcal{P})$ is a smooth del Pezzo surface, then $X_f(\mathcal{P})$ is toric surface.

Proof. Smooth non-toric del Pezzo surfaces have a finite automorphism group. On the other hand, by Lemma 3.5 a shearing tropical mutation surface has infinite automorphism group. Hence shearing tropical mutation surface cannot be both smooth and non-toric. □

Definition 3.7. The coarse moduli space of detropicalizations of $\mathcal{M}$, denoted $\mathfrak{Det}(\mathcal{M})$, is the set of isomorphism classes of detropicalizations of $\mathcal{M}$.

Theorem 3.8. Let $s \geq 1$. The coarse moduli space of detropicalizations of the polyptych lattice $\mathcal{M}_s$ is given by the quotient

$$\mathfrak{Det}(\mathcal{M}_s) \ \cong\ \mathbb{A}^{s-1}/D_{2s},$$

where D_{2s} denotes the dihedral group generated by the cyclic subgroup μ_s of s-th roots of unity and a reflection s.

The group D_{2s} acts on the coordinate space $\mathbb{A}^{s-1} = \operatorname{Spec} \mathbb{K}[b_1, \ldots, b_{s-1}]$ by

$$\zeta \cdot (b_1, \ldots, b_{s-1}) = (\zeta b_1,\ \zeta^2 b_2,\ \ldots,\ \zeta^{s-1} b_{s-1}) \qquad (\zeta \in \mu_s),$$

and

$$s \cdot (b_1, \ldots, b_{s-1}) = (b_{s-1},\ b_{s-2},\ \ldots,\ b_1).$$

Proof. Let $f(y) = a_s y^s + a_{s-1} y^{s-1} + \cdots + a_1 y + a_0$ be a degree s polynomial with $a_s, a_0 \neq 0$. By Proposition 3.2, rescaling y and $f(y)$ by a torus action and preserves the isomorphism class of the associated affine tropical mutation variety. Rescaling the equation by $\lambda := a_0^{-1}$ normalizes the constant term to 1, and replacing y by $c\,y$ with $c^s = a_0/a_s$ normalizes the leading coefficient. Thus f may be reduced to the form

$$f(y) = y^s + b_{s-1} y^{s-1} + \cdots + b_1 y + 1.$$

Furthermore, by Proposition 3.2, the isomorphism class is preserved by reversing the order of the coefficient.

$$(b_1, \ldots, b_{s-1}) \longmapsto (b_{s-1},\, b_{s-2},\, \ldots,\, b_1).$$

The normal form is unique up to the action of $y \mapsto \zeta y$ with $\zeta \in \mu_s$, which transforms the coefficients by

$$(b_1, \ldots, b_{s-1}) \longmapsto (\zeta b_1,\, \zeta^2 b_2,\, \ldots,\, \zeta^{s-1} b_{s-1}).$$

Therefore the set of isomorphism classes is parameterized by orbits of this D_{2s} action, i.e. by the quotient of $\mathbb{A}^{s-1}/D_{2s}$. □

4. Singularities of tropical mutation varieties

In this section we study the singularities of tropical mutation varieties from both geometric and combinatorial perspectives.

Definition 4.1. Let $\mathcal{P}$ be a polytope in $\mathcal{M}$. The *PL vertex* of $\mathcal{P}$, consists of those vertices of $\mathcal{P}$ that remain vertices in every chart image P_α.

Definition 4.2. A log pair (X, B) is said to be *log Calabi–Yau pair* if $K_X + B = 0$ and (X, B) is log canonical. The complement of the boundary $U := X \setminus B$ is called *log Calabi Yau variety.* When X has a divisor B such that (X, B) is a log Calabi-Yau pair, then we will call X a *Calabi-Yau type variety.*

Proposition 4.3. Let $\mathcal{P}$ be a normal polytope in the degenerable polyptych lattice

$$\mathcal{M} := (\{M_\alpha\}_{\alpha \in I}, \{\mu_{\alpha,\beta}\}_{\alpha,\beta \in I}).$$

Consider the tropical mutation variety associated to $\mathcal{P}$, and let $\mathcal{F}_i$ be a facet of $\mathcal{P}$ with corresponding irreducible divisor D_i. Let $\mathcal{X}_\alpha \to \mathbb{A}^1$ be a degeneration of $X_{A_\mathcal{M}}(\mathcal{P})$ to the toric variety $T(P_\alpha)$, which is trivial away from the origin. Then:

(1) D_i degenerates to a (possibly reducible) component $D_{i,0}$ of the toric boundary;
(2) the tropical mutation variety $X_{A_\mathcal{M}}(\mathcal{P})$ has rational singularities along the boundary. Moreover, if there exists a chart α such that P_α is a smooth (resp. Gorenstein) polytope, then $X_{A_\mathcal{M}}(\mathcal{P})$ is smooth (resp. Gorenstein) along the boundary;
(3) the singularities of on the boundary of the tropical mutation surface appear at PL vertices of $\mathcal{P}$;
(4) $(X_{A_\mathcal{M}}(\mathcal{P}), B)$ is a log Calabi–Yau pair; and
(5) when $\mathcal{M}$ has rank two, singularities on the boundary are at most cyclic quotient singularities.

Proof. Let $D_i \subset \mathrm{Supp}(B)$ be an irreducible component of the boundary. By Proposition 2.7, there exist a projective embedding given by adapted basis

$$\Phi_\mathcal{P} : X_{A_\mathcal{M}}(\mathcal{P}) \longrightarrow \mathbb{P}^n.$$

Each divisor D_i is defined using lattice points $m_1, \ldots, m_\ell$ lying in the facet $\mathcal{F}_i$. In coordinates,

$$\Phi_{\mathcal{P},i} : U_{A_\mathcal{M}} \longrightarrow D_i, \qquad t \longmapsto [0 : \cdots : \theta_{m_1}(t) : \cdots : \theta_{m_\ell}(t) : 0 : \cdots : 0].$$

On the central fiber, this degenerates to a possibly non-irreducible divisor $D_{i,0}$, defined by

$$\Phi_{P_\alpha,i} \colon \mathbb{G}_m \longmapsto D_{i,0}, \qquad t \longmapsto [0 : \cdots : \chi_{m_{1,\alpha}}(t) : \cdots : \chi_{m_{\ell,\alpha}}(t) : 0 : \cdots : 0],$$

with $m_{j,\alpha} := \pi_\alpha(m_j)$. By [9, Lemma 7.9], the graded algebra of $A_\mathcal{M}$ with respect to the valuation $\mathfrak{v}_{\alpha,\rho}$ is the semigroup algebra of P_α. By [19, Theorem 1], the adapted basis θ_{m_j} degenerates to $\chi_{m_{j,\alpha}}$.

For (2), recall that normal toric varieties have at worst rational singularities (cf. [6, Theorem 11.4.2]), and that smoothness and Gorensteinness of a projective toric variety $T(P_\alpha)$ are determined by a polytope (cf. [6, Definition 2.4.3], [6, Proposition 8.2.12]). Since tropical mutation varieties degenerate to a toric variety $T(P_\alpha)$, and the toric degeneration $\mathcal{X}_\alpha \to \mathbb{A}^1$ preserves singularities in a Zariski neighborhood (cf. [16, Section 9.1]), the claim follows.

For (3), if a vertex of the polytope is not a vertex of some chart image P_α, then by (2) the corresponding point of $X_{A_\mathcal{M}}(\mathcal{P})$ is smooth. Thus singularities occur precisely at the PL vertices.

For (4), the log Calabi–Yau condition of the boundary $(X_{A_\mathcal{M}}(\mathcal{P}), B)$ follows from the openness of log Calabi–Yau pairs (cf. [11, Lemma 8.42]). Indeed, in (1) we showed that the tropical mutation boundary degenerates to a toric boundary, and toric pairs satisfy the required hypotheses. Hence the result extends to the tropical mutation case.

For (5), by the classification of two-dimensional log canonical pairs, the only possible singularities on the reduced boundary are cyclic quotient singularities (cf. [21, Section 3.40]). ☐

Proposition 4.4. Let $U_{A_\mathcal{M}}$ be an affine tropical mutation variety with at most log terminal singularities and $X_{A_\mathcal{M}}(\mathcal{P})$ is $\mathbb{Q}$-factorial. Then $X_{A_\mathcal{M}}(\mathcal{P})$ is of Fano type and hence a Mori dream space.

Proof. By Theorem 2.5 (4), B supports an ample divisor so that there exist $a_i \geq 0$ with $\sum a_i D_i$ ample. Choose an integer $N \gg 0$ so that $a_i/N < 1$ for all i. Set $\Delta := -K_{X_{A_\mathcal{M}}(\mathcal{P})} - \frac{1}{N}A$. so that $-K_{X_{A_\mathcal{M}}} - \Delta$ is ample and $(X_{A_\mathcal{M}}(\mathcal{P}), \lceil\Delta\rceil)$ is a log canonical pair. We claim $(X_{A_\mathcal{M}}(\mathcal{P}), \Delta)$ is klt so that this pair is log Fano pair. Take a log resolution $f : Y \to X_{A_\mathcal{M}}(\mathcal{P})$ then

$$K_Y = f^*K_{X_{A_\mathcal{M}}(\mathcal{P})} + \sum a_i E_i, \quad \tilde{B} := f^*B - \sum b_i E_i$$

for $\tilde{B}$ the strict transformation of B then

$$K_Y = f^*(K_{X_{A_\mathcal{M}}(\mathcal{P})} + \lceil\Delta\rceil) + \sum (a_i - b_i)E_i - \lceil\tilde{\Delta}\rceil.$$

Since $(X_{A_\mathcal{M}}, \lceil\Delta\rceil)$ is log canonical, we have $a_i - b_i \geq -1$. On the other hand, the relation $\tilde{\Delta} = f^*\Delta - \sum t_i D_i$ implies $t_i < b_i$ at the log canonical center, so that $a_i - t_i > a_i - b_i \geq -1$. At the klt center in affine tropical mutation varieties, it is automatically klt. Hence $(X_{A_\mathcal{M}}(\mathcal{P}), \Delta)$ is a klt pair.

