SOME TIME-INVARIANT STOPPING RULE PROBLEMS

Thomas S. Ferguson
Mathematics Department,UCLA, Los Angeles CA 90024

James B. MacQueen
Anderson Graduate School of Management,UCLA, Los Angeles, CA 90024

Summary: Let $X_1, X_2, \ldots$ be an i.i.d. sequence. We consider three stopping rule problems for stopping the sequence of partial sums, $S_n = \sum_1^n X_i$, each of which has a time- invariance for the payoff that allows us to describe the optimal stopping rule in a particularly simple form, depending on one or two parameters. For certain distributions of the X_n, the optimal rules are found explicitly. The three problems are: (1) stopping with payoff equal to the absolute value of the sum with a cost of time, $Y_n = |S_n| - nc$, (2) stopping with payoff equal to the maximum of the partial sums with a cost of time, $Y_n = \max\{S_0, S_1, \ldots, S_n\} - nc$, and (3) deciding when to give up trying to attain a goal or set a record, $Y_n = I(S_n \geq a) - nc$, or $Y_n = I(S_n > a) - nc$. For each of these problems, the corresponding problems repeated in time, where the objective is to maximize the rate of return, can also be solved.

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1. Introduction.

The general optimal stopping problem may be described as follows. We are given a probability space, $(\Omega, \mathcal{F}, P)$, an increasing sequence of σ-fields, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$, and a sequence of real-valued random variables $\{Y_0, Y_1, Y_2, \ldots, Y_\infty\}$ where Y_n is $\mathcal{F}_n$-measurable for $n = 0, 1, 2, \ldots$, and Y_∞ is $\mathcal{F}_\infty$- measurable where $\mathcal{F}_\infty$ is the σ-field generated by $\cup\mathcal{F}_n$. The problem is to find an extended-valued stopping rule N (a random variable with values in $\{0, 1, 2, \ldots, \infty\}$ such that the event $\{N = n\}$ is in $\mathcal{F}_n$ for $n = 0, 1, 2, \ldots$) to maximize EY_N. If we define

$$V_n = \operatorname{ess\,sup}_{N \geq n} E\{Y_N | \mathcal{F}_n\} \tag{1.1}$$

where the essential supremum is taken over all stopping rules N such that $N \geq n$ a.s., then the stopping rule given by the principle of optimality is

$$N^* = \min\{n \geq 0 : Y_n = V_n\}. \tag{1.2}$$

The following assumptions are basic for the theory of optimal stopping.

$$A_1. \quad E\sup_{n\geq 0} Y_n < \infty.$$

$$A_2. \quad \limsup_{n\to\infty} Y_n \leq Y_\infty \quad \text{a.s.}$$

In all the applications that follow, Y_∞ is taken to be $-\infty$.

The following theorems are known. See, for example, Theorem 4.1 and Theorem 4.5′ of Chow, Robbins and Siegmund [2], and the theorems of section 1 of Klass [14]. A description of all optimal stopping rules under assumptions A_1 and A_2 may be found in Klass's paper.

Theorem 1. *Under assumptions A_1 and A_2, the stopping rule N^* is optimal.*

Theorem 2. *(The optimality equation.) Under assumption A_1,*

$$V_n = \max\{Y_n, E(V_{n+1}|\mathcal{F}_n)\} \quad \text{a.s.}$$

In the applications in this paper, $\mathcal{F}_n$ is taken to be the σ-field generated by $X_1, \ldots, X_n$, where the X_j are i.i.d. random variables. Even with this strong assumption, only a handful of stopping rule problems have been solved. Among these are the following.

(1) The search problem of MacQueen and Miller [16], Derman and Sacks [4] and Chow and Robbins [1]: X_j represents the value of the jth object found. The reward for stopping at stage n is $Y_n = X_n - nc$ or $Y_n = \max(X_1, \ldots, X_n) - nc$, where c is a given positive constant representing the cost per search. Karlin [11] considers this problem with a discount rather than a cost.

(2) The burglar problem of Haggstrom [10] and Dubins and Teicher [7]: X_j represents the return of the jth burglary. The rewards are $Y_n = \beta^n S_n$, where $0 < \beta < 1$ represents the probability of being caught in a burglary, and $S_n = X_1 + \ldots + X_n$.

(3) The problem of stopping a sum with negative drift of Darling, Liggett and Taylor [3]: the X_j are assumed to have negative expectation and the rewards are taken to be the positive part of the partial sums, $Y_n = (S_n)^+$.

(4) The problem of stopping a sum during a success run of Starr [17] and Ferguson [9]: the reward is the sum of the observations since the last failure minus a cost proportional to time.

