OPTIMAL INVESTMENT POLICIES FOR THE HORSE RACE MODEL

Thomas S. Ferguson and C. Zachary Gilstein
UCLA and Bell Communications

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Abstract. Optimal investment policies for maximizing the expected value of the utility of an investor's fortune are presented for investment models in which there are m investment opportunities exactly one of which will pay off, similar to betting on a winner in a horse race. It is assumed the investor knows for $i = 1, \ldots, m$ the probabilities p_i that outcome i is the winner, and the respective odds z_i that will be paid if outcome i wins, where z_i represents the number of dollars returned on a dollar bet. The investor's problem is to choose the amount to invest on each outcome in order to maximize the expected utility of his resulting wealth. This model combines and extends the models of Kelly and Murphy. Using the Karush-Kuhn-Tucker Theorem, the optimal strategies for the log, power and exponential utility functions are derived.

1. Introduction.

In Ferguson and Gilstein [6], conditions were given under which the myopic rule is optimal for making investment decisions in a general multistage investment model. Usually, evaluation of the myopic rule requires approximation methods associated with concave programming problems. In this paper, we examine one class of investment models for which the myopic rule may be evaluated explicitly. The distinguishing feature of this class of models is that of the $m \geq 2$ possible investment opportunities exactly one will pay off, a situation referred to as the horse race in Baldwin [2].

The investor is given the opportunity to bet on m possible outcomes, one and only one of which will occur, each paying odds of a specified amount. The investor knows the probabilities p_i $i = 1, \ldots, m$ of the outcomes and the respective odds z_i that will be paid if outcome i occurs, $i = 1, \ldots, m$. Here, z_i represents the number of dollars returned on a dollar bet. We assume $p_i > 0$ and $z_i > 0$ for all i since an outcome giving zero return will not be bet on and can be removed from the set of betting opportunities. The investor's problem is to choose the amount to invest on each possible outcome in order to maximize the expected utility of his resulting wealth. The problem of maximizing the expected value of a log utility function was considered in models of Kelly [7] and Murphy [8], both in the context of this problem. The model considered here bridges these two models and generalizes their solutions to the power and exponential utility functions.

We describe the optimal rules for log, power, and exponential utility functions. We first consider the class of log and power utility functions

$$U_\gamma(x) = \begin{cases} (x^\gamma - 1)/\gamma & \text{for } \gamma \neq 0, \\ \log(x) & \text{for } \gamma = 0. \end{cases} \tag{1}$$

where $x \geq 0$ represents the investor's fortune. The parameter γ measures the investor's aversion to risk; the value $\gamma = 1$ being the linear utility function and smaller values of

γ indicating greater aversion to risk. These are the utility functions that have constant relative risk aversion, where the relative risk aversion of a utility, u, is defined by Arrow[1] and Pratt [9] to be $-xu''(x)/u'(x)$. They are fairly representative, containing the linear utility ($\gamma = 1$) and the logarithmic utility ($\gamma = 0$) as special cases. Since utility functions are determined only up to change of location and scale, the form of $U_\gamma(x)$ has been chosen so that $\lim_{\gamma\to 0} U_\gamma(x) = U_0(x)$. As was noted by Kelly [7] and Bellman and Kalaba [3], the optimal investment for an investor with a utility function in this class is proportional to the investor's fortune.

The second class of utility functions we consider is the exponential class

$$W_\theta(x) = 1 - e^{-\theta x}, \tag{2}$$

where the fortune x may assume negative as well as positive values and $\theta > 0$. Models with this utility function were considered in Ferguson [5] for investors whose main concern was not going bankrupt. Larger values of θ lead to more cautious investment decisions. The optimal investments for an investor with a utility function in this class are independent of fortune.

In Section 2, we describe the investment models and the optimal investment policies for both the U_γ utility functions and the W_θ utility functions. In Section 3, we describe the optimal investment policies for several different utility functions in each class. In Section 4, we present the proofs of the theorems given in Section 2.

2. Optimal Investment Strategies.

2.1 The Utility Functions, U_γ.

As mentioned above, for the utility functions U_γ, the optimal investments are proportional to the investor's wealth. Thus, the investor must choose an investment policy $\boldsymbol{c} = (c_0, c_1, \ldots, c_m)$, where for $i = 1, \ldots, m$, $c_i \geq 0$ represents the proportion of the investor's wealth bet on outcome i, and $c_0 = 1 - \sum_1^m c_i \geq 0$ represents the porportion of the investor's wealth not invested. If the investor uses investment policy $\boldsymbol{c}$ and if outcome i occurs, the return V_i per unit wealth is assumed to be of the form

$$V_i = 1 - \alpha + \alpha c_0 + c_i z_i, \qquad \text{for } i = 1, \ldots, m.$$

The parameter α, $0 \leq \alpha \leq 1$, represents a bound on the downside risk of the investment and provides a connection between the investment model of Kelly and that of Murphy. Regardless of the outcome, the investor cannot lose more than α times his fortune. Kelly's model occurs when $\alpha = 1$: $V_i = c_0 + c_i z_i$. The amount not invested is held at no interest and money bet on outcomes that did not occur is lost. The case $\alpha = 0$ corresponds to Murphy's model: $V_i = 1 + c_i z_i$. In this case, the investor may invest in several different opportunities, one and only one of which will show an increase, the others remaining constant. Thus, the final holding is equal to the initial holding plus the incremental increase due to investment in outcome i.