Fano type varieties are Mori dream spaces (cf. [4, Corollary 1.3.1]), the claim follows. ☐

Combining with the above result and Proposition 3.1 with Proposition 4.3 (5), $X_f(\mathcal{P})$ is a Mori dream space.

For tropical mutation surfaces associated with $\mathcal{M}_s$, we obtain a more explicit description of their singularities in terms of the PL vertices of the polytope $\mathcal{P}$.

Proposition 4.5. Let $X_f(\mathcal{P})$ be the shearing tropical mutation variety associated to a polytope $\mathcal{P}$ with chart images P_α. Fix a PL vertex m_k lying the interior of a linear domain, and let $\pi_\alpha(m_k)$ denote its image in the toric chart P_α. Let U_k (resp. $U_{k\alpha}$) for the affine chart of $X_f(\mathcal{P})$ (resp. of $T(P_\alpha)$) centered at m_k (resp. $\pi_\alpha(m_k)$). Denote by

$$\mathfrak{m} = \left(\tfrac{\theta_1}{\theta_k}, \ldots, \tfrac{\theta_m}{\theta_k}\right), \qquad \mathfrak{m}_\alpha = \left(\tfrac{\chi_1}{\chi_k}, \ldots, \tfrac{\chi_m}{\chi_k}\right)$$

the maximal ideals of $\mathcal{O}_{X_f(\mathcal{P})}(U_k)$ and $\mathcal{O}_{P(T_\alpha)}(U_{k\alpha})$, respectively. Then the completed local rings at $\mathfrak{m}$ and $\mathfrak{m}_\alpha$ are canonically isomorphic:

$$\widehat{\mathcal{O}_{X_f(\mathcal{P})}(U_k)} \;\cong\; \widehat{\mathcal{O}_{P(T_\alpha)}(U_{k\alpha})}.$$

Proof. Observe that

$$\mathcal{O}(U_k) \;\cong\; \mathbb{K}\left[\tfrac{\theta_1}{\theta_k}, \ldots, \tfrac{\theta_m}{\theta_k}\right],$$

since each affine chart is defined by the image of the affine embedding $\left(\frac{\theta_1}{\theta_k}, \ldots, \frac{\theta_m}{\theta_k}\right)$.

Once a linear domain is fixed, the adapted basis $\{\theta_i\}$ and the toric characters $\{\chi_i\}$ differ only by multiplication with $f(y)^{\pm 1}$, which is invertible after localization. Hence, for the Hilbert basis $\chi_1, \ldots, \chi_t$ associated to the vertex m_k, they are contains in the same linear domain and so as the corresponding adapted basis $\theta_1, \dots, \theta_t$. Thus one has an isomorphism of completed algebras:

$$\widehat{\mathbb{K}\left[\frac{\chi_1}{\chi_k}, \dots, \frac{\chi_t}{\chi_k}\right]} \cong \widehat{\mathbb{K}\left[\frac{\theta_1}{\theta_k}, \dots, \frac{\theta_t}{\theta_k}\right]}.$$

It remains to check that, after the completion local $\mathbb{K}\left[\frac{\theta_1}{\theta_k}, \dots, \frac{\theta_t}{\theta_k}\right]_{\mathfrak{m}}$-algebra, the ring $\mathbb{K}\left[\frac{\theta_1}{\theta_k}, \dots, \frac{\theta_m}{\theta_k}\right]$ is isomorphic to the completion above. If m_i lies in the same linear domain as m_k, then by [6, Proposition. 2.1.8] θ_i is generated by $\{\theta_1, \dots, \theta_t\}$. If m_i lies in a different domain, then the relation $x_1x_2 = f(y)$, with y invertible, shows that the adapted generators differ from the toric ones only by multiplication with a unit.

Therefore the completed local rings coincide, and the claim follows. □

5. The complexity of tropical mutation surface pairs

In this section, we compute the complexity of the tropical mutation pair $(X_f(\mathcal{P}), B)$ and prove that it is a cluster type pair.

Definition 5.1. (cf. [7, Definition 2.14]) Let X be a projective $\mathbb{Q}$-factorial variety with Picard rank $\rho(X)$. Let B be a Weil divisor such that (X, B) a log canonical pair and $-(K_X + B)$ is nef. The *complexity* of the pair (X, B) is defined as

$$c(X, B) := \dim X + \rho(X) - |B|.$$

Definition 5.2.
(1) Let (X, B) be a projective log Calabi–Yau surface pair. A point $p \in X$ is said to be *nodal* if it lies in the intersection of two irreducible components of the log Calabi–Yau boundary.
(2) Let (T, B_T) be a toric pair. The points $\{p_1, \dots, p_s\}$ are said to be *collinear* if p_i is not nodal and they all lie on a single irreducible torus-invariant divisor of B_T, with repetitions permitted.

The following lemma highlights the distinction between collinear blow ups and nodal blow ups.

Lemma 5.3. Let T be a toric surface with reduced toric boundary B. Let $p_1, \dots, p_s$ be s distinct collinear points of T, and let $\pi\colon \widetilde{T} \to T$ be the blow up at these points. Let $\widetilde{B}$ denote the strict transform of B in $\widetilde{T}$. Then:

(1) the pair $(\widetilde{T}, \widetilde{B})$ is a log Calabi–Yau pair of complexity s;
(2) the $\mathbb{G}_m$–action on T lifts to $\widetilde{T}$; and
(3) the weighted blow up at nodal point $\pi\colon (X, B) \to (\widetilde{T}, \tilde{B_T})$ preserves the complexity. Here B is the total transformation of B_T by π.

Proof. For (1), a collinear blow up does not change the number of boundary components, but increases the Picard rank by s. For (2), the $\mathbb{G}_m$–action lifts since the stabilizer subgroup persists under blow up. For (3), the log Calabi–Yau property is preserved under weighted blow ups [22, Proposition 6.39], and the claim then follows from (1) and (2). □

We count the $\mathbb{A}^1$-curves in the affine tropical mutation surface U_f for computing the complexity of the pair $(X_f(\mathcal{P}), B)$. Then we study how each C_i meets the tropical mutation boundary B. The following elementary facts will be used.

Lemma 5.4. Let U_f be an affine tropical mutation variety with $f(y) = \prod_{i=1}^{\gamma}(y - \alpha_i)^{\beta_i}$. Then U_f contains exactly 2γ embedded copies of $\mathbb{A}^1$.

Explicitly, these $\mathbb{A}^1$-curves are given by

$$C_{2i-1} = \{(x_1, x_2, y) \in U_f \mid x_1 = 0, y = \alpha_i\} \quad \text{and} \quad C_{2i} = \{(x_1, x_2, y) \in U_f \mid x_2 = 0, y = \alpha_i\}.$$

We refer to these curves as *interior curves*. Throughout this article, the notation C_i will be used both for the curves on U_f and for their images under the projective embedding into $\mathbb{P}^m$.

Definition 5.5. Let $\mathcal{P} \subset \mathcal{M}_s$ be a polytope.

(1) A face $\mathcal{E} \subset \mathcal{P}$ is called a *sink* if $\mathcal{E}$ consists of the points of $\mathcal{P}$ having the maximal y coordinate.
(2) A face $\mathcal{E} \subset \mathcal{P}$ is called a *source* if $\mathcal{E}$ consists of the points of $\mathcal{P}$ having the minimal y coordinate.

By the convexity of the polytope, the sink and source are uniquely defined. A source or sink corresponds to an irreducible component of the tropical mutation boundary. We also use the terms source and sink to refer to the corresponding strata of the tropical mutation boundary.

Proposition 5.6. Let $X_f(\mathcal{P})$ be a tropical mutation variety and $\{C_1, \dots C_{2\gamma}\}$ be interior curves. Then:

(1) when i is odd, then C_i intersect with sink; and
(2) when i is even then C_i intersect with source.