In this paper, we consider three other problems which, like the above four, can be solved using the principle of optimality and some form of time invariance, which reduce the problems to Markov decision problems with a 1-dimensional state space. In section 2, we treat the problem of maximizing the absolute value of the partial sum with a constant cost of time, $Y_n = |S_n| - nc$. In section 3, we seek to maximize the probability of obtaining a fixed goal with a constant cost of time, $Y_n = I\{S_n \geq a\} - nc$, where a is a fixed positive number representing the goal. This is a variation of the how-to-gamble-if-you-must problem of Dubins and Savage [5]. In section 4, we treat the problem of deciding when to give up trying to increase the maximum of the partial sums when there is a constant cost of time, $Y_n = \max(S_1, \ldots, S_n) - nc$. This problem is treated in Dubins and Schwartz [6] for symmetric Bernoulli random walk. In each case, the optimal stopping rule is seen to have a simple form, and in some special cases the details are worked out fairly explicitly.

If a stopping rule problem is repeated in time, for example, in problems of replacing deteriorating equipment or reordering stock, it makes more sense to maximize the rate of return,

$$E(Z_N)/E(N + c_1), \tag{1.3}$$

where Z_n represents the reward for stopping at n, and where $c_1 > 0$ represents the set-up time for the problem, so that $E(N + c_1)$ is the expected total time. Such problems can be solved when the corresponding stopping rule problems with returns $Y_n = Z_n - cn$ can be solved for arbitrary $c > 0$. All three problems treated in this paper have this feature.

The method of solution is to solve for that value of λ such that

$$\sup_{N \geq 0} E\{Z_N - \lambda(N + c_1)\} = 0. \tag{1.4}$$

Then, the maximum of (1.3) over all stopping rules $N \geq 0$ is equal to λ and is achieved by the same stopping rule that achieves the supremum in (1.4).

2. Stopping the absolute partial sum.

In this section, we take $X_1, X_2, \ldots$ to be a sample from a known distribution having mean μ and finite second moment. For a given positive constant, c, the reward for stopping after the nth observation is $Y_n = |S_n| - nc$ for $n = 1, 2, \ldots$. We can generalize this reward function slightly to

$$Y_n = g(S_n) - nc \quad \text{for} \quad n = 1, 2, \ldots, \tag{2.1}$$

provided $g(x)$ is linear on both $(-\infty, 0]$ and $[0, \infty)$:

$$g(x) = -a_1 x I(x < 0) + a_2 x I(x > 0), \tag{2.2}$$

where a_1 and a_2 are both positive. To complete the definition of the stopping rule problem, we put

$$Y_0 = 0 \quad \text{and} \quad Y_\infty = -\infty. \tag{2.3}$$

To avoid the possibility of obtaining arbitrarily large positive values of Y_n, it is assumed that

$$-a_1\mu < c \quad \text{and} \quad a_2\mu < c. \tag{2.4}$$

First we check that conditions A_1 and A_2 are satisfied. For this we use the following result of Kiefer and Wolfowitz [12]: *If $X_1, X_2, \ldots$ are i.i.d. with finite mean $\mu < 0$, then $E\sup_{n\ge 0} S_n < \infty$ if, and only if, $E(X_1^+)^2 < \infty$.* Note that

$$\sup_{n\ge 0} Y_n = \max\{\sup_{n\ge 0}(a_2 S_n - nc), \sup_{n\ge 0}(-a_1 S_n - nc)\}.$$

The result of Kiefer and Wolfowitz can be applied to both terms of the maximum, since we are assuming that the variance of the X_n is finite and (2.4) implies there is a negative drift to the sums in both cases. A_1 then follows since the expectation of the maximum of two random variables with finite expectation is finite. A_2 follows easily from the strong law of large numbers since a.s.

$$\begin{aligned}\limsup_{n\to\infty} Y_n &= \lim_{n\to\infty} n \max\{a_2 S_n/n - c, -a_1 S_n/n - c\} \\ &= -\infty = Y_\infty.\end{aligned}$$

Thus, the rule N^* given by (1.2) is optimal. The main result of this section is the form of this rule as given in the following theorem.

Theorem 3. *If $Eg(X) \leq c$, then $N^* \equiv 0$. If $Eg(X) > c$, then the optimal rule has the form*

$$N^* = \min\{n \geq 1 : S_n \leq -\gamma_1 \quad \text{or} \quad S_n \geq \gamma_2\} \tag{2.5}$$

for some $\gamma_1 > 0$ and $\gamma_2 > 0$.

