7.4 Solving Games. There are many interesting games that are more complex and that require a good deal of thought and ingenuity to find the solution. There is one tool for solving such games that is basic. This is the infinite game analog of the principle of indifference mentioned in Chapter 3. Search for strategies that make the opponent indifferent among all his "good" pure strategies.

To be more specific, consider the game (X, Y, A) with $X = Y = [0,1]$ and $A(x, y)$ continuous. Let v denote the value of the game and let P denote the distribution that represents the optimal strategy for Player I. Then, $A(P, y)$ must be equal to v for all "good" y, which here means for all y in the support of Q for any Q that is optimal for Player II. (A point y is in the support of Q if the Q probability of the interval $(y-\epsilon, y+\epsilon)$ is positive for all $\epsilon > 0$.) So to attempt to find the optimal P, we guess at the set of "good" points and search for a distribution P such that $A(P, y)$ is constant on that set, sometimes called an equalizer strategy. The first example shows what is involved in this.

Example 1. Meeting Someone at the Train Station. A young lady is due to arrive at a train station at some random time, T, distributed uniformly between noon and 1 PM. She is to wait there until one of her two suitors arrives to pick her up. Each suitor chooses a time in [0,1] to arrive. If he finds the young lady there, he departs immediately with her; otherwise, he leaves immediately, disappointed. If either suitor is successful in meeting the young lady, he receives 1 unit from the other. If they choose the same time to arrive, there is no payoff. Also, if they both arrive before the young lady arrives, the payoff is zero. (She takes a taxi at 1 PM.)

Solution: Denote the suitors by I and II, and their strategy spaces by $X = [0,1]$ and $Y = [0,1]$. Let us find the function $A(x, y)$ that represents I's expected winnings if I chooses $x \in X$ and II chooses $y \in Y$. If $x < y$, I wins if $T < x$ and loses if $x < T < y$. The probability of the first is x and the probability of the second is $y - x$, so $A(x, y)$ is $x - (y - x) = 2x - y$ when $x < y$. When $y < x$, a similar analysis shows $A(x, y) = x - 2y$, Thus,

$$A(x,y) = \begin{cases} 2x - y & \text{if } x < y \\ x - 2y & \text{if } x > y \\ 0 & \text{if } x = y. \end{cases} \tag{1}$$

This payoff function is not continuous, nor is it upper semicontinuous or lower semicontinuous. It is symmetric in the players so if it has a value, the value is zero and the players have the same optimal strategy.

Let us search for an equalizer strategy for Player I and assume it has a density $f(x)$ on [0,1]. We would have

$$\begin{aligned} A(f,y) &= \int_0^y (2x - y) f(x)\,dx + \int_y^1 (x - 2y) f(x)\,dx \\ &= \int_0^y (x + y) f(x)\,dx + \int_0^1 (x - 2y) f(x)\,dx = \text{ constant} \end{aligned} \tag{2}$$

Taking a derivative with respect to y yields the equation

$$2yf(y) + \int_0^y f(x)\,dx - 2\int_0^1 f(x)\,dx = 0 \tag{3}$$

and taking a second derivative gives

$$2f(y) + 2yf'(y) + f(y) = 0 \qquad \text{or} \qquad \frac{f'(y)}{f(y)} = \frac{3}{2y}. \tag{4}$$

This differential equation has the simple solution,

$$\log f(y) = \frac{3}{2}\log(y) + c \qquad \text{or} \qquad f(y) = ky^{-3/2} \tag{5}$$

for some constants c and k. Unfortunately, $\int_0^1 y^{-3/2}\,dy = \infty$, so this cannot be used as a density on [0,1].

If we think more about the problem, we can see that it cannot be good to come in very early. There is too little chance that the young lady has arrived. So perhaps the "good" points are only those from some point a on. That is, we should look for a density $f(x)$ on $[a,1]$ that is an equalizer from a on. So in (2) we replace the integrals from 0 to integrals from a and assume $y > a$. The derivative with respect to y gives (3) with the integrals starting from a rather than 0. And the second derivative is (4) exactly. We have the same solution (5) but for $y > a$. This time the resulting $f(y)$ on $[a,1]$ is a density if

$$k^{-1} = \int_a^1 x^{-3/2}\,dx = -2\int_a^1 dx^{-1/2} = \frac{2(1-\sqrt{a})}{\sqrt{a}} \tag{6}$$

We now need to find a. That may be done by solving equation (3) with the integrals starting at a.

$$2yky^{-3/2} + \int_a^y kx^{-3/2}\,dx - 2 = 2ky^{-1/2} - 2k(y^{-1/2} - a^{-1/2}) - 2 = 2ka^{-1/2} - 2 = 0$$

So $ka^{-1/2} = 1$, which implies $1 = 2(1-\sqrt{a})$ or $a = 1/4$, which in turn implies $k = 1/2$. The density

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1/4 \\ (1/2)x^{-3/2} & \text{if } 1/4 < x < 1 \end{cases} \tag{7}$$

is an equalizer for $y > 1/4$ and is therefore a good candidate for the optimal strategy. We should still check at points y less than $1/4$. For $y < 1/4$, we have from (2) and (7)

$$A(f,y) = \int_{1/4}^1 (x-2y)(1/2)x^{-3/2}\,dx = \int_{1/4}^1 \frac{1}{2\sqrt{x}}\,dx - 2y = \frac{1}{2} - 2y.$$

So

$$A(f,y) = \begin{cases} (1-4y)/2 & \text{for } y < 1/4 \\ 0 & \text{for } y > 1/4 \end{cases} \tag{8}$$

This guarantees I at least 0 no matter what II does. Since II can use the same strategy, The value of the game is 0 and (7) is an optimal strategy for both players.

Example 2. Competing Investors. Two investors compete to see which of them, starting with the same initial fortune, can end up with the larger fortune. The rules of the competition require that they invest only in fair games. That is, they can only invest in games whose expected return per unit invested is 1.

Suppose the investors start with 1 unit of fortune each (and we assume money is infinitely divisible). Thus no matter what they do, their expected fortune at the end is equal to their initial fortune, 1.

Thus the players have the same pure strategy sets. They both choose a distribution on $[0,\infty)$ with mean 1, say Player I chooses F with mean 1, and Player II chooses G with mean 1. Then Z_1 is chosen from F and Z_2 is chosen from G independently, and I wins if $Z_1 > Z_2$, II wins if $Z_2 > Z_1$ and it is a tie if $Z_1 = Z_2$. What distributions should the investors choose?

The game is symmetric in the players, so the value if it exists is zero, and both players have the same optimal strategy. Here the strategy spaces are very large, much larger than in the Euclidean case. But it turns out that the solution is easy to describe. The optimal strategy for both players is the uniform distribution on the interval (0,2):

$$F(z) = \begin{cases} z/2 & \text{for } 0 \le z \le 2 \\ 1 & \text{for } z > 2 \end{cases}$$

This is a distribution on $[0,\infty]$ with mean 1 and so it is an element of the strategy space of both players. Suppose Player I uses F. Then the probability that I loses is

$$\mathrm{P}(Z_1 < Z_2) = \mathrm{E}[\mathrm{P}(Z_1 < Z_2 | Z_2)] \le \mathrm{E}[Z_2/2] = (1/2)\mathrm{E}[Z_2] = 1/2.$$

So the probability I wins is at least 1/2. Since the game is symmetric, Player II by using the same strategy can keep Player I's probability of winning to at most 1/2.