Proof. Using adapted bases $\{\theta_0, \dots, \theta_m\} \subset \Gamma(U_f, \mathcal{P})$, we embed C_{2i-1} into projective space $\mathbb{P}^m$ as in Proposition 2.7, where the first coordinate θ_0 corresponds to the adapted basis element correspond to the origin of $\mathcal{P}$. In the affine chart $\theta_0 = 1$, we have

$$C_{2i-1}(t) = (1, \theta_1(t, 0, \alpha_i), \dots, \theta_m(t, 0, \alpha_i)), \quad t \in \mathbb{A}^1. \tag{5.1}$$

Let k be the maximal y–coordinate of a lattice point of P_2. Passing to homogenize coordinate with respect to $\mathbb{P}^1$, we obtain

$$C_{2i-1}([\tau_0 : \tau_1]) = \left[\tau_1^k : \tau_1^k \theta_1\left(\frac{\tau_0}{\tau_1}, 0, \alpha_i\right) : \dots : \tau_1^k \theta_m\left(\frac{\tau_0}{\tau_1}, 0, \alpha_i\right)\right], \quad [\tau_0 : \tau_1] \in \mathbb{P}^1. \tag{5.2}$$

Taking $[\tau_0 : \tau_1] = [1 : 0]$, we obtain an intersection point on the tropical mutation boundary B and the only nonvanishing terms in (5.2) correspond to the adapted basis elements consists the maximal y-weight. Hence C_{2i-1} intersect with sink. When sink is divisor, then Equation (5.2) shows that C_{2i-1} intersects sink at colinear points.

The argument for (2) is analogous, with the roles of maximal and minimal vertical coordinates interchanged. □

Proposition 5.7. Let $\mathcal{P}$ be a polytope on the shearing polyptych lattice $\mathcal{M}_s$. Let $X_f(\mathcal{P})$ be the associated tropical mutation variety and let B denote the tropical mutation boundary and $\{C_1, \dots, C_{2\gamma}\}$ be the interior curves. Then

(1) if the sink D_k is a divisor, then the interior curves C_{2i-1} meet D_k transversally;
(2) if the sink $\{p\}$ is a nodal point, then for some coprime integers (p, q), the weighted blow up
$$\pi\colon \big(Y, B_Y\big) \longrightarrow \big(X_f(\mathcal{P}), B\big),$$
where $Y = \mathrm{Bl}_{(p,q)}(X_f(\mathcal{P}))$ and B_Y is the total transform of the weighted blow up. Then the strict transforms $\widetilde{C}_{2i-1}$ meet B_Y transversally.

Proof. By (5.2), in this case the interior curves do not intersect at the nodal point. The tangent direction of D_j is parallel to the facet $\mathcal{F}_i$, whereas the tangent direction of C_{2i-1} is determined by the adapted basis element corresponding to the second smallest y–coordinate. Since these directions are distinct, the intersection is transversal.

For the second statement, Proposition 4.5 implies that the type of singularities remains constant under toric degeneration. Moreover, (5.2) shows that each C_i is locally isomorphic to a curve connecting a source to a sink. Such curves can be separated by a weighted blow up that inserts the ray corresponding to e_2 in both P_1 and P_2. The weight is uniquely determined in the chart image, and applying the weighted blow up with this weight separates the curves at the nodal point. □

Theorem 5.8. Let $\mathcal{P} \subset \mathcal{M}_s$ be a polytope in the shearing polyptych lattice, and $X_f(\mathcal{P})$ denote the corresponding tropical mutation surface with tropical mutation boundary B. Then the following statements hold:

(1) there exists a crepant resolution of singularities of U_f

$$\psi\colon (\widetilde{X_f(\mathcal{P})}, \widetilde{B}) \longrightarrow (X_f(\mathcal{P}), B);$$

(2) if the sink is a divisor, then there exists a toric pair (T, B_T) together with an s collinear blow up

$$\pi : \widetilde{X_f(\mathcal{P})} \longrightarrow T;$$

(3) if the sink is a nodal point, then there exists a nodal weighted blow up

$$\phi : T' \longrightarrow \widetilde{X_f(\mathcal{P})},$$

where T' is obtained by blowing up s collinear points on a toric variety T.

Proof. The interior singularities are Du Val, hence they admit a crepant resolution; therefore, (1) holds.

For (2), after resolving all singularities, there are in total s exceptional curves in $\widetilde{U_f}$ with γ many (-1)-curves and $(s-\gamma)$ many (-2)-curves. Since B is an anticanonical divisor, the adjunction formula implies that the strict transforms of the interior curves $\widetilde{C_k}$ are (-1)-curves. By Castelnuovo's contraction theorem, (-1)-curves are contractible, and after contracting them inductively, the (-2)-curves become (-1)-curves, so eventually all exceptional curves are contractible:

$$(\widetilde{X_f(\mathcal{P})}, \widetilde{B}) \longrightarrow (T, B_T),$$

with B_T is the image of $\tilde{B}$. After contracting all exceptional curves, the complement $T \setminus B_T$ is isomorphic to $\mathbb{G}_m^2$. By the negativity lemma, the pair (T, B_T) is log Calabi–Yau. Hence (T, B_T) is a log Calabi–Yau surface of complexity zero, i.e., toric.

For the last statement, applying Proposition 5.7, we obtain a log Calabi–Yau boundary $\widetilde{B}$ such interior curves and $\widetilde{B}$ meet transversally. Denote this variety by T'. By (1), so that T' is the s collinear blow up of the fixed torus invariant divisor on the toric boundary B_T. □

Definition 5.9. Let $X_f(\mathcal{P})$ be a shearing tropical mutation variety. A toric variety $T(P)$ with a crepant birational map

$$\varphi : X_f(\mathcal{P}) \dashrightarrow T(\mathcal{P})$$

is called a *toric model* of $X_f(\mathcal{P})$ if it can be obtained as a composition of the following operations:

(1) ψ^{-1} : resolution of interior canonical singularities;
(2) ϕ^{-1} : a weighted blow up at a sink if sink is a nodal point;
(3) π contraction of s exceptional curves on the fixed boundary.

Definition 5.10. (cf. [8, Definition 2.23]) A log Calabi–Yau pair (X, B) is of *cluster type* if there exists a toric log Calabi–Yau pair (T, B_T), and a crepant birational map $\varphi\colon (T, B_T) \dashrightarrow (X, B)$ that extracts only log canonical places of (X, B). A variety X is of *cluster type* if it admits a log Calabi–Yau pair (X, B) of cluster type.

Cluster type pair generalize toric pair for having common features, such as log rationality and constructibility (cf. [17, Theorem 1.2]), yet cluster type pair may have complexity strictly greater than 0.

Corollary 5.11. Let $\mathcal{P}$ be a polytope in the rank-two shearing polyptych lattice $\mathcal{M}_s$ for $s > 0$, and let $(X_f(\mathcal{P}), B)$ be a tropical mutation pair associated to degree s polynomial $f \in \mathbb{K}[y]$ Then

(1) B support an effective ample divisor;

(2) $\mathbb{G}_m \leq \mathrm{Aut}(X_f(\mathcal{P}), B)$;
(3) B supports least two irreducible components;
(4) $\big(X_f(\mathcal{P}), B\big)$ is a cluster type pair; and
(5) the complexity of $(X_f(\mathcal{P}), B)$ is equal to the number of distinct roots of f.

Proof. (1) and (2) are the consequence of Theorem 2.5 and Lemma 3.5. For (3), the surface $X_f(\mathcal{P})$ is obtained as an s-collinear blow up of a toric surface, possibly after a single additional blow up at a nodal point. Since the toric boundary of the surface has at least three irreducible components, the resulting tropical mutation surface pair $(X_f(\mathcal{P}), B)$ has at least two boundary components.

For (3) and (4), in the proof of Theorem 5.8, we constructed a crepant birational map to the toric model

$$(X_f(\mathcal{P}), B) \dashrightarrow (T, B_T),$$

which only contracts divisors of coefficient 1. Hence $(X_f(\mathcal{P}), B)$ is a cluster type pair.

Furthermore, we have a birational factorization

$$(T, B_T) \xleftarrow{\pi} (T', B'_T) \xrightarrow{\phi} (\widetilde{X_f(\mathcal{P})}, \widetilde{B}) \xrightarrow{\psi} (X_f(\mathcal{P}), B).$$

Here π reduces the complexity by s; since (T, B_T) is a toric pair, the complexity of (T', B'_T) is s. ϕ does not change the complexity by Lemma 5.3. Finally, ψ corresponds to blow up the interior $\sum_{i=1}^{\gamma}(\beta_i - 1)$ times, which increases the complexity by exactly this amount. Putting these together, the resulting complexity of $(X_f(\mathcal{P}), B)$ is γ. $\square$

6. Cluster type surfaces as tropical mutation surfaces

Corollary 5.11 imply that a tropical mutation surface pair associated with a shearing polyptych lattice is a log Calabi–Yau pair whose boundary both supports an ample divisor. In this section, we show that log Calabi–Yau surface pairs with these properties are precisely tropical mutation surface pairs.

In this section, we assume that all surfaces are normal, projective, and $\mathbb{Q}$–factorial.

The following proposition is a standard result in intersection theory (cf. [10, Corollary 10.1]).

Proposition 6.1. Let $\mathcal{P}$ be a polytope in the shearing polyptych lattice $\mathcal{M}_s$. Assume $\mathcal{P}$ is defined by the facets $\mathcal{F}_1, \dots \mathcal{F}_n$ and let $D_1, \dots D_n$ be the corresponding irreducible component of the tropical mutation boundary B. Let $X_f(\mathcal{P})$ be the tropical mutation variety associated with $\mathcal{P}$. Let P_α be one of the chart images of $\mathcal{P}$.