Proof. First define the function $V_0(z)$ as the optimal return when one starts with an initial fortune, z, that is,

$$V_0(z) = \sup_{N \geq 0} E\{g(z + S_N) - Nc\}. \tag{2.6}$$

Next note that due to the structure of the reward function, the problem at stage n is the same as the one at stage zero starting with $z = S_n$. This time-invariance allows us to write V_n in terms of $V_0(z)$ as follows.

$$\begin{aligned} V_n &= \operatorname{ess\,sup}_{N \geq n} E\{g(S_N) - Nc | \mathcal{F}_n\} \\ &= \operatorname{ess\,sup}_{N \geq n} E\{g(S_n + S_{n,N-n}) - (N-n)c | \mathcal{F}_n\} - nc \\ &= V_0(S_n) - nc \qquad \text{a.s.} \end{aligned} \tag{2.7}$$

where $S_{n,k} = X_{n+1} + \ldots + X_{n+k}$. The optimal rule is

$$\begin{aligned} N^* &= \min\{n \geq 0 : g(S_n) - nc = V_n\} \\ &= \min\{n \geq 0 : g(S_n) = V_0(S_n)\} \\ &= \min\{n \geq 0 : \phi(S_n) = 0\}, \end{aligned} \tag{2.8}$$

where $\phi(z)$ is defined as

$$\phi(z) = V_0(z) - g(z) = \sup_{N \geq 0} E\{g(z + S_N) - g(z) - Nc\}. \tag{2.9}$$

We will complete the proof by showing that $\phi(z)$ is continuous unimodal with a maximum at $z = 0$, that $\phi(z) = 0$ for $|z|$ sufficiently large, and that $\phi(0) > 0$ if and only if $Eg(X) > c$, thus implying (2.5).

We first note that $\phi(z)$ is nondecreasing for $z \in (-\infty, 0]$, and nonincreasing for $z \in [0, \infty)$: For $z \geq 0$,

$$g(z + S_n) - g(z) = \begin{cases} a_2 S_n & \text{for } z + S_n \geq 0 \\ -(a_1 + a_2)z - a_1 S_n & \text{for } z + S_n \leq 0. \end{cases}$$

For fixed S_n this is a nonincreasing function of z. Hence from (2.9), $\phi(z)$ is a nonincreasing function of z for $z \geq 0$. The case $z \leq 0$ follows by symmetry. Continuity holds since $g(z + S_n) - g(z)$ is continuous in z uniformly in S_n.

Next note that $\phi(z)$ is equal to zero for $|z|$ sufficiently large. To see this, one can apply the optimality equation and (2.6) to find

$$\phi(z) = \max\{0, E(V_0(z + X) - g(z)) - c\}.$$

Then, for all $z \geq 0$,

$$\phi(z) = \max\{0, E(\phi(z + X) + a_2 X) - c\}.$$

If $\phi(z) > 0$ for all $z > 0$, then $E(\phi(z + X)) > c - a_2\mu = \epsilon > 0$, say, for all $z > 0$, which since ϕ is nonincreasing for $z > 0$ implies that $\phi(z) > \epsilon$ for all sufficiently large z, which in turn implies that $E(\phi(z + X)) > 2\epsilon$, for sufficiently large z, etc. Eventually this would imply that $E(\phi(z + X)) > \phi(0)$, a contradiction. By symmetry, one may also contradict $\phi(z) > 0$ for all $z < 0$.

Finally, if $Eg(X) > c$, the simple stopping rule $N \equiv 1$ shows that $\phi(0) > 0$. If $Eg(X) \leq c$, then the problem is monotone with a reward structure of the form $Y_n = Z_n - W_n$ with $E\sup|Z_n| < \infty$ and W_n nonnegative and nondecreasing a.s., so that the one-stage look-ahead rule is optimal. (See Chow et al. [2], Theorem 4.4.) This rule stops without taking any observations, so that $V_0 = 0$ and hence $\phi(0) = 0$.

Examples. We give some examples in which the optimal rule can be explicitly evaluated. From Theorem 3, the problem reduces to finding γ_1 and γ_2 to maximize

$$\begin{aligned} EY_N &= Eg(S_N) - cEN \\ &= -a_1 E\{S_N I(S_N \leq -\gamma_1)\} + a_2 E\{S_N I(S_N \geq \gamma_2)\} - cEN \end{aligned} \tag{2.10}$$

where N is the rule (2.5).

(1) *Bernoulli trials.* Let $P(X=1)=\pi$ and $P(X=-1)=1-\pi$. By (2.4), $\mu=2\pi-1$ is assumed to be between $-c/a_1$ and c/a_2. Assume also that $Eg(X)=a_1(1-\pi)+a_2\pi>c$ so that both γ_1 and γ_2 are positive in the optimal rule. In this case, we may take γ_1 and γ_2 to be integers, and the boundaries will be hit exactly, so that (2.10) becomes

$$EY_N = a_1\gamma_1 P(S_N=-\gamma_1) + a_2\gamma_2 P(S_N=\gamma_2) - cEN. \tag{2.11}$$