The utility of V_i is taken to be $U_\gamma(V_i)$ where U_γ is given by (1). Since outcome i occurs with probability p_i and $\sum p_i = 1$, the expected utility of investment policy $\boldsymbol{c}$ is equal to

$$\Phi_\gamma(\boldsymbol{c}) = \begin{cases} \frac{1}{\gamma}\sum_1^m p_i(1-\alpha+\alpha c_0+c_i z_i)^\gamma - \frac{1}{\gamma} & \text{for } \gamma \neq 0 \\ \sum_1^m p_i \log(1-\alpha+\alpha c_0+c_i z_i), & \text{for } \gamma = 0. \end{cases} \tag{3}$$

The problem is to choose $\boldsymbol{c}$ to maximize $\Phi_\gamma(\boldsymbol{c})$ subject to the constraints

$$c_i \geq 0 \qquad \text{for} \quad i = 0, 1, \ldots, m, \quad \text{and} \quad \sum_0^m c_i = 1. \tag{4}$$

The investor may invest only nonnegative amounts and cannot invest more than he has.

This model may be generalized to include an additional parameter, r, representing the interest rate on money not invested. The equation for the return given outcome i becomes $V_i = 1-\alpha+r\alpha c_0+c_i z_i$, for $i = 1, \ldots, m$. However, by factoring out $S = 1-\alpha+r\alpha$ and changing variables, we may write $V_i/S = 1-\beta+\beta c_0+c_i(z_i/S)$, where $\beta = r\alpha/S$. For the utility functions U_γ, the optimal investment vector for maximizing the expected utility of V_i is the same as that for maximizing the expected utility of V_i/S. Thus, the generalized model may be solved by solving the specialized model with the returns z_i replaced by z_i/S, and α replaced by $r\alpha/S$. A similar substitution may be used to derive the optimal investments for the more general utility functions $U(x) \sim (A + x/(1-\gamma))^\gamma$, $A > 0$, $\gamma \neq 0, 1$ (see Rubenstein [10]).

If $\gamma \geq 1$, the function Φ_γ is convex on its domain of definition so that it assumes its maximum at one of the extreme points of the constraint set. The solution of the problem is then trivial. Find a subscript j such that $p_j(1-\alpha+z_j)^\gamma$ is a maximum and put $c_j = 1$ (invest everything in opportunity j), unless $p_j(1-\alpha+z_j)^\gamma < 1$, in which case put $c_0 = 1$, (invest nothing). The solution for $\gamma < 1$ is given below in Theorem 1.

The statement of the theorem is facilitated by the introduction of some notation. First, we assume that the investment opportunities have been ordered in order of decreasing expected returns per unit invested,

$$p_1 z_1 \geq p_2 z_2 \geq \ldots \geq p_m z_m. \tag{5}$$

There is an optimal rule that chooses investments according to this order; more precisely, it has the property that if no investment is made in opportunity j, then no investment is made in opportunities i with $i > j$.

When $\alpha \sum_1^m z_i^{-1} < 1$, the optimal c_0 will be zero, i.e. the whole fortune will be invested. This is because the investor can gain a positive amount surely by investing $c_j = z_j^{-1}/\sum_1^m z_i^{-1}$ in opportunity j for $j = 1, \ldots, m$.

For $j = 0, 1, \ldots, m$, let Q_j be defined as

$$Q_j = \begin{cases} \frac{\alpha(1-\sum_1^j p_i)}{1-\alpha\sum_1^j z_i^{-1}} & \text{if } \alpha\sum_1^j z_i^{-1} < 1, \\ +\infty & \text{if } \alpha\sum_1^j z_i^{-1} \geq 1. \end{cases} \tag{6}$$

Note that $Q_0 = \alpha$. With the variables ordered as in (5), let $0 \le k \le m$ be the smallest integer such that

$$Q_k = \min_{0 \le j \le m} Q_j. \tag{7}$$

Note that if $\alpha = 0$ then $k = 0$. In Lemma 1 of Section 4, we will see that the Q_j are decreasing for $j \le k$ and nondecreasing for $j \ge k$. This implies that if $Q_1 \ge \alpha$ then $k = 0$.

For $\gamma < 1$, let R_j be defined by

$$R_j = \frac{\sum_1^j (p_i z_i^\gamma)^{1/(1-\gamma)}}{1 + (1-\alpha)\sum_1^j z_i^{-1}}, \qquad \text{for } j = 0, 1, \ldots, m. \tag{8}$$

Note that $R_0 = 0$. Let $0 \le k' \le m$ be the smallest integer such that

$$R_{k'} = \max_{0 \le j \le m} R_j. \tag{9}$$

In Lemma 2, we will see that the R_j are increasing for $j \le k'$ and nonincreasing for $j \ge k'$.

Theorem 1. *Given probabilities $p_1, \ldots, p_m > 0$, $\sum_1^m p_i = 1$, odds $z_1, \ldots, z_m > 0$, $0 \le \alpha \le 1$, and $\gamma < 1$, the function $\Phi_\gamma(\boldsymbol{c})$ of (3) is maximized subject to the constraints (4) by the vector $\boldsymbol{c} = \boldsymbol{c}^*$ chosen as follows: Order the subscripts as in (5) and find k as in (7) and k' as in (9). Then,*

A. If $((1-\alpha)R_k)^{1-\gamma} < Q_k$, then $\alpha > 0$ and

$$c_j^* = \begin{cases} 1 - T/(1+\alpha T) & \text{for } j = 0, \\ (1/(1+\alpha T))[(p_j z_j/Q_k)^{1/(1-\gamma)} - 1]/z_j & \text{for } j = 1, \ldots, k. \\ 0 & \text{for } j = k+1, \ldots, m, \end{cases}$$

where $T = \sum_{i=1}^k [(p_i z_i/Q_k)^{1/(1-\gamma)} - 1]/z_i$.