Then, the toric degeneration $X_f(\mathcal{P}) \rightsquigarrow T(P_\alpha)$ induces an isometry between the intersection product of the tropical mutation boundary and the toric boundary

$$\Phi : N^1(X_f(\mathcal{P})) \longrightarrow N^1(T(P_\alpha)),$$
$$\left\langle \sum_{i=1}^n a_i D_i, \sum_{i=1}^n b_i D_i \right\rangle_{X_f(\mathcal{P})} = \left\langle \Phi\left(\sum_{i=1}^n a_i D_i\right), \Phi\left(\sum_{i=1}^n b_i D_i\right) \right\rangle_{T(P_\alpha)}.$$

This proposition shows that, by choosing an appropriate chart image of a polytope, one can recover the intersection theoretic data of the tropical mutation boundary from the associated toric degenerations. More precisely, when a divisor degenerates as $D_i \rightsquigarrow D_{i,0} + D'_{i,0}$, the contributions of the degenerated components allow us to determine all intersection numbers of the boundary of $X_f(\mathcal{P})$.

Proposition 6.2. Let U_f be an affine tropical mutation variety and $X_f(\mathcal{P})$ be a tropical mutation variety. Let m_k be a vertex of $\mathcal{P}$ in the linear domain of the polyptych lattice. Then

(1) the weighted nodal blow up of $X_f(\mathcal{P})$ at the node corresponding to m_k yields a tropical mutation surface compactifying U_f; and

(2) the toric model $T(P)$ is uniquely determined by the ray structure of the chart images of $\mathcal{P}$.

Proof. To prove (1), observe that the interior is unaffected by the nodal operation, so it suffices to construct a corresponding polytope $\mathcal{P}'$ that yields the desired compactification. By Proposition 4.5, the neighborhood of such a vertex is locally isomorphic to its toric degeneration. Combinatorially, we define a new polytope $\mathcal{P}'$ by inserting a new ray v in the chart image P_1 and the mutation image of v in P_2. We claim that the tropical mutation variety $X_f(\mathcal{P}')$ coincides with the weighted blow up of $X_f(\mathcal{P})$ at the corresponding nodal point.

If the chosen vertex is neither a sink nor a source, the claim follows immediately: inserting the ray induces the corresponding weighted blow up without affecting the vertex intersecting C_i.

If the chosen vertex is a sink or a source, then by Proposition 5.7 the interior curves C_i are locally isomorphic to the curves joining the sink and source, and the ray operation modifies the vertices compatibly under the weighted blow up.

For (2) Let $\widetilde{X_f(\mathcal{P})}$ be the resolution of the canonical singularities in the interior of $X_f(\mathcal{P})$. If $\widetilde{X_f(\mathcal{P})}$ is obtained by an s–collinear blow up along a divisor D_k, then the configuration of self-intersection numbers of the tropical mutation surfaces is

$$(D_1^2, \dots, D_k^2, \dots, D_n^2).$$

In this case, its toric model has the self-intersection sequence

$$(D_1^2, \dots, D_k^2 + s, \dots, D_n^2).$$

Since a toric variety is uniquely determined by its ray structure, and these self-intersection numbers determines the ray structure of the polytope. From Proposition 6.1, charts contains enough information to determine the intersection number of the tropical mutation boundary, the claim follows.

If $\widetilde{X_f(\mathcal{P})}$ is not obtained by such a blow up, then performing a weighted blow up corresponding to the insertion of the ray e_2 in the first chart as in (1) reduces the situation to the case of an s–collinear blow up of a toric variety. Hence the toric model $T(\mathcal{P})$ is uniquely determined by the ray structures of the charts P_1 and P_2. ☐

Proposition 6.3. Let $T(P)$ be a toric surface with toric boundary B_T, and let $D_1 \subset B_T$ be a boundary component. Then the s collinear blow up of $T(P)$ along D_1 and possibly with a contraction of D_1, followed by the contraction of all interior (-2)–curves, yields a tropical mutation pair (X, B).

Proof. Let $T(P)$ be a toric variety with ray structure $\Sigma = \{v_1, \dots, v_n\}$. We will construct a tropical mutation variety $X_f(\mathcal{P})$ with s collinear blow up along D_1, correspond to the ray v_1.

Note that the choice of f is determined by the point of s collinear points blowing up on $D_1^\circ \simeq \mathbb{G}_m$ together with data of the polarization as in (5.2).

We enumerate the rays counterclockwise with $v_1 = e_2$ with v_j lies in the fourth quadrant (positive x– and negative y–coordinates), so that v_i lies in the first quadrant for $i > j$, while for some index l with v_l in the second quadrant (negative x– and positive y–coordinates), the vectors v_i lie in the third or fourth quadrants for $l < i \leq j$.

Let $\mu^T := \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$. Define the first chart of the polytope is

$$\Sigma_1 = \{(\mu^T)^{-1}v_1, \dots, (\mu^T)^{-1}v_j, (\mu^T)^{-1}v_{j+1}, v_{j+1}, \dots, v_n\},$$

and, up to orientation, the second chart is

$$\Sigma_2 = \{(\mu^T)^{-1}v_1, \dots, (\mu^T)^{-1}v_l, v_l, v_{l+1}, \dots, v_n\}.$$

Since the global $\mathrm{GL}_2(\mathbb{Z})$–action preserves intersection products and e_2 is an eigenvector of $(\mu^T)^{-1}$, all intersection products are preserved, except those involving D_1 and the degenerate components.

It is straightforward to compare Σ and Σ_1. The self–intersection of D_1 changes by

$$D_1^2 - s = (D_{1,0})^2.$$

Let D_{j+1} degenerate into two irreducible components of the torus invariant divisor $D_{j+1,0}$ and $D_{\mu^{-1}j+1,0}$. Each of them correspond to the ray $v_{j+1}, (\mu^T)^{-1}v_{j+1}$. Then $D_{j+1}^2 = (D_{j+1,0}+D_{\mu^{-1}j+1,0})^2$ so that intersection product preserved. One can compute the self–intersection number of the divisor $D_{j+1,0}+D_{\mu^{-1}j+1,0}$ by using the two wall relations [6, Equation (6.4.4)] involving v_{j+1} and $(\mu^T)^{-1}v_{j+1}$ in Σ_1. Comparing with the original wall relation of v_{j+1} in Σ, one verifies that the self–intersection numbers of D_{j+1} and $(D_{j+1,0} + D_{\mu^{-1}j+1,0})$ agree.

By Proposition 3.1, tropical mutation surfaces do not contain interior (-2)-curves, so contract them in the case there is. The contraction preserve tropical mutation varieties as in Proposition 6.2, in case we need to contract D_1, we can remove $(\mu^T)^{-1}v_1$. □

We provide a birational geometric characterization of shearing tropical mutation surfaces.

Proposition 6.4. Let (X, B) be an index one log Calabi–Yau surface pair with B supports an effective ample divisor; and $\mathbb{G}_m \leq \mathrm{Aut}(X, B)$. Then (X, B) is a cluster type pair.

Proof. As B supports an effective ample divisor, X is a Fano type variety (cf. Proposition 4.4); hence X is rationally connected by [20, Corollary 1.6], and X is rational. Note that B is also not irreducible. The complement of the boundary is affine as the boundary supports an ample divisor. As U is affine and admit effective $\mathbb{G}_m$-action, there exists a $\mathbb{G}_m$–semi-invariant function f on U that satisfies $t \cdot f = \chi(t)f$ for some character χ. Extending f to the boundary and considering the limits as t to 0 and t to ∞, we see that f has zeros and poles on distinct irreducible components of the boundary. Hence the boundary cannot be a single irreducible divisor. Hence by, by Looijenga's classification of anticanonical cycles on rational surfaces [23], it follows that B is a cycle of rational curves.

Let $\mu\colon (W, B_W) \to (X, B)$ be a dlt modification, which is an isomorphism over the klt locus. In this case, the dlt modification is obtained by blowing up boundary nodes. We first observe that all such operations on B preserve the property that the boundary supports an ample divisor.

Let A be an ample divisor supported on B, and set $A_W := \mu^*(A)+\sum \epsilon_i E_i$, where the E_i are μ–exceptional divisors and the coefficients $\epsilon_i \in \mathbb{Q}_{>0}$ are sufficiently small. By the Nakai–Moishezon criterion, A_W is ample.

For the $\mathbb{G}_m$–action on W, by Rosenlicht's theorem (cf. [25, Theorem 2]), there exists a rational quotient $W \dashrightarrow \mathbb{P}^1$ whose general fiber is $\mathbb{P}^1$. Moreover, since B_W has ample support, the complement $U_W := W \setminus B_W$ is an affine variety $\mathrm{Spec}(R)$. Then, by [24, Corollary 5.5.4], the affine quotient $\mathrm{Spec}(R^{\mathbb{G}_m})$ exists. It follows that the indeterminacy locus is contained in the boundary.

To resolve the indeterminacy of the rational map $W \dashrightarrow C$, we perform a sequence of blow ups with centers contained in B_W. By the preceding discussion, such nodal blow ups preserve the property that B_W supports an ample divisor. Therefore, after replacing X by a suitable birational model W, we may assume that the map $W \dashrightarrow \mathbb{P}^1$ is in fact a morphism $W \to \mathbb{P}^1$ whose general fiber is $\mathbb{P}^1$.