Formulas for these quantities are well known from the problem of gambler's ruin (Feller [8], Chap 9). Let $t=(1-\pi)/\pi$. Then,

$$P(S_N=\gamma_2) = \begin{cases} \dfrac{1-t^{-\gamma_1}}{t^{\gamma_2}-t^{-\gamma_1}} & \text{if } \ \pi\neq 1/2 \\ \dfrac{\gamma_1}{\gamma_1+\gamma_2} & \text{if } \ \pi = 1/2. \end{cases} \tag{2.12}$$

and

$$EN = \begin{cases} ((\gamma_2-\gamma_1)P(S_N=\gamma_2)-\gamma_1)/\mu & \text{if } \pi\neq 1/2 \\ \gamma_1\gamma_2 & \text{if } \pi=1/2. \end{cases} \tag{2.13}$$

These expressions may be used in (2.11) to find the values of γ_1 and γ_2 that maximize EY_N.

In the case $\pi=1/2$ an interesting result emerges, namely, the optimal values of γ_1 and γ_2 are equal. This is in spite of the fact that the problem is not symmetric since a_1 need not be equal to a_2. For $\pi=1/2$,

$$EY_N = \gamma_1\gamma_2[(a_1+a_2)/(\gamma_1+\gamma_2)-c]$$

From this, one sees that for fixed $\gamma_1+\gamma_2$, EY_N is maximized by taking γ_1 and γ_2 as close together as possible; that is, $\gamma_1=\gamma_2$ if $\gamma_1+\gamma_2$ is even, and $\gamma_1=\gamma_2\pm 1$ if $\gamma_1+\gamma_2$ is odd. A more detailed analysis shows that if a maximum occurs for $\gamma_1+\gamma_2=2k+1$ odd, that is, if

$$\begin{aligned} k^2[(a_1+a_2)/(2k)-c] &\le k(k+1)[(a_1+a_2)/(2k+1)-c] \\ &\ge (k+1)^2[(a_1+a_2)/(2k+2)-c], \end{aligned}$$

then there is equality throughout, so that the maximum also occurs for $\gamma_1 + \gamma_2 = 2k$ and $2(k+1)$. Therefore, the maximum of EY_N occurs for $\gamma_1 = \gamma_2$. An easy calculation shows the optimal value of γ_1 and γ_2 to be

$$\gamma_1 = \gamma_2 = \text{the integer nearest to} \quad (a_1 + a_2)/(4c).$$

This also holds if $(a_1 + a_2)/2 \leq c$.

(2) *The double exponential distributions.* For an example with continuous variables, consider the three parameter double exponential distribution with parameters $\mu_1 > 0$, $\mu_2 > 0$, and $0 < \pi < 1$, and with density,

$$f(x) = \begin{cases} (1-\pi)\exp\{x/\mu_1\}/\mu_1 & \text{for } x < 0 \\ \pi \exp\{-x/\mu_2\}/\mu_2 & \text{for } x > 0. \end{cases} \tag{2.14}$$

This distribution has mean $\mu = \pi\mu_2 - (1-\pi)\mu_1$. To evaluate (2.10), we make use of the following formulas, due to Kemperman [13] p.67, in which $t = -\mu/\mu_1\mu_2 = (1-\pi)/\mu_2 - \pi/\mu_1$.

$$P(S_N \geq \gamma_2) = \begin{cases} \dfrac{\pi(1-\pi)(\mu_1+\mu_2) - \pi\mu_2\exp\{-t\gamma_1\}}{(1-\pi)\mu_1\exp\{t\gamma_2\} - \pi\mu_2\exp\{-t\gamma_1\}} & \text{if} \quad t \neq 0, \\ \dfrac{\mu_1 + \gamma_1}{\mu_1 + \gamma_1 + \mu_2 + \gamma_2} & \text{if} \quad t = 0. \end{cases} \tag{2.15}$$

The expectation of N can be obtained from this through use of the formulas $EN = ES_N/\mu$ if $\mu \neq 0$, and $EN = ES_N^2/\sigma^2$ if $\mu = 0$, where $\sigma^2 = 2\mu_1\mu_2$ is the variance of (2.14) when $\mu = 0$. The values of ES_N and ES_n^2 may be computed easily, since from the lack of memory property of the exponential distribution, the distribution of $S_N - \gamma_2$ given $S_N \geq \gamma_2$ is exponential with mean μ_2 and variance $\sigma_2^2 = \mu_2^2$. Similarly, given $S_N \leq -\gamma_1$, the distribution of $-(S_N + \gamma_1)$ is exponential with mean μ_1 and variance $\sigma_1^2 = \mu_1^2$. Thus, letting $z_j = \gamma_j + \mu_j$, we have

$$EN = \frac{1}{\mu}[z_2 P(S_N \geq \gamma_2) - z_1 P(S_N \leq -\gamma_1)] \qquad \text{if} \quad \mu \neq 0, \tag{2.16}$$

and for $\mu = 0$,

$$\begin{aligned} EN &= \frac{1}{\sigma^2}[(z_2^2 + \sigma_2^2)P(S_N \geq \gamma_2) + (z_1^2 + \sigma_1^2)P(S_N \leq -\gamma_1)] \\ &= \frac{1}{\sigma^2}[z_1 z_2 + \frac{\sigma_2^2 z_1 + \sigma_1^2 z_2}{z_1 + z_2}]. \end{aligned} \tag{2.17}$$