B. If $((1-\alpha)R_k)^{1-\gamma} \ge Q_k$, then

$$c_j^* = \begin{cases} 0 & \text{for } j = 0, \\ (p_j z_j^\gamma)^{1/(1-\gamma)}/R_{k'} - (1-\alpha) z_j^{-1} & \text{for } j = 1, \ldots, k', \\ 0 & \text{for } j = k'+1, \ldots, m. \end{cases}$$

It is interesting to note as a corollary that no resources are invested, that is $c_0^* = 1$, if and only if $\max_i p_i z_i \le \alpha$. Since α represents a bound on the downside risk to the investor, the proportion of the investor's fortune at risk is equal to α, so unless an investment opportunity has expected return in excess of α, the investor should invest nothing. The use of this theorem is illustrated by examples discussed in Section 3 and the proof is given in Section 4.

2.2 The Utility Functions, W_θ, $\theta > 0$.

As before, the investor has m investment opportunities one and only one of which will pay off. Opportunity i has probability p_i of paying off at odds z_i for $i = 1, \ldots, m$. The investor wishes to choose a vector of bets, $\boldsymbol{b} = (b_1, \ldots, b_m)$, that will maximize the expected utility of return. However, we no longer require that the total amount bet, $\sum_1^m b_i$, be no greater than his present fortune, X_0. That is, we allow the investor to borrow unlimited amounts of money at prevailing interest rates.

The return, V_i, when the investor uses the bet vector $\boldsymbol{b}$ and opportunity i pays off is assumed to be of the form

$$V_i = (X_0 - \sum_{i=1}^{m} b_j)r + b_i z_i,$$

where $r > 0$ represents the growth factor so that $r - 1$ is the interest rate, taken to be known. The case $r < 1$ may be used to treat investment problems in periods of high inflation. Note that if $\sum_1^m b_j$ exceeds X_0, the first term on the right is negative. This represents interest and capital the investor must pay on borrowed money. The expectation of the utility of the of the return may be written as $EW_\theta(V) = 1 - Ee^{-\theta V}$. The problem of maximizing the expectation of the utility of the return is equivalent to the problem of choosing the vector $\mathbf{b}$ to minimize

$$\begin{aligned} Ee^{-\theta V} &= \sum_{i=1}^{m} p_i e^{-\theta[(X_0 - \sum_1^m b_j)r + b_i z_i]} \\ &= e^{-\theta X_0 r} e^{\theta r \sum_1^m b_j} \sum_{i=1}^{m} p_i e^{-\theta b_i z_i}, \end{aligned}$$

subject to the restriction $b_j \geq 0$ for $j = 1, \ldots, m$. Note that the factor $\exp\{-\theta X_0 r\}$ is independent of $\boldsymbol{b}$ so that the optimal investment will be independent of the investor's fortune. The solution to this problem is given in the following theorem.

Theorem 2. *Let θ, r, $p_1, \ldots, p_m$ and $z_1, \ldots, z_m$ be positive numbers such that $\sum_1^m p_i = 1$, and define*

$$\Phi(\boldsymbol{b}) = e^{\theta r \sum_1^m b_i} \sum_{i=1}^{m} p_i e^{-\theta b_i z_i}.$$

If $r \sum_1^m z_i^{-1} < 1$, then $\Phi(\boldsymbol{b})$ can be made arbitrarily close to zero by choosing $b_j = N z_j^{-1}$ for $j = 1, \ldots, m$ with N arbitrarily large. If $r \sum_1^m z_i^{-1} \geq 1$, then $\Phi(\boldsymbol{b})$ is minimized subject to $\boldsymbol{b} \geq 0$ by $\boldsymbol{b} = \boldsymbol{b}^*$ obtained as follows: Order the subscripts as in (5) and let k be the largest positive integer less than or equal to m such that

$$p_k z_k \left(1 - r \textstyle\sum_{i=1}^k z_i^{-1}\right) > r \left(1 - \textstyle\sum_{i=1}^k p_i\right).$$

If no such k exists, let $k = 0$. Then $k < m$ and

$$b_j^* = \frac{1}{\theta z_j} \log\left(\frac{p_j z_j (1 - r \sum_1^k z_i^{-1})}{r(1 - \sum_1^k p_i)}\right) \qquad \text{for } j = 1, \ldots, k,$$

and $b_j^* = 0$ for $j = k+1, \ldots, m$. In particular, $b_m^* = 0$.

As a corollary, it may be noted that no money is bet, that is $b_i^* = 0$ for all i, if and only if $\max_i p_i z_i \leq r$. In addition, there is a discontinuity in the optimal return as a function of r at the point $r_0 = 1/\sum_1^m z_i^{-1}$. For $r < r_0$, $\inf_{\boldsymbol{b}} \Phi(\boldsymbol{b}) = 0$. For $r = r_0$, k of Theorem 2 becomes $m-1$ and the optimal investment policy reduces to

$$b_j^* = \frac{1}{\theta z_j} \log(p_j z_j / p_m z_m) \qquad \text{for } j = 1, \ldots, m.$$

This leads to the optimal return of $\Phi(\boldsymbol{b}^*) = r_0^{-1} \exp\{r_0 \sum_1^m z_j^{-1} \log(p_j z_j)\} > 0$. In this formula for the b_j^*, $p_m z_m$ can be replaced by any smaller number without changing the value of $\Phi(\boldsymbol{b}^*)$. This gives other optimal strategies. An example illustrating the use of this theorem is given in the next section and the proof is given in Section 4.