By resolving interior singularities and contracting all (-1)–curves in the fibers, we obtain the relative minimal model $S \to \mathbb{P}^1$, S is isomorphic to a Hirzebruch surface. The anticanonical divisor B_S may consist of a cycle of rational curves or an irreducible curve of arithmetic genus one. However, the only curves preserved under the $\mathbb{G}_m$–action are the zero section, the infinity section, and the ruling curves. Hence the boundary cannot be irreducible, and (S, B_S) is a toric pair. Therefore, (X, B) is a cluster type pair. □

Remark 6.5. The first part of the argument completely fails once the $\mathbb{Q}$-factorial hypothesis is dropped. Let S be the projective cone over an elliptic curve, and let E denote the zero section. Then (S, E) forms a log Calabi–Yau pair by the adjunction formula. Furthermore E is a hyperplane section and supports an ample divisor. However, the vertex of the cone is not a $\mathbb{Q}$-factorial singularity, it is a strictly log canonical singularity. The $\mathbb{G}_m$–action on S preserves the elliptic curve E, yet the log Calabi–Yau boundary consists of a single irreducible component. Moreover, S itself is not rational.

Theorem 6.6. Let (X, B) be an index one log Calabi–Yau surface pair, such that B supports an effective ample divisor and $\mathbb{G}_m \leq \mathrm{Aut}(X, B)$. Then (X, B) is a tropical mutation surface pair.

Proof. When there is an effective $\mathbb{G}_m^2$-action, the resulting pair is simply a toric pair. So in this cases it is trivial.

First of all, Since (X, B) is a cluster type pair, there exists a crepant birational map

$$\varphi\colon (T, B_T) \dashrightarrow (X, B)$$

from a toric pair, obtained by contracting a log canonical center of X. Without loss of generality, we may assume that (X, B) is dlt, by replacing it with its dlt modification $(\tilde{X}, \tilde{B})$ obtained by resolving the interior klt singularities of X. Since this process preserves the $\mathbb{G}_m$–action equivariantly, we may further assume that φ is the contraction of log canonical centers. The interior singularities are canonical, since collinear blow ups do not affect the log Calabi–Yau structure of the pair. Hence, we may contract them crepantly. By Proposition 5.1 of [7], these singularities are specifically of A-type singularities.

The map of the pair defines a map from the open torus to the log Calabi–Yau variety. We claim that the restriction

$$\varphi|_{\mathbb{G}_m^2} : \mathbb{G}_m^2 \dashrightarrow U$$

is an inclusion of varieties. Indeed, if this were not the case, there would be indeterminacy on the torus; however, by construction of the dlt model, φ^{-1} only extracts log canonical centers and is otherwise an isomorphism. Thus φ induces a torus inclusion $\mathbb{G}_m^2 \hookrightarrow U$, realizing a $\mathbb{G}_m$-action on U. This action extends to a subgroup embedding

$$\mathbb{G}_m \hookrightarrow \mathrm{Aut}(\mathbb{G}_m^2)$$

The outer automorphism group of $\mathbb{G}_m^2$ is discrete, hence the $\mathbb{G}_m$–action extends uniquely to T equivariantly.

Now consider a divisor that contains a log canonical center of X. Since the $\mathbb{G}_m$–action preserves the boundary, this center must be $\mathbb{G}_m$–invariant, otherwise, log canonical center cannnot be discrete.

Therefore, there are at most two divisors of B containing log canonical centers of φ. By an elementary transformation, we can assume we extracted log canonical center from single irreducible divisor from B. This is an inverse operation of s collinear blow of of some toric variety. Now apply Proposition 6.3, the statement follows. □

7. Toric degenerations of tropical mutation varieties

In this section, we describe the global tropical mutation varieties $\mathcal{X}_f(\mathcal{P})$ in terms of divisorial fans. As a first step, we construct the global tropical mutation variety associated with the shearing polyptych lattice

Example 7.1. Let $X_f(\mathcal{P})$ be the shearing tropical mutation surface associated to an integral polytope $\mathcal{P} \subset \mathcal{M}_s$. We construct a global tropical mutation variety $\mathcal{X}_f(\mathcal{P})$ by $\mathrm{Proj}_{\mathbb{P}^1} \mathcal{R}_f^{\mathcal{P}}$ for

$$\mathcal{R}_f^{\mathcal{P}} := \mathbb{K}[\tau_0, \tau_1][X_{a,b} \mid (a, b) \in P_2] \Big/ \left\langle x_1 x_2 - \prod_{i=1}^{\gamma} (\tau_0 y - \tau_1 \alpha_i)^{\beta_i} \right\rangle, \qquad X_{a,b} := \begin{cases} x_1^a y^b, & b \geq 0, \\ x_2^{-b} y^a, & b < 0, \end{cases}$$

defined over $\mathbb{K}[\tau_0, \tau_1]$. We denote by $\mathcal{R}_{f,[\alpha:\beta]}^{\mathcal{P}}$ the specialization of $\mathcal{R}_f^{\mathcal{P}}$ at $[\tau_0 : \tau_1] = [\alpha : \beta]$.

The general fibers are mutually isomorphic, and the projective variety $\mathrm{Proj}\, \mathcal{R}_f^{\mathcal{P}}$ admits two toric degenerations. Comparing the two limits with the table in Definition 2.11, we obtain

$$\mathcal{R}^{\mathcal{P}}_{f,[0:1]} \cong \mathbb{K}[x^b y^a t \mid (a,b) \in P_2], \qquad \mathcal{R}^{\mathcal{P}}_{f,[1:0]} \cong \mathbb{K}[x^b y^a t \mid (a,b) \in P_1].$$

These graded rings give the desired toric varieties appear as fibers.

We now relate $\mathcal{R}^{\mathcal{P}}_f$ to the Ilten pencil interpolating the polytopes P_1^* and P_2^*. Recall that $P_1^*, P_2^* \subset N_{\mathbb{R}}$ are integral polytopes that are combinatorially mutation equivalent (cf. [18, Definition 2.1]). Ilten proved that there exists a flat family $X(\mathcal{S}_s) \to \mathbb{P}^1$ with special fibers $T(P_1)$ and $T(P_2)$ at $\{0\}$ and $\{\infty\}$, respectively (cf. [14, Theorem 2.8]). This projective family is constructed via Ilten-Vollmert framework [15] by finding divisorial fan $\mathcal{S}_s$. By varying the p-divisors of the affine pieces changes the general fibers while keeping the special fibers fixed. This choice of p-divisors corresponds to the choice of f in the global tropical mutation variety.

We recall some basic properties of convex polytopes and their duals, which will be used to show the existence of the Ilten pencil interpolating between P_1 and P_2 in the chart of the polytope $\mathcal{P}$. The following lemma is well known in the literature (cf. [26, Theorem 2.11]).

Lemma 7.2. Let P be a convex integral polytope and containing the origin in the interior. Let P^* be the polar dual of P. Then

(1) P^* is rational convex polytope;
(2) $(rP)^* = \frac{1}{r}P^*$. for the rational number $r \in \mathbb{Q}$; and
(3) $P = P^{**}$.

Proposition 7.3. Let $\mathcal{P} \subset \mathcal{M}_s$ be a polytope in the rank–two shearing polyptych lattice $\mathcal{M}_s$, and let μ_s be the associated mutation. Let $P_1 \subset M_1$ and $P_2 \subset M_2$ be the chart images of $\mathcal{P}$, and let N_1, N_2 be the dual lattices of M_1, M_2, respectively. Write $P_1^* \subset N_1$ and $P_2^* \subset N_2$ for the polar duals of P_1, P_2. Then there exist a width vector e_1 and a factor H_s^1 such that

$$\mu_{e_1,H_s^1}(P_1^*) \;=\; P_2^*,$$

for μ_{e_1,H_s^1} the combinatorial mutation. Namely P_1^* and P_2^* are related by a combinatorial mutation.

Proof. Let $P_1 \subset \mathbb{R}^n$ be a rational polytope with $0 \in \mathrm{int}(P_1)$, and let P_1^* be its polar dual. By Lemma 7.2(2), $(\lambda P_1)^* = \lambda^{-1} P_1^*$ for all $\lambda > 0$. Choose $k \in \mathbb{Z}_{>0}$ so that kP_1^* is lattice integral, and replace P_1 by kP_1. Thus we may assume that P_1 is rational and P_1^* is integral.

Following [1, Proposition 2.19], fix the factor and width vector

$$F := H_s^1 := \mathrm{Conv}\left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\s\end{pmatrix} \right\}, \qquad w := e_1 := \begin{pmatrix}1\\0\end{pmatrix}.$$

Suppose $\mu_{e_1,H_s^1}(P_1^*)$ is a combinatorial mutation of P_1^* of given data. Then there is a piecewise linear map

$$\phi_{e_1,H_s^1} \colon M_1 \longrightarrow M_2, \qquad u \longmapsto u - \min(0, su_2)\, e_1,$$

such that

$$\phi_{e_1,H_s^1}(P_1) = (\mu_{e_1,H_s^1}(P_1^*))^*,$$

again by [1, Proposition 2.19]. Notice that $\phi_{e_1,H_s^1} = \mu_s$ defined in Example 2.11. Hence by Lemma 7.2

$$P_2 = (\mu_{e_1,H_s^1}(P_1^*))^*, \qquad P_2^* = \mu_{e_1,H_s^1}(P_1^*).$$

□

This proposition provides an Ilten pencil $X(\mathcal{S}_s)$ interpolating between $T(P_1)$ and $T(P_2)$. We now study the relation to the tropical mutation variety.