To evaluate (2.10), note that $E\{S_N I(S_N \geq \gamma_2)\} = E\{S_N | S_N \geq \gamma_2\} P(S_N \geq \gamma_2) = z_2 P(S_N \geq \gamma_2)$, so that (2.10) becomes

$$EY_N = a_1 z_1 P(S_N \leq -\gamma_1) + a_2 z_2 P(S_N \geq \gamma_2) - cEN.$$

The optimal values of γ_1 and γ_2 can be explicitly evaluated when $\mu = 0$, or equivalently $t = 0$ or $\pi = \mu_1/(\mu_1 + \mu_2)$. In this case,

$$EY_N = (a_1 + a_2) z_1 z_2/(z_1 + z_2) - cEN.$$

Note that this expectation, and hence the optimal values of γ_1 and γ_2, depend on a_1 and a_2 only through the sum, $a_1 + a_2$. To find γ_1 and γ_2 to maximize this, let $A = \sigma^2(a_1 + a_2)/c = 2(a_1 + a_2)\mu_1\mu_2/c$, and write

$$EY_N = \frac{c}{\sigma^2}\left[\frac{Az_1z_2 - \sigma_2^2 z_1 - \sigma_1^2 z_2}{z_1 + z_2} - z_1 z_2\right].$$

Setting the partial derivatives of this expression with respect to z_1 and z_2 to zero leads to the equations

$$\begin{aligned} Az_1 &= \sigma_1^2 - \sigma_2^2 + (z_1 + z_2)^2 \\ Az_2 &= \sigma_2^2 - \sigma_1^2 + (z_1 + z_2)^2. \end{aligned}$$

Adding and subtracting these equations yields two linear equations as the only positive solutions, from which we derive

$$\begin{aligned} \gamma_1 &= A/4 + (\mu_1^2 - \mu_2^2)/A - \mu_1 \\ \gamma_2 &= A/4 - (\mu_1^2 - \mu_2^2)/A - \mu_2 \end{aligned}$$

as the solution, provided both are positive. If either is negative or zero, then so is the other and the optimal rule is $N \equiv 0$.

In the special case $\mu_1 = \mu_2$, the Laplace distribution, we find that the optimal values of γ_1 and γ_2 are equal:

$$\gamma_1 = \gamma_2 = \theta\left(\frac{(a_1 + a_2)\theta}{2c} - 1\right)^+,$$

where θ is the common value of μ_1 and μ_2.

(3) *The double geometric distributions.* As a generalization of the Bernoulli case, details may also be worked out for the three parameter double geometric distributions, with parameters $0 \le p_1 < 1$, $0 \le p_2 < 1$, $0 < \pi < 1$, and probability mass function,

$$f(x) = \begin{cases} (1-\pi)(1-p_1)p_1^{-x-1} & \text{for } x = -1, -2, \ldots \\ \pi(1-p_2)p_2^{x-1} & \text{for } x = 1, 2, \ldots . \end{cases} \tag{2.18}$$

Note that either of p_1 or p_2 may be zero. If both are zero, the distribution becomes the Bernoulli with probability π of 1 and probability $1-\pi$ of -1.

By (2.4), the mean, $\mu = \pi/(1-p_2) - (1-\pi)/(1-p_1)$, is assumed to be between $-c/a_1$ and c/a_2. The distribution of $S_N - \gamma_2$ given $S_N \ge \gamma_2$ is geometric so that (2.10) becomes

$$EY_N = a_1(\gamma_1 + \mu_1)P(S_N \le -\gamma_1) + a_2(\gamma_2 + \mu_2)P(S_N \ge \gamma_2) - cEN, \tag{2.19}$$

where $\mu_j = p_j/(1-p_j)$. Expressions for the quantities $P(S_N \ge \gamma_2)$ and EN may be derived in a manner similar to those that Kemperman used to obtain the expressions (2.15). Let $t = (1-\pi+\pi p_1)/(\pi + (1-\pi)p_2)$.