3. Examples.

It is worthwhile stating Theorem 1 for the case $\alpha = 1$, and $\gamma = 0$ (log utility), as a corollary. This result is found in Kelly's original paper in a slightly different form. The resulting optimal policy is known as the Kelly betting system. It is particularly important because it is a system of bets which, used in repeated play, gives the maximal rate at which the investor's fortune tends to infinity. Moreover, the result has a simple statement that gives one a better idea of what is going on.

Corollary. *Assume in Theorem 1 that $\alpha = 1$ and $\gamma = 0$. Let*

$$Q_j = \begin{cases} \frac{1-\sum_1^j p_i}{1-\sum_1^j z_i^{-1}} & \text{if } \sum_1^j z_i^{-1} < 1, \\ +\infty & \text{if } \sum_1^j z_i^{-1} \geq 1, \end{cases}$$

and let k be the smallest integer such that $Q_k = \min_j Q_j$. Then,

$$c_j^* = \begin{cases} Q_k & \text{for } j = 0, \\ p_j - z_j^{-1} Q_k & \text{for } j = 1, \ldots, k. \\ 0 & \text{for } j = k+1, \ldots, m. \end{cases}$$

The optimal expected return per unit bet is $\sum_1^k p_j \log(p_j z_j)$.

Both cases A and B of Theorem 1 can occur. Case B occurs if and only if $k = m$ or equivalently, $Q_k = 0$, or equivalently, $\sum_1^m z^{-1} \leq 1$. The optimal investment policy in Case B is $c_0^* = 0$ and $c_j^* = p_j$ for $j = 1, \ldots, m$. Thus it is optimal to invest the whole fortune in proportion to the probabilities of success, independent of the z_j! This particular result for the horse race model was also noticed in Proposition 2 of Breiman [4] as well as in the paper of Kelly.

Here is an example to illustrate this phenomenon. Suppose $m = 3$, with probabilities $p_1 = .1$, $p_2 = .3$, and $p_3 = .6$, and with odds $z_1 = 100$, $z_2 = 10$ and $z_3 = 1.5$. The subscripts have been ordered according to (5). Since $\sum_1^3 z_i^{-1} = 0.77666\cdots < 1$, we have case B where our entire fortune is invested. Ten percent is invested in the most favorable outcome, thirty percent in the next most favorable and sixty percent in the least favorable outcome. Sixty percent of the time we will end up with only ninety percent of our original fortune (since $p_3 z_3 = 0.9$). Moreover, if someone tells us that they can raise the return on the first outcome to 200 or even 1000, we simply say "Thank you!"; we do not increase our investment in that outcome!

Let us also consider a more typical example of Theorem 1 for various values of α and γ. Table 1 contains the values of the optimal investment vector, $\boldsymbol{c}$, for utility function U_γ for the case

$$p_1 = 0.2 \quad p_2 = 0.3 \quad p_3 = 0.4 \quad p_4 = 0.1$$
$$z_1 = 7.0 \quad z_2 = 4.0 \quad z_3 = 2.0 \quad z_4 = 6.0$$

and several values of α and γ. Note that the variables have been ordered in decreasing value of $p_j z_j$. As an example of how Table 1 was computed, consider the case $\alpha = 0.5$ and $\gamma = 0.9$. First note that the k found in (7) is equal to 4 since $1 - \alpha \sum_1^4 z_i^{-1} = .47 > 0$ and so $Q_4 = 0$ while $Q_3 > 0$. Next we compute $R_4 = 1.874$ and note that $R_4^{1-\gamma} > Q_4 = 0$, so that case B is applicable and $c_0^* = 0$. To find k', one computes say $R_1 = 1.928$, $R_2 = 2.374$, and $R_3 = 1.982$. So $k' = 2$ and $c_3^* = 0$ and $c_4^* = 0$. Then c_1^* and c_2^* are computed using the formula for case B.

Table 1. Optimal Investments for U_γ Utility Functions

		$\gamma = -1$	$\gamma = 0.0$	$\gamma = 0.9$
	c_0^*	0	0	0
	c_1^*	.199	.269	.870
$\alpha = 0.0$	c_2^*	.303	.368	.130
	c_3^*	.404	.324	0
	c_4^*	.094	.039	0
	c_0^*	0	0	0
	c_1^*	.182	.235	.799
$\alpha = 0.5$	c_2^*	.286	.334	.201
	c_3^*	.421	.362	0
	c_4^*	.110	.070	0
	c_0^*	.913	.824	.025
	c_1^*	.040	.082	.713
$\alpha = 1.0$	c_2^*	.047	.094	.262
	c_3^*	0	0	0
	c_4^*	0	0	0

Since a greater value of γ indicates a smaller aversion to risk, as γ increases the investor invests more of his wealth and a greater proportion of his investment is on outcomes with the higher expected return. Also as α increases the investor chooses to invest less of his wealth since more is at risk. If $\sum_1^m z_i^{-1} < 1/\alpha$, the investor will invest all of his wealth since in this case $k = m$ and $Q_m = 0$. It is easy to see that if $\min_i p_i z_i > \alpha$, then $\sum_1^m z_i^{-1} < 1/\alpha$. Notice that in the example, since $\min_i p_i z_i = 0.6$, the investor will invest all of his wealth for any value of $\alpha < 0.6$ regardless of how risk averse the investor is. While the above condition is sufficient for the investor to invest all his wealth, it is not necessary. The condition in Theorem 1 which separates Case A from Case B is a necessary and sufficient condition for the investor to invest all of his wealth.