Theorem 7.4. Let $\mathcal{P} \subset \mathcal{M}_s$ be a polytope, and P_1, P_2 be the chart images of $\mathcal{P}$, and let P_1^*, P_2^* be their polar duals. Let tropical mutation variety $X_f(\mathcal{P})$ with $f(y) = \prod_{i=1}^{\gamma}(y-\alpha_i)^{\beta_i}$, and $X(\mathcal{S}_s)$ be the Ilten pencil interpolating P_1 and P_2. Then the following hold:

(1) there exists an isomorphism as $\mathbb{P}^1$-variety $X(\mathcal{S}_s) \cong \mathcal{X}_{(y+1)^s}(\mathcal{P})$; and
(2) there exists a divisorial fan $\mathcal{S}_f$ on $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathcal{X}_f(\mathcal{P}) \cong X(\mathcal{S}_f)$.

Proof. Let $N \cong \mathbb{Z}\langle e_1, e_2\rangle$. Then H_s^1 is contained in the subspace $\mathbb{Z}\langle e_2\rangle$. Set $N' = \mathbb{Z}\langle e_2\rangle$ and let $M := \mathrm{Hom}(N', \mathbb{Z})$. Denote by P_1', P_2' the images in M'. Let Γ be a diagonal divisor in $\mathbb{P}^1 \times \mathbb{P}^1$.

By Proposition 7.3, there exists an Ilten pencil that interpolating $T(P_1)$ and $T(P_2)$, so there is polyhedra $\Delta_0, \Delta_\infty \in N'$ such that

$$\mathcal{D}_s^+ \ := \ \Delta_0 \otimes [0 \times \mathbb{P}^1] - H_s^1 \otimes \Gamma, \qquad \mathcal{D}_s^- \ := \ -H_s^1 \otimes \Gamma + \Delta_\infty \otimes [\infty \times \mathbb{P}^1],$$

$$\mathcal{S}_s := \{\mathcal{D}_s^+, \mathcal{D}_s^-\},$$

and $X(\mathcal{S}_s)$ is the Ilten pencil. Fix a point $[\tau_0 : \tau_1] \in \mathbb{P}^1$, and define $X(\mathcal{S}_{s,[\tau_0:\tau_1]})$ to be the variety associated with the restriction of the divisorial fan $\mathcal{S}_s$ to the fiber over $[\tau_0 : \tau_1]$. Then, by construction,

$$X(\mathcal{S}_{s,0}) \cong T(P_1), \qquad X(\mathcal{S}_{s,\infty}) \cong T(P_2).$$

From the gluing of the divisorial fan we obtain

$$X(\mathcal{D}_s^+) \cap X(\mathcal{D}_s^-) = X(\mathcal{D}_s^+ \cap \mathcal{D}_s^-) = \mathrm{Spec}_{\mathbb{P}^1} \bigoplus_{b \in P_1'} \Gamma\big(\mathbb{P}^1, \mathcal{O}(-H_s^1(b) \cdot \Gamma)\big)\, x^b.$$

Evaluating the p-divisor $H_s^1 \otimes [\tau_0 : \tau_1]$ at each fiber with $b \in P_1'$, we have

$$H_s^1(b) \otimes [\tau_0 : \tau_1] \ = \ \min(0, sb) \cdot [\tau_0 : \tau_1].$$

$$x^b(\tau_0 y - \tau_1)^{-\min(0,sb)} \in \Gamma(\mathbb{P}^1, \mathcal{O}(-H_s^1(b)) \cdot \Gamma) x^b.$$

Thus as $\mathbb{K}[\tau_0, \tau_1]$-algebra, $\bigoplus_{b \in P_1'} \Gamma\big(\mathbb{P}^1, \mathcal{O}(-H_s^1(b) \cdot \Gamma)\big)\, x^b$ is generated by $1, x_1 := x$ and $x_2 := (\tau_0 y - \tau_1)^s x^{-1}$. There is a relation

$$x_1 x_2 \ = \ (\tau_0 y - \tau_1)^s.$$

Moreover, for $b \in kP_1$, one has

$$y^a(\tau_0 y - \tau_1)^{-\min\{0,sb\}} x^b \ \in \ \Gamma(\mathbb{P}^1, \mathcal{D}_s^+(b)) x^b$$

for some $a \in \mathbb{Z}$. At the special fibers, we get graded ring

$$\mathbb{K}\big[x^b y^a t^k \ \mid \ (a,b) \in kP_1,\ a, b \in \mathbb{Z},\ k > 0\big], \qquad \mathbb{K}\big[x^b y^a t^k \ \mid \ (a,b) \in kP_2,\ a, b \in \mathbb{Z},\ k > 0\big],$$

and by specialization to the central fiber we recover $y^a x^b \in kP_2$. It follows that

$$X(\mathcal{D}_s^+) = \ \mathrm{Spec}_{\mathbb{P}^1}\ \mathbb{K}[\tau_0, \tau_1]\big[\, X_{a,b} \ \mid \ (a,b) \in P_2,\ a \geq 0\,\big],$$

$$X(\mathcal{D}_s^-) = \ \mathrm{Spec}_{\mathbb{P}^1}\ \mathbb{K}[\tau_0, \tau_1]\big[\, X_{a,b} \ \mid \ (a,b) \in P_2,\ a \leq 0\,\big],$$

$$X_{a,b} := \begin{cases} x_1^{\,b}\, y^{\,a}, & b \geq 0, \\ x_2^{-b}\, y^{\,a}, & b < 0. \end{cases}$$

Upon gluing and affine pieces with a compatible grading with the projective degeneration, we obtain

$$X(\mathcal{S}_s) \ = \ \mathrm{Proj}_{\mathbb{P}^1}\, \mathbb{K}[\tau_0, \tau_1][\, X_{a,b} t \ \mid \ (a,b) \in P_2\,] / \langle\, x_1 x_2 - (\tau_0 y - \tau_1)^s \,\rangle.$$

This corresponds precisely to the choice $f = (y+1)^s$ in Example 7.1.

For part (2) we use the same polyhedra Δ_0 and Δ_∞ as above. Let

$$f(y) := \prod_{i=1}^{\gamma}(y-\alpha_i)^{\beta_i}, \qquad H_1^1 := \operatorname{Conv}\left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}.$$

For each $i = 1, \dots, \gamma$, define the graph divisor

$$\Gamma_i \ := \ \left\{ ([x:y],[\tau_0:\tau_1]) \in \mathbb{P}^1_x \times \mathbb{P}^1_\tau \;\middle|\; \tau_0 y - \alpha_i \tau_1 x = 0 \right\}.$$

Using these divisors, we define the divisorial fan

$$\mathcal{S}_f := \left\{ \Delta_0 \otimes [0 \times \mathbb{P}^1] \ - \ \sum_{i=1}^{\gamma} \beta_i\, H_1^1 \otimes \Gamma_i, \quad -\sum_{i=1}^{\gamma} \beta_i\, H_1^1 \otimes \Gamma_i \ + \ \Delta_\infty \otimes [\infty \times \mathbb{P}^1] \right\}.$$

For $b \in P_1'$, we write $H_1^1(b) = \min\{0, b\}$. Then the contribution from the coefficient along Γ_i gives the global section

$$\prod_{i=1}^{\gamma}(\tau_0 y - \alpha_i\tau_1)^{\min\{0,-\beta_i b\}} x^b \ \in \ \Gamma\Bigg(\mathbb{P}^1, \mathcal{O}\Bigg(- \sum_{i=1}^{\gamma} \beta_i\, H_1^1(b)\cdot[\tau_0 : \alpha_i\tau_1]\Bigg)\Bigg)\, x^b.$$

Proceeding as in part (1), we obtain a graded $\mathbb{K}[\tau_0,\tau_1]$–algebra whose $\operatorname{Proj}_{\mathbb{P}^1}$ gives the interpolating family between the toric varieties:

$$\mathcal{R}_f(P_1,P_2) := \mathbb{K}[\tau_0,\tau_1][X_{a,b}t \mid (a,b) \in P_2] \Big/ \left\langle x_1x_2 - \prod_{i=1}^{\gamma}(\tau_0 y - \tau_1\alpha_i)^{\beta_i} \right\rangle, \quad X_{a,b} := \begin{cases} x_1^{\,b} y^{\,a}, & b \geq 0, \\ x_2^{-b} y^{\,a}, & b < 0. \end{cases}$$

This coincides with $\mathcal{R}_f^{\mathcal{P}}$ in Example 7.1.

□

8. Cox rings of tropical mutation surfaces

In this section, we study the Cox rings of tropical mutation varieties. We give explicit presentations for the Cox rings of tropical mutation surfaces $X_f(\mathcal{P})$, describing their generators and relations in terms of the data defining the polytope $\mathcal{P}$.