$$P(S_N \ge \gamma_2) = \begin{cases} \dfrac{(1-p_1/t)(1-p_2 t) - (1-p_2 t)(1-p_1)t^{-\gamma_1}}{(1-p_2)(1-p_1/t)t^{\gamma_2} - (1-p_2 t)(1-p_1)t^{-\gamma_1}} & \text{if} \quad \mu \ne 0, \\ \dfrac{\gamma_1 + \mu_1}{\gamma_1 + \mu_1 + \gamma_2 + \mu_2} & \text{if} \quad \mu = 0. \end{cases} \tag{2.20}$$

The expectation of N may also be computed as for the double exponential case. In fact, formulas (2.16) and (2.17) apply here as is, except that here we have

$$\begin{aligned} \mu &= \pi/(1-p_2) - (1-\pi)/(1-p_1) \\ \mu_2 &= p_j/(1-p_j) \qquad j = 1,2 \\ z_j &= \gamma_j + p_j/(1-p_j) \qquad j = 1,2 \\ \sigma_j^2 &= p_j/(1-p_j)^2 \qquad j = 1,2 \\ \sigma^2 &= 2(1-p_1p_2)/[(2-p_1-p_2)(1-p_1)(1-p_2)]. \end{aligned} \tag{2.21}$$

When $p_1 = p_2 = 0$, these formulas reduce to (2.12) and (2.13). These expressions may be used in (2.20) to find the values of γ_1 and γ_2 that maximize EY_N.

3. Setting a record and attaining a goal.

In this section, $X_1, X_2, \ldots$ are i.i.d. with no moment assumptions required. Starting with initial fortune, $T_0 = z$, a series of investments with net returns $X_1, X_2, \ldots$ is made so that the fortune at stage n is $T_n = z + S_n$. A given positive number a, the target, and a positive number c, the cost of observation, are fixed. In the problem of setting a record, the reward for stopping at n is $Y_n = I(T_n > a) - nc$. In the problem of attaining a goal, the reward for stopping at n is $Y_n = I(T_n \geq a) - nc$. These problems are of the form (2.1) with $g(x) = I(x > a - z)$ and $g(x) = I(x \geq a - z)$ respectively. We take $Y_\infty = -\infty$ to complete the description of the problem.

The following analysis is analogous to that of the previous section, but because the function g is discontinuous, a somewhat different argument is needed. It is clear that assumption A_1 is satisfied since Y_n is bounded above by 1. Assumption A_2 is also satisfied since we take Y_∞ to be $-\infty$. Hence, the rule N^* is optimal.

Theorem 4. *In the problem of setting a record (resp. attaining a goal), there is an optimal rule of the form*

$$N^* = \min\{n \geq 0 : T_n \leq a - \gamma \quad \text{or} \quad T_n > a\} \tag{3.1}$$

$$(\text{resp.} \quad N^* = \min\{n \geq 0 : T_n < a - \gamma \quad \text{or} \quad T_n \geq a\})$$

for some number $\gamma \geq 0$.

Proof. First consider the problem of setting a record. The analysis follows the proof of Theorem 3 through equations (2.6), (2.7), (2.8) and (2.9). We complete the proof by showing that $\phi(z)$ is zero for $z > a$, is nondecreasing for $z \leq a$, is zero for z sufficiently small, and is left continuous.

For $z > a$, $V_0(z) = 1$ since this can be attained by $N \equiv 0$, and clearly no larger return can be attained; hence, $\phi(z) = 0$. For $z \leq a$, $g(z + S_n) - g(z) = I(z + S_n > a)$ is nondecreasing in z for fixed S_n. Hence, $\phi(z)$ is nondecreasing as well. That $\phi(z)$ is zero for z sufficiently small follows using the optimality equation as in the corresponding part of the proof of Theorem 3.

To see that ϕ is left continuous, note that for $z < a$,

$$\phi(z) = \sup_{N \geq 0}[P(T_N > a) - Nc],$$

so that for $z \leq a$ and $\epsilon > 0$,

$$0 \leq \phi(z) - \phi(z-\epsilon) \leq P(S_N > a - z) - P(S_N > a - z + \epsilon)$$
$$= P(a - z < S_N \leq a - z + \epsilon)$$

where N is the optimal rule for the problem starting at z. Since the right side converges to zero as $\epsilon \to 0$, ϕ is left continuous at z.