The optimal $\boldsymbol{b}^*$ for the utility function W_θ and the same values of the p_i and z_i as above may be found in Table 2 for $\theta = 1$ and for various value of r. For $r \geq \max_i p_i z_i = 1.4$, nothing is invested (i.e. all $c_i = 0$. For arbitrary $\theta > 0$, the optimal $\boldsymbol{b}^*$ can easily be found from this by dividing the results by θ. Note that these investments are in absolute dollar terms and the amount invested is independent of the investor's fortune.

Table 2. Optimal Investments for W_θ Utility Functions

r	r_0	1.0	1.1	1.2	1.3	1.4
b_1^*	.121	.076	.053	.030	.013	0
b_2^*	.173	.094	.054	.014	0	0
b_3^*	.144	0	0	0	0	0
b_4^*	0	0	0	0	0	0

$$r_0 = 1/\sum_1^4 z_i^{-1} = 0.9438,\ \theta = 1$$

4. Proofs of Theorems 1 and 2.

In this section we give the proofs of the two theorems presented in Section 2. The proofs involve verifying the conditions of the Karush-Kuhn-Tucker Theorem which gives sufficient conditions under which a point is a local maximum of a nonlinear function subject to equality and inequality constraints. Since the function considered here is convex, a local maximum is also a global maximum.

4.1 Proof of Theorem 1.

Let m' be defined as the largest integer $m' \leq m$ such that $\alpha \sum_1^{m'} z^{-1} < 1$, $0 \leq m' \leq m$. Then Q_j has the form

$$Q_j = \begin{cases} \frac{\alpha(1-\sum_1^j p_i)}{1-\alpha\sum_1^j z_i^{-1}} & \text{for } j = 0, 1, \ldots, m' \\ = \infty & \text{for } j = m'+1, \ldots, m. \end{cases}$$

The proof of Theorem 1 is facilitated by the use of two lemmas

Lemma 1. *With the variables ordered as in (5), there is a unique k, $0 \le k \le m'$, such that*

a. $Q_j < Q_{j-1}$ *for* $j = 1, \ldots, k$ *and* $Q_j \ge Q_{j-1}$ *for* $j = k+1, \ldots, m$,

b. $Q_j < p_j z_j$ *for* $j = 1, \ldots, k$ *and* $Q_j \ge p_j z_j$ *for* $j = k+1, \ldots, m$,

c. If $k < m$, *then* $p_{k+1} z_{k+1} \le Q_k$.

Proof. Easy algebra gives the equations for $j = 1, \ldots, m'$

$$Q_j - Q_{j-1} = \frac{\alpha(Q_{j-1} - p_j z_j)}{z_j(1 - \alpha \sum_1^j z_i^{-1})} \tag{10}$$

and

$$Q_j - Q_{j-1} = \frac{\alpha(Q_j - p_j z_j)}{z_j(1 - \alpha \sum_1^{j-1} z_i^{-1})} \tag{11}$$

If $Q_j < Q_{j-1}$, then from (8) and (5) $Q_{j-1} < p_j z_j \le p_{j-1} z_{j-1}$, so that from (9) with j replaced by $j-1$, $Q_{j-1} < Q_{j-2}$. This gives statements a and b for $j = 1, \ldots, m'$. The rest of a and b follows easily from (9) and the definition of Q_j for $j > m'$. To show statement c, suppose that $p_{k+1} z_{k+1} > Q_k$. Then

$$\begin{aligned}
1 - \alpha \textstyle\sum_1^{k+1} z_i^{-1} &= (1 - \alpha \textstyle\sum_1^k z_i^{-1}) - \alpha z_{k+1}^{-1} \\
&= \frac{\alpha(1 - \sum_1^k p_i)}{Q_k} - \alpha z_{k+1}^{-1} \\
&> \frac{\alpha(1 - \sum_1^k p_i)}{p_{k+1} z_{k+1}} - \alpha \frac{1}{z_{k+1}} \ge 0.
\end{aligned}$$

Hence $m' > k$, and since (8) then holds for $j = k+1$, we have $Q_{k+1} < Q_k$, a contradiction. □

Lemma 2. *With the variables ordered as in (5), there exists a unique integer k', $1 \le k' \le m$, such that*

a. $R_j > R_{j-1}$ *for* $j = 1, \ldots, k'$ *and* $R_j \le R_{j-1}$ *for* $j = k'+1, \ldots, m$.

b. $((1-\alpha)R_j)^{1-\gamma} < p_j z_j$ *for* $j \le k'$ *and* $((1-\alpha)R_j)^{1-\gamma} \ge p_j z_j$ *for* $j > k'$.

c. If $k' < m$, *then* $p_{k'+1} z_{k'+1} \le ((1-\alpha)R_{k'})^{1-\gamma}$.

d. If $((1-\alpha)R_k)^{1-\gamma} \ge Q_k$, *where k is as in Lemma 1, then* $k' \le k$.