Lemma 8.1. Let

$$A_f = \mathbb{K}[x_1,x_2,y^{\pm 1}] \Big/ \left(x_1x_2 - \prod_{i=1}^{\gamma}(y-\alpha_i)^{\beta_i} \right), \qquad U_f = \operatorname{Spec} A_f.$$

Then

$$\operatorname{Cl}(U_f) \ \cong \ \mathbb{Z}^{\gamma-1} \oplus \mathbb{Z}/\gcd(\beta_1,\dots,\beta_\gamma).$$

Let $X_f(\mathcal{P})$ be a compactification of U_f with irreducible boundary divisors $D_1,\dots,D_n$, and $\{C_1,\dots C_{2\gamma}\}$ are interior curves. Then the is a surjective map

$$\mathbb{Z}\langle D_1,\dots,D_n,\, C_1,\dots,C_{2\gamma}\rangle \twoheadrightarrow \operatorname{Cl}\big(X_f(\mathcal{P})\big).$$

Proof. Set $f(y) = \prod_{i=1}^{\gamma}(y-\alpha_i)^{\beta_i}$. Consider the localization $A_f[f(y)^{-1}] \ \cong\ \mathbb{K}[x_2^{\pm1}, y^{\pm1}]_f$, which is a UFD. Under this localization, the interior curves are removed. Hence $\operatorname{Cl}(U_f)$ is generated by the classes $\{C_k\}$ to the relations coming from the zeros of the principal divisors of x_1, x_2, and $y-\alpha_i$:

$$C_{2i-1} + C_{2i} \ \sim \ 0 \quad (i=1,\dots,\gamma), \qquad \sum_{i=1}^{\gamma} \beta_i\, C_{2i-1} \ \sim \ 0.$$

It follows that

$$\mathrm{Cl}(U_f) \;\cong\; \mathbb{Z}^{\gamma-1} \oplus \mathbb{Z}/\gcd(\beta_1, \dots, \beta_\gamma).$$

Finally, by [12, Proposition 6.5], the following exact sequence gives desired surjection

$$\bigoplus_i \mathbb{Z}\langle D_i \rangle \;\longrightarrow\; \mathrm{Cl}(X_f(\mathcal{P})) \;\longrightarrow\; \mathrm{Cl}(U_f) \;\longrightarrow\; 0.$$

□

The next lemma computes the orders of vanishing and poles of rational functions, yielding a description of the class group and the effective cone of the tropical mutation surfaces.

Lemma 8.2. Let $\mathcal{P}$ be a polytope in the rank two shearing polyptych lattice $\mathcal{M}_s$, and let $X_f(\mathcal{P})$ be the associated tropical mutation surface.

Assume $\mathcal{P}$ is defined by the tropical points $p_i = (a_i, b_i, c_i)$ for $i = 1, \dots, n$, with $c_i \geq 0$ for $1 \leq i \leq j$ and $c_i < 0$ for $j + 1 \leq i \leq n$. Let $D_1, \dots, D_n$ be the prime components of the tropical mutation boundary, and $C_1, \dots, C_{2\gamma}$ are interior curves. Then there is an isomorphism $\mathbb{K}(X_f(\mathcal{P})) \simeq \mathbb{K}(x, y)$ with

$$\begin{aligned}\deg(x) &= (a_1, \dots, a_n, 0, \beta_1, \dots, 0, \beta_\gamma),\\ \deg(y) &= (-c_1, \dots, -c_n, 0, \dots, 0),\\ \deg(y - \alpha_i) &= (-c_1, \dots, -c_j, 0, \dots, 1, 1, 0, \dots),\end{aligned}$$

where $\deg = (\mathrm{ord}_{D_1}, \dots, \mathrm{ord}_{D_n}, \mathrm{ord}_{C_1}, \dots, \mathrm{ord}_{C_{2\gamma}})$.

Proof. This follows directly from Proposition 2.7 (2) by computing the orders of vanishing of x_2, y, and $y - \alpha_i$ along the boundary divisors $\{D_1, \dots, D_n\}$. The chosen isomorphism $\mathbb{K}(X_f(\mathcal{P})) \simeq \mathbb{K}(x, y)$ corresponds to identifying $\mathbb{K}(x, y) \cong \mathbb{K}(x_2, y)$. □

Corollary 8.3. Let A_f be the detropicalized algebra associated to f of the shearing polyptych lattice $\mathcal{M}_s$ of rank two. Let $\mathcal{P}$ be a polytope in $\mathcal{M}_s$. Then, the class group of $X_f(\mathcal{P})$ is isomorphic to

$$\mathbb{Z}\langle D_1, \dots, D_n, C_1, \dots, C_{2\gamma}\rangle / \left\langle \sum_{i=1}^n c_i D_i,\ \sum_{i=1}^n a_i D_i + \sum_{i=1}^\gamma \beta_i C_{2i},\ \sum_{i=1}^j -c_i D_i + C_{2k-1} + C_{2k} \ \mid\ k = 1, \dots, \gamma \right\rangle.$$

Proof. We have a surjective homomorphism

$$\bigoplus_{i=1}^n \mathbb{Z}\langle D_i \rangle \oplus \bigoplus_{i=1}^{2\gamma} \mathbb{Z}\langle C_i \rangle \;\twoheadrightarrow\; \mathrm{Cl}(X_f(\mathcal{P})).$$

The kernel is generated by principal divisors $\mathrm{div}(g) \in \mathbb{Z}\langle D_1, \dots, D_n, C_1, \dots, C_{2\gamma}\rangle$ for $g \in \mathbb{K}(x, y)$. g has the form

$$g := x^{k_1} y^{k_2} \prod (y - \alpha_i)^{k_i},$$

since otherwise g vanishes somewhere on $U_f \setminus (C_1 \cup \cdots \cup C_{2\gamma})$. The result then follows from the degree formulas in Lemma 8.2. □

Lemma 8.4. Let U_f be the affine tropical mutation variety associated with $f(y)$. Let $\mathcal{P}$ be a polytope in $\mathcal{M}_s$, and let $X_f(\mathcal{P})$ be the corresponding compactification. Let $D_1, \dots, D_n$ be the irreducible components of the tropical mutation boundary B, and let $C_1, \dots, C_{2\gamma}$ are interior curves. Let $\{F_i\}$ be effective irreducible divisors whose support does not contain any of the D_j or C_j. For each i, let $f_i \geq 0$. Then:

(1) there exists $p \in \mathbb{K}[x, y]$ with $\mathrm{div}(p) = \sum f_i F_i$;

(2) there exists $r \in \mathbb{K}(X_f(\mathcal{P}))$ with

$$\mathrm{div}(r) = \sum f_i F_i + \sum d_j D_j, \quad d_j \leq 0;$$

(3) the effective cone of $X_f(\mathcal{P})$ is generated by $\{D_1, \dots, D_n, C_1, \dots, C_{2\gamma}\}$.

Proof. By Lemma 8.1, the open subset $U_f \setminus (C_1 \cup \cdots \cup C_{2\gamma})$ has trivial class group, so every divisor is principal. Thus $\sum f_i F_i = \mathrm{div}(p)$ for some $p \in \mathbb{K}[x^{\pm}, y^{\pm}]_f$, by multiplying appropriate power of x, y, and $y - \alpha_i$, we assume p lies in $\mathbb{K}[x, y]$ and not divisible by $x, y, (y - \alpha_i)$ for all i. This proves (1).

To extend p to $X_f(\mathcal{P})$, write $p(x, y) = \sum_{l_1, l_2} e_{l_1, l_2} x^{l_1} y^{l_2}$ and let k be the minimal exponent of y among the terms of p. Then

$$r := \frac{p}{y^k}$$

has the same zeros along the F_i, since y is invertible on U_f. There is no zeros around the boundary because

$$\mathrm{ord}_{D_j}(p) = \min\{\mathrm{ord}_{D_j}(x^{l_1} y^{l_2})\} \leq \mathrm{ord}_{D_j}(y^k) = -kc_j.$$

$$\mathrm{ord}_{D_j}(r) = \mathrm{ord}_{D_j}(p) - (-kc_j) \leq 0.$$

by Proposition 2.7 (2). This proves (2).

For (3), let D be linearly equivalent to an effective divisor. Then for some rational g_0 we can write

$$D + \mathrm{div}(g_0) = \sum e_i D_i + \sum e'_j C_j + \sum f_i F_i > 0.$$

Using r as above, the divisor of $1/r$ cancels the $\sum f_i F_i$ while only adding multiples of D_j. Thus

$$D + \mathrm{div}(g_0/r) = \sum e''_i D_i + \sum e'_j C_j > 0,$$

showing the effective cone is generated by $\{D_1, \dots, D_n, C_1, \dots, C_{2\gamma}\}$. □

Theorem 8.5. Let A_f be the detropicalized algebra with isomorphism

$$A_f \cong \mathbb{K}[x_1, x_2, y^{\pm}]/\langle x_1 x_2 - \prod_{i=1}^{\gamma} (y - \alpha_i)^{\beta_i} \rangle$$

Let $\mathcal{P} \subset \mathcal{M}_s$ be a polytope with defined by tropical points $p_1 := (a_1, b_1, c_1), \dots, p_n := (a_n, b_n, c_n)$, with $c_1, \dots, c_j$ nonnegative and $c_{j+1}, \dots c_n$. Let $X_f(\mathcal{P})$ be the tropical mutation surface associated with $\mathcal{P}$ then

(1) $\mathrm{Cox}(X_f(\mathcal{P})) \cong \mathbb{K}[w_1, \dots w_{n+2\gamma}]/\langle w_{n+2i-1} w_{n+2i} + \alpha_i w_1^{c_1} \cdots w_j^{c_j} - w_{j+1}^{-c_{j+1}} \cdots w_n^{-c_n} \mid i \in \{1, \dots, \gamma\} \rangle$;
(2) $\mathrm{Cox}(X_f(\mathcal{P}))$ is a complete intersection ring.