In the problem of attaining a goal, the optimal rule of the form stated in Theorem 4, might not be the rule N^* of equation (1.2). Instead we use another form of an optimal rule given by the principle of optimality, $N^{*\prime} = \min\{n \geq 0 : Y_n < V_n'\}$ where $V_n' = \operatorname{ess\,sup}_{N \geq n+1} E\{Y_N | \mathcal{F}_n\}$. The proof proceeds as in the problem of setting a record except that now we must show that the function,

$$\phi(z) = \sup_{N \geq 1}\{P(S_N \geq a - z) - cEN\} - I(z \geq a),$$

is right continuous, and the argument for this is more complex.

$$0 \leq \phi(z+\epsilon) - \phi(z) \leq P(S_N \geq a - z - \epsilon) - P(S_N \geq a - z)$$
$$= P(a - z - \epsilon \leq S_N < a - z)$$
$$\leq P(a - z - \epsilon \leq S_M < a - z)$$

where $N = N(\epsilon)$ is an optimal rule for the problem at $z + \epsilon$, and where $M = M(\epsilon)$ is the rule that stops at the first n such that S_n is not in the interval $(b - z, a - z - \epsilon)$ for some fixed sufficiently small b. The events $A_\epsilon = \{a - z - \epsilon \leq S_{M(\epsilon)} < a - z\}$ are decreasing as ϵ decreases to zero, with limit

$$A = \cap_{\epsilon > o} A_\epsilon \subset B = \{b - z < S_n < a - z \quad \text{for all} \quad n\}.$$

Hence, as $\epsilon \to 0$, $P(A_\epsilon) \to P(A) \leq P(B) = 0$.

Examples. (1) *The Bernoulli case.* When the X_n are integer valued, we may assume that all numerical quantities involved are integers. We consider the problem of attaining a goal starting with a fortune of z and search for a stopping rule of the form

$$N = \min\{n \geq 0 : T_n \leq a - \gamma \quad \text{or} \quad T_n \geq a\}$$
$$= \min\{n \geq 0 : S_n \leq a - z - \gamma \quad \text{or} \quad S_n \geq a - z\} \tag{3.2}$$

to maximize EY_N, where $a - \gamma < z < a$. From the Markov property, there is an optimal value of γ that is independent of z. When the distribution of the X_n is Bernoulli with probability π of 1 and probability $1-\pi$ of -1, the terms in $EY_N = P(S_N \geq a-z) + cEN$ may be found from (2.12) and (2.13) and the optimal value of γ may easily be found by numerical methods. We illustrate for the symmetric case ($\pi = 1/2$) where the optimal γ can be explicitly evaluated. From (2.12) and (2.13),

$$\begin{aligned} EY_N &= (\gamma + z - a)/\gamma - c(a-z)(\gamma - z - a) \\ &= 1 + c(a-z)^2 - (a-z)(c\gamma + 1/\gamma). \end{aligned}$$

Therefore, the optimal stopping rule is (3.2), where

$$\begin{aligned} \gamma &= \text{ the positive integer that minimizes } \quad c\gamma + 1/\gamma \\ &= \text{ the integer closest to } \quad (1/4 + 1/c)^{1/2}. \end{aligned}$$

This stopping rule is optimal for all z. For example, if $c \geq 1/2$ then $\gamma = 1$ and it is optimal to take no observations.

(2) *The double exponential.* When the distribution of the X_n is continuous, there is no difference between the problems of attaining a goal and setting a record except when $z = a$. We take the problem of setting a record, and, since the optimal value of γ is independent of z, we put $z = a$ and consider a stopping rule of the form

$$N = \min\{n \geq 0 : S_n \leq -\gamma \quad \text{or} \quad S_n > 0\}.$$

Assuming the distribution of the X_n to be double exponential of the form (2.14), we may write an explicit expression for EY_N using (2.15), (2.16) and (2.17) with $\gamma_1 = \gamma$ and $\gamma_2 = 0$, and find the value of γ to maximize it by numerical methods. For the special case $\mu = 0$, we find for $\gamma > 0$,

$$EY_N = \frac{z_1}{z_1 + \mu_2} - \frac{c}{\sigma^2}[z_1\mu_2 + \frac{\sigma_2^2 z_1 + \sigma_1^2 \mu_2}{z_1 + \mu_2}], \tag{3.3}$$

where $z_1 = \gamma + \mu_1$. Using the values of σ^2 and σ_j^2 given in section 3 for the double exponential, it is easy to find the optimal rule. If $c \geq \pi = \mu_1/(\mu_1 + \mu_2)$, then EY_N is negative for all $\gamma > 0$, so the optimal γ is 0 and the optimal rule is $N \equiv 0$. If $c < \pi$, then EY_N has a unique maximum at

$$\gamma = (\mu_1^2 - \mu_2^2 + 2\mu_1\mu_2/c)^{1/2} - \mu_1 - \mu_2.$$

(3) *The double geometric.* For the distribution (2.18) and integral γ, we obtain the same formula for EY_N when $\mu = 0$ as in (3.3) but with the values of the moments as given in (2.21). The optimal value of γ is then the nonnegative integer closest to

$$(\sigma_1^2 - \sigma_2^2 + \sigma^2/c + 1/4)^{1/2} - \mu_1 - \mu_2.$$

4. Stopping the maximum of a partial sum.

We assume that $X_1, X_2, \ldots$ are i.i.d. with $E(X^+)^2 < \infty$ and $\mu = EX < c$, where $c > 0$ is the cost per observation (μ may be $-\infty$). The return for stopping at n is

$$Y_n = \max(S_0, S_1, \ldots, S_n) - nc = M_n - nc$$

where $S_0 = 0$. Thus, the problem treated in this section is in spirit much like the problem of setting a record, but instead of a zero/one reward for failing/achieving a new record, the reward is the actual numerical value of the record.