Proof. Easy algebra gives the equations for $j = 1, \ldots, m$,

$$\begin{aligned}
R_{j-1} - R_j &= \frac{(1-\alpha)R_{j-1} - (p_j z_j)^{1/(1-\gamma)}}{z_j(1 + (1-\alpha)\sum_1^j z_i^{-1})} \\
&= \frac{(1-\alpha)R_j - (p_j z_j)^{1/(1-\gamma)}}{z_j(1 + (1-\alpha)\sum_1^{j-1} z_i^{-1})}
\end{aligned}$$

From these equations, statements a, b, and c follow as in the proof of Lemma 1 without the complication of m'. Futhermore, k' cannot be zero since $R_0 = 0$ and $R_1 > 0$. To prove statement d, assume $((1-\alpha)R_k)^{1-\gamma} \geq Q_k$ and consider two cases. If $((1-\alpha)R_k)^{1-\gamma} \geq p_k z_k$, then $k' \leq k$ from parts b of Lemmas 1 and 2. If $((1-\alpha)R_k)^{1-\gamma} < p_k z_k$, then $((1-\alpha)R_k)^{1-\gamma} \geq Q_k \geq p_{k+1} z_{k+1}$, which implies $k = k'$ by b. $\square$

Proof of Theorem 1.

Since $c_0 = 1 - \sum_1^m c_i$, $\Phi_\gamma(\boldsymbol{c})$ may be written

$$\Phi_\gamma(\boldsymbol{c}) = \begin{cases} \frac{1}{\gamma}\sum_{j=1}^m p_j(1-\alpha\sum_{i=1}^m c_i + c_j z_j)^\gamma - \frac{1}{\gamma} & \text{for } \gamma \neq 0 \\ \sum_{j=1}^m p_j \log(1-\alpha\sum_{i=1}^m c_i + c_j z_j) & \text{for } \gamma = 0, \end{cases}$$

Where we take $\boldsymbol{c} = (c_1, c_2, \ldots, c_m)$ as the variable. Since the functions $\Phi_\gamma(\boldsymbol{c})$ are concave over their convex domain, $c_1, \ldots, c_n \geq 0$, $\sum_1^m c_i \leq 1$, there exists a maximum that satisfies the conditions of the Karush-Kuhn-Tucker (KKT) Theorem. From this, it is sufficient to show:

(1) $c_j^* \geq 0$, for $j = 1, \ldots, m$.

(2) $\sum_{i=1}^m c_i^* \leq 1$.

(3) There exists a number $\lambda \geq 0$ such that for $j = 1, \ldots, m$, $\frac{\partial}{\partial c_j}\Phi_\gamma(\boldsymbol{c})\Big|_{\boldsymbol{c}^*} \leq \lambda$, with equality if $c_j^* > 0$, and where $\lambda = 0$ if $\sum_1^m c_i < 1$.

The partial derivatives of Φ_γ may be written

$$\frac{\partial}{\partial c_j}\Phi_\gamma(\boldsymbol{c}) = p_j z_j(1-\alpha\sum_{i=1}^m c_i + c_j z_j)^{\gamma-1} - \alpha\mu(\boldsymbol{c}),$$

where

$$\mu(\boldsymbol{c}) = \sum_{j=1}^m p_j(1-\alpha\sum_{i=1}^m c_i + c_j z_j)^{\gamma-1}$$

Suppose the subscripts are ordered as in (5) and that k is chosen as in Lemma 1.

Part A: $((1-\alpha)R_k)^{1-\gamma} < Q_k$. In this case, Q_k and hence α are positive. First we show that $c_0^* > 0$, or equivalently that $1-(1-\alpha)T > 0$. This is trivial if $k = 0$ (since then $T = 0$), and if $k > 0$,

$$\begin{aligned} T &= Q_k^{-1/(1-\gamma)}\sum_1^k (p_i z_i^\gamma)^{1/(1-\gamma)} - \sum_1^k z_i^{-1} \\ &< ((1-\alpha)R_k)^{-1}\sum_1^k (p_i z_i^\gamma)^{1/(1-\gamma)} - \sum_1^k z_i^{-1} = 1/(1-\alpha) \end{aligned}$$

from the definition of R_k. From this, we can see that the first KKT condition is satisfied since $1/(1+T) > 1/(1+\frac{\alpha}{1-\alpha}) = 1-\alpha > 0$ and for $j = 1, \ldots, k$,

$$p_j z_j \geq p_k z_k > Q_k$$

from Lemma 1. The second KKT condition follows from the computation

$$\begin{aligned}\textstyle\sum_1^m c_j^* &= (1-\alpha(1-c_0^*))\big[Q_k^{-1/(1-\gamma)}\textstyle\sum_1^k (p_j z_j^\gamma)^{1/(1-\gamma)} - \sum_1^k z_j^{-1}\big]\\ &= \big[1-\frac{\alpha T}{1+\alpha T}\big]T = 1-c_0^* < 1.\end{aligned}$$

This implies that in the third KKT condition we must have $\lambda = 0$. This condition breaks into two parts

$$\frac{\partial}{\partial c_j}\Phi_\gamma(\boldsymbol{c})\Big|_{\boldsymbol{c}^*} = 0 \qquad \text{for } j = 1,\dots,k, \tag{12}$$

and

$$\frac{\partial}{\partial c_j}\Phi_\gamma(\boldsymbol{c})\Big|_{\boldsymbol{c}^*} \le 0 \qquad \text{for } j = k+1,\dots,m. \tag{13}$$