Proof. As in Lemma 8.4, every effective divisor on $X_f(\mathcal{P})$ is linearly equivalent to a positive integral linear combination of

$$\{D_1, \dots, D_n,\, C_1, \dots, C_{2\gamma}\}.$$

Consequently, the Cox ring

$$\mathrm{Cox}\left(X_f(\mathcal{P})\right) \;\cong \bigoplus_{D \in \mathrm{Cl}(X_f(\mathcal{P}))} \Gamma(X_f(\mathcal{P}),\, \mathcal{O}(D))$$

consisting of rational functions whose poles are supported on the boundary divisors D_i and C_j, with multiplicity bounded by the coefficients assigned. Write $g = p/q$ with $p, q \in \mathbb{K}[x, y]$ coprime. If q is not product of x, y, or $(y - \alpha_i)$, then q vanishes along some divisor of U_f outside interior curves. So $g \notin \Gamma(X_f, \mathcal{O}(D))$. Thus the denominator of the element of the linear system is divisible by x, y, or $(y - \alpha_i)$. In particular, we obtain a surjective ring homomorphism

$$\Phi : \mathbb{K}[w_1, \ldots, w_{n+2\gamma}] \longrightarrow \mathrm{Cox}(X_f(\mathcal{P})), \qquad \deg(w_i) = \begin{cases} D_i & \text{for } 1 \le i \le n, \\ C_{i-n} & \text{for } n+1 \le i \le n+2\gamma. \end{cases}$$

The kernel of Φ is generated by homogeneous relations reflecting the linear equivalences among effective divisors. These equivalences arise from principal divisors associated with $\{D_1, \ldots, D_n, C_1, \ldots, C_{2\gamma}\}$. Note that the function x_2 does not induce any nontrivial relation among the effective divisors, as in the cases of toric varieties do not have any relations.

$$\sum a_i D_i + \sum_{i=1}^{\gamma} \beta_i C_{2i-1} = 0.$$

We can construct a nontrivial a kernel of Φ with principal divisors $y, (y - \alpha_i)$. The vector space

$$\Gamma\left(X_f(\mathcal{P}), \mathcal{O}_{X_f(\mathcal{P})}(C_{2i-1} + C_{2i})\right) = \mathrm{span}\left\{1, \frac{y}{y - \alpha_i}\right\},$$

is two-dimensional. Now observe that there exist $\gamma + 2$ linear equivalences of the form:

$$C_{2i-1} + C_{2i} \sim \sum_{i=1}^{j} c_i D_i \sim \sum_{i=j+1}^{n} -c_i D_i \quad \text{for each } i$$

all representing the same class in $\mathrm{Cl}(X_f(\mathcal{P}))$ mapping into $\Gamma(X_f(\mathcal{P}), \mathcal{O}_X(C_{2i-1}+C_{2i}))$. Thus, $\gamma+2$ effective divisors are mapped to the same two-dimensional space of global sections. This implies there are γ many relationships in $\ker(\Phi)$. Using Lemma 8.2, the kernel is generated by following relations:

$$w_{n+2i-1} w_{n+2i} = w_1^{c_1} \cdots w_j^{c_j} (y - \alpha_i) = w_{j+1}^{-c_{j+1}} \cdots w_n^{-c_n} - \alpha_i w_1^{c_1} \cdots w_j^{c_j} \text{ for } \quad 1 \le i \le \gamma.$$

For (2), assume that $x_{n+1}, \ldots, x_{n+2\gamma-1}$ are nonzero. Then $x_{n+2}, \ldots, x_{n+2\gamma}$ are then uniquely determined as a rational function of $(x_1, \ldots, x_n,\, x_{n+1}, \ldots, x_{n+2\gamma-1})$. Hence these $n+\gamma$ variables form a transcendence basis of the fraction field, and the Krull dimension of the Cox ring is $n + \gamma$.

On the other hand, the Cox ring is the quotient of a polynomial ring in $n+2\gamma$ variables by γ independent relations. Since the dimension count agrees, these relations form a regular sequence. Thus the Cox ring is a complete intersection. □

Corollary 8.6. The surface $X_f(\mathcal{P})$ is toric if and only if f is equivalent to $(y+1)^s$, and either:

(1) all but one of the coefficients c_i are $+1$ and the others are nonpositive, or
(2) all but one of the coefficients c_i are -1 and the others are nonnegative.

Proof. Under the conditions (1) or (2), it is easy to see that the Cox ring is isomorphic to polynomial ring. On the other hand, if it does not satisfies above conditions, Cox ring has a singularity at the origin, so that Cox ring is not isomorphic to polynomial ring. □

References

[1] M. E. Akhtar. Mutations of Laurent polynomials and lattice polytopes, Jul 2015. doi:https://doi.org/10.25560/28115. Ph.D. thesis at Imperial College London.

[2] K. Altmann and L. Petersen. Cox rings of rational complexity-one T-varieties. *J. Pure Appl. Algebra*, 216(5):1146–1159, 2012. doi:10.1016/j.jpaa.2011.12.018.

[3] D. Anderson. Okounkov bodies and toric degenerations. *Math. Ann.*, 356(3):1183–1202, 2013. doi:10.1007/s00208-012-0880-3.

[4] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010. doi:10.1090/S0894-0347-09-00649-3.

[5] A. Cook, L. Escobar, M. Harada, and C. Manon. Gorenstein-Fano polytopes and compactifications of rank 2 polyptych lattices, 2024, arXiv:2408.01788.

[6] D. A. Cox, J. B. Little, and H. K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011. doi:10.1090/gsm/124.

[7] J. Enwright, F. Figueroa, and J. Moraga. Log Calabi-Yau pairs of birational complexity zero, 2024, arXiv:2404.05878.

[8] J. Enwright, J. Li, and J. I. Yáñez. Complexity one varieties are cluster type, 2025, arXiv:2504.17369.

[9] L. Escobar, M. Harada, and C. Manon. Geometric families of degenerations from mutations of polytopes, 2024, arXiv:2408.01785v2.

[10] W. Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. doi:10.1515/9781400882526. The William H. Roever Lectures in Geometry.

[11] M. Gross, P. Hacking, S. Keel, and K. M. Canonical bases for cluster algebras. *J. Amer. Math. Soc.*, 31:497–608, 2018. doi:10.1090/jams/890.

[12] R. Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[13] J. Hausen and H. Süß. The Cox ring of an algebraic variety with torus action. *Adv. Math.*, 225(2):977–1012, 2010. doi:10.1016/j.aim.2010.03.010.

[14] N. O. Ilten. Mutations of Laurent polynomials and flat families with toric fibers. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 8:Paper 047, 7, 2012. doi:10.3842/SIGMA.2012.047.

[15] N. O. Ilten and R. Vollmert. Deformations of rational T-varieties. *J. Algebraic Geom.*, 21(3):531–562, 2012. doi:10.1090/S1056-3911-2011-00585-7.

[16] S. Ishii. *Introduction to Singularities*. Springer Tokyo, 1997.

[17] L. Ji and J. Moraga. Toricity in families of Fano varieties, 2024, arXiv:2409.03564.

[18] A. Kasprzyk, B. Nill, and T. Prince. Minimality and mutation-equivalence of polygons. *Forum Math. Sigma*, 5:Paper No. e18, 48, 2017. doi:10.1017/fms.2017.10.

[19] K. Kaveh and C. Manon. Khovanskii bases, higher rank valuations, and tropical geometry. *SIAM J. Appl. Algebra Geom.*, 3(2):292–336, 2019. doi:10.1137/17M1160148.

[20] S. Keel and J. McKernan. Rational curves on quasi-projective surfaces. *Mem. Amer. Math. Soc.*, 140(669):viii+153, 1999. doi:10.1090/memo/0669.

[21] J. Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. doi:10.1017/CBO9781139547895. With a collaboration of Sándor Kovács.

[22] J. Kollár, K. E. Smith, and A. Corti. *Rational and nearly rational varieties*, volume 92 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2004. doi:10.1017/CBO9780511734991.

[23] E. Looijenga. Rational surfaces with an anticanonical cycle. *Ann. of Math. (2)*, 114(2):267–322, 1981. doi:10.2307/1971295.

[24] S. Mukai. Geometric realization of T-shaped root systems and counterexamples to Hilbert's fourteenth problem. In *Algebraic transformation groups and algebraic varieties*, volume 132 of *Encyclopaedia Math. Sci.*, pages 123–129. Springer, Berlin, 2004. doi:10.1007/978-3-662-05652-3_7.

[25] M. Rosenlicht. Some basic theorems on algebraic groups. *Am. J. Math.*, 78:401–443, 1956. doi:10.2307/2372523.

[26] G. M. Ziegler. *Lectures on polytopes*, volume 152 of *Grad. Texts Math.* Berlin: Springer-Verlag, 1995. doi:10.1007/978-1-4613-8431-1.

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA
Email address: tomokioda0723@math.ucla.edu