Since for $j = 1,\dots,k$,

$$\begin{aligned}p_j z_j(1-\alpha\textstyle\sum_1^k c_i^* + c_j^* z_j)^{\gamma-1} &= p_j z_j(1-\alpha(1-c_0^*) + c_j^* z_j)^{\gamma-1}\\ &= (1-\alpha(1-c_0^*)^{\gamma-1} p_j z_j (p_j z_j/Q_k)^{-1}\\ &= (1-\alpha(1-c_0^*)^{\gamma-1} Q_k\end{aligned}$$

and

$$\begin{aligned}\mu(\boldsymbol{c}^*) &= \sum_{j=1}^k p_j(1-\alpha\sum_1^k c_i^* + c_j^* z_j)^{\gamma-1} + \sum_{j=k+1}^m p_j(1-\alpha\sum_1^k c_i^*)^{\gamma-1}\\ &= (1-\alpha(1-c_0^*))^{\gamma-1} Q_k \sum_{j=1}^k z_j^{-1} + (1-\alpha(1-c_0^*))^{\gamma-1}\sum_{j=k+1}^m p_j\\ &= (1-\alpha(1-c_0^*))^{\gamma-1} Q_k/\alpha,\end{aligned}$$

we see that (12) follows. For $j = k+1,\dots,m$,

$$\begin{aligned}\frac{\partial}{\partial c_j}\Phi_\gamma(\boldsymbol{c})\Big|_{\boldsymbol{c}^*} &= p_j z_j(1-\alpha(1-c_0^*))^{\gamma-1} - \alpha\mu(\boldsymbol{c}^*)\\ &= (1-\alpha(1-c_0^*))^{\gamma-1}(p_j z_j - Q_k)\\ &\le (1-\alpha(1-c_0^*))^{\gamma-1}(p_{k+1} z_{k+1} - Q_k)\end{aligned}$$

since the $p_j z_j$ are nonincreasing. Now (13) follows from Lemma 1c.

Part B: $((1-\alpha)R_k)^{1-\gamma} \ge Q_k$. If $\alpha = 1$ in this case, then R_k is clearly increasing, which implies that $k' = m$. Let k' be chosen as in Lemma 2, which implies here $1 \le k' \le k$. Now since for $j = 1,\dots,k'$,

$$c_j^* = \frac{1}{z_j}\left[\frac{(p_j z_j)^{1/(1-\gamma)}(1+(1-\alpha)\sum_1^{k'} z_i^{-1})}{\sum_1^{k'}(p_i z_i^\gamma)^{1/(1-\gamma)}} - (1-\alpha)\right],$$

we have $c_j^* > 0$ for $j = 1, \ldots, k'$ from the definition of k'. Thus $c_j^* \geq 0$ for all j and the first KKT condition is proved. The second KKT condition follows from the computation

$$\sum_{j=1}^{k'} c_j^* = (1 + (1-\alpha)\sum_1^{k'} z_i^{-1}) - (1-\alpha)\sum_1^{k'} z_i^{-1} = 1.$$

The third KKT condition again breaks into two pieces. For some $\lambda \geq 0$,

$$\frac{\partial}{\partial c_j}\Phi_\gamma(\boldsymbol{c})\Big|_{\boldsymbol{c}^*} = \lambda \qquad \text{for } j = 1, \ldots, k', \tag{14}$$

and

$$\frac{\partial}{\partial c_j}\Phi_\gamma(\boldsymbol{c})\Big|_{\boldsymbol{c}^*} \leq \lambda \qquad \text{for } j = k'+1, \ldots, m. \tag{15}$$

For $j = 1, \ldots, k'$,

$$\frac{\partial}{\partial c_j}\Phi_\gamma(\boldsymbol{c})\Big|_{\boldsymbol{c}^*} = p_j z_j (1 - \alpha + c_j^* z_j)^{\gamma-1} - \alpha\mu(\boldsymbol{c}^*),$$

where for $\alpha < 1$

$$\mu(\boldsymbol{c}^*) = \sum_{j=1}^{k'} p_j (1 - \alpha + c_j^* z_j)^{\gamma-1} + (1-\alpha)^{\gamma-1}\sum_{k'+1}^{m} p_j.$$

For $\alpha = 1$, the last term does not appear because $k' = m$, so the formula is valid in this case also. Now for $j = 1, \ldots, k'$,

$$p_j z_j (1 - \alpha + c_j^* z_j)^{\gamma-1} = p_j z_j (z_j (p_j z_j^\gamma)^{1/(1-\gamma)}/R_{k'})^{\gamma-1} = R_{k'}^{1-\gamma},$$

so that

$$\mu(\boldsymbol{c}^*) = R_{k'}^{1-\gamma}\sum_1^{k'} z^{-1} + (1-\alpha)^{\gamma-1}(1 - \sum_1^{k'} p_j),$$

and hence

$$\begin{aligned}\frac{\partial}{\partial c_j}\Phi_\gamma(\boldsymbol{c})\Big|_{\boldsymbol{c}^*} &= R_{k'}^{1-\gamma}(1 - \alpha\sum_1^{k'} z_j^{-1}) - (1-\alpha)^{\gamma-1}\alpha(1 - \sum_1^{k'} p_j) \\ &= (1-\alpha)^{\gamma-1}(1 - \alpha\sum_1^{k'} z_j^{-1})[((1-\alpha)R_{k'})^{1-\gamma} - Q_{k'}]\end{aligned}$$

is a constant independent of j, call it λ. This gives (14). Finally, note that for $j = k'+1, \ldots, m$,

$$\begin{aligned}\frac{\partial}{\partial c_j}\Phi_\gamma(\boldsymbol{c})\Big|_{\boldsymbol{c}^*} &= p_j z_j (1-\alpha)^{\gamma-1} - \alpha\mu(\boldsymbol{c}^*) \\ &= p_j z_j (1-\alpha)^{\gamma-1} - R_{k'}^{1-\gamma} + \lambda \\ &\leq \lambda\end{aligned}$$

since $p_j z_j$ is nonincreasing in j and $p_{k'+1} z_{k'+1} \leq ((1-\alpha) R_{k'})^{1-\gamma}$ from Lemma 2c. This gives (15) and completes the proof. $\square$

Proof of Theorem 2.

If $r \sum_1^m z_i^{-1} < 1$, and $b_j = N z_j^{-1}$ for $j = 1, \ldots, m$, then

$$\Phi(\boldsymbol{b}) = \exp\{-N\theta(1 - r \sum_1^m z_i^{-1})\} \to 0 \qquad \text{as } N \to \infty.$$

Assume now that $r \sum_1^m z_i^{-1} \geq 1$. Since

$$\frac{\partial^2}{\partial \eta^2} \Phi(\eta \boldsymbol{b} + (1-\eta)\boldsymbol{b}') \geq 0 \qquad \text{for } \boldsymbol{b} \geq 0,\ \boldsymbol{b}' \geq 0, \text{ and } 0 \leq \eta \leq 1,$$

Φ is convex over its domain. Therefore, it is sufficient to show that $\boldsymbol{b}^*$ satisfies the KKT conditions. This amounts to showing

(a) $b_j^* \geq 0$, for $j = 1, \ldots, m$.

(b) $\left.\frac{\partial}{\partial b_j} \Phi(\boldsymbol{b})\right|_{\boldsymbol{b}^*} \geq 0$ with equality if $b_j^* > 0$, for $j = 1, \ldots, m$.

Suppose the subscripts are ordered as in (5) and that k is chosen as in the statement of the theorem. Note that $k < m$ since $r \sum_1^m z_i^{-1} \geq 1$. The inequalities (a) are automatically satisfied for $j = k+1, \ldots, m$. For $j = 1, \ldots, k$, b_j^* is positive provided

$$p_j z_j (1 - r \sum_1^k z_i^{-1}) > r(1 - \sum_1^k p_i).$$

But $p_j z_j \geq p_k z_k$ for $j \leq k$, so (a) follows from the definition of k.

To check condition (b), note that for $j = 1, \ldots, m$,

$$\frac{\partial}{\partial b_j} \Phi(\boldsymbol{b}) = \theta \exp\{\theta r {\textstyle\sum_1^m} b_i\} (r {\textstyle\sum_1^m} p_i e^{-\theta b_i z_i} - p_j z_j e^{-\theta b_j z_j}).$$

We must show that this, evaluated at $\boldsymbol{b}^*$, is zero for $j = 1, \ldots, k$, and is nonnegative for $j = k+1, \ldots, m$. For $j \leq k$,

$$\exp\{-\theta b_j^* z_j\} = \frac{r(1 - \sum_1^k n p_i)}{p_j z_j (1 - r \sum_1^k z_i^{-1})}.$$

Hence,

$$\begin{aligned} r \sum_{j=1}^m p_j e^{-\theta b_j^* z_j} &= r \sum_{j=1}^k \frac{r(1 - \sum_1^k p_i)}{z_j (1 - r \sum_1^k z_i^{-1})} + r \sum_{j=k+1}^m p_j \\ &= r(1 - {\textstyle\sum_1^k} p_i) \left(\frac{r \sum_1^k z_j^{-1}}{1 - r \sum_1^k z^{-1}} + 1 \right) \\ &= \frac{r(1 - \sum_1^k p_i)}{1 - r \sum_1^k z_i^{-1}}, \end{aligned}$$

so that $\left.\frac{\partial}{\partial b_j}\Phi(\boldsymbol{b})\right|_{\boldsymbol{b}^*} = 0$, for $j = 1, \ldots, k$. For $j = k+1, \ldots, m$, $\left.\frac{\partial}{\partial b_j}\Phi(\boldsymbol{b})\right|_{\boldsymbol{b}^*} \geq 0$, if

$$r\sum_{i=1}^{m} p_i e^{-\theta b_i^* z_i} - p_j z_j = \frac{r(1-\sum_1^k p_i)}{1 - r\sum_1^k z_i^{-1}} - p_j z_j \geq 0,$$

or equivalently if

$$p_j z_j(1 - r\textstyle\sum_1^k z_i^{-1}) \leq r(1 - \textstyle\sum_1^k p_i).$$

By the definition of k, we must have

$$p_{k+1} z_{k+1}(1 - r\textstyle\sum_1^{k+1} z_i^{-1}) \leq r(1 - \textstyle\sum_1^{k+1} p_i)$$

which reduces to

$$p_{k+1} z_{k+1}(1 - r\textstyle\sum_1^k z_i^{-1}) \leq r(1 - \textstyle\sum_1^k p_i).$$

Since the $p_j z_j$ are nonincreasing,, we must have

$$p_j z_j(1 - r\textstyle\sum_1^k z_i^{-1}) \leq r(1 - \textstyle\sum_1^k p_i)$$

for $j = k+1, \ldots, m$. $\square$

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