To check condition A_1, note that

$$\begin{aligned}\sup_{n\geq 0} Y_n &= \sup_{n\geq 0}(\max(S_0, S_1, \ldots, S_n) - nc)\\ &= \sup_{n\geq 0}(S_n - nc)\\ &= \sup_{n\geq 0}(\sum_1^n (X_j - c)).\end{aligned}$$

Thus, A_1 follows from the theorem of Kiefer and Wolfowitz.

To check condition A_2, we must show that for any number B no matter how large negative, there exists an integer $N(B)$ a.s. such that $Y_n < B$ for all $n > N(B)$. Note that since $S_n/n \to \mu$ a.s. and $\mu < c$, there exists for every number B an integer $M(B)$ a.s. such that $S_n - nc < B$ for all $n > M(B)$. Then, for $n > M(B)$,

$$Y_n \leq \max[\max(0, S_1, \ldots, S_{M(B)}) - nc, B].$$

For n sufficiently large, the first term of the max can also be made less than B so that $Y_n < B$, proving A_2.

Theorem 5. *In the problem of stopping the maximum of a partial sum, there is an optimal rule of the form*

$$N = \min\{n \geq 0 : M_n - S_n \geq \gamma\} \tag{4.1}$$

for some $\gamma \geq 0$.

Proof. To evaluate the rule given by the principle of optimality, we need to evaluate

$$\begin{aligned} V_n^* &= \operatorname{ess\,sup}_{N \geq n} E\{M_N - N|\mathcal{F}_n\} \\ &= \operatorname{ess\,sup}_{N \geq n}[M_n - nc + E\{\max(0, M'_{N-n} - M_n + S_n) - (N-n)c|\mathcal{F}_n\}] \\ &= M_n - nc + \phi(M_n - S_n) \quad \text{a.s.} \end{aligned}$$

where $M'_k = \max(0, S'_1, \ldots, S'_k), S'_k = S_{n-k} - S_n$, and

$$\phi(z) = \sup_{N \geq 0} E\{\max(0, M'_n - z) - Nc\}.$$

Therefore, the rule given by the principle of optimality becomes

$$N^* = \min\{n \geq 0 : \phi(M_n - S_n) \leq 0\}.$$

It is easy to see that $\phi(z)$ is nonincreasing and uniformly continuous in z, and that $\phi(z) = 0$ for sufficiently large z. Hence, there is some $\gamma \geq 0$ such that N^* has the form given in the theorem.

Below, we give some examples in which the optimal rule can be explicitly evaluated. For the continuous time analog of the stopping rule N, the joint Laplace transform of M_N and N has been derived by Taylor [18] for Brownian motion, and by Lehoczky [15] for more general diffusion processes.

Examples. (1) *The Bernoulli case.* If $P(X = 1) = \pi$ and $P(X = -1) = 1 - \pi$, then for the rule N of Theorem 5, $M_N - S_N = \gamma$ with probability 1, so that

$$\begin{aligned} EY_N &= EM_N - cEN \\ &= E(M_N - S_N) + ES_N - cEN \\ &= \gamma + \mu EN - cEN. \end{aligned}$$

To compute EN, we note that the movement of the Markov chain $Z_n = M_n - S_n$ is like Bernoulli random walk with reflection at $-1/2$ and absorption at γ.

When $\pi = 1/2$, there is a simple argument for evaluating EN. Z_n starts at zero and if it ever tries to go to -1 it gets put back to zero. Eventually it hits γ and stops. Construct Z_n' from Z_n as follows: if Z_j has tried to go to -1 an even number of times for $1 \le j \le n$, then $Z_n' = Z_n$; otherwise, $Z_n' = -Z_n - 1$. Then it is easy to see that Z_n' is a symmetric random walk starting at zero and stopping when it hits γ or $-\gamma - 1$. Hence, from the gambler's ruin problem, $EN = \gamma(\gamma + 1)$.

Thus, when $\pi = 1/2$ so that $\mu = 0$, we have $EY_N = \gamma - c\gamma(\gamma + 1)$. The optimal rule is given by the largest integer γ that is less than or equal to $1/(2c)$.

(2) *The double exponential distribution.* For the distribution with density (2.14), we have

$$\begin{aligned} EY_N &= E(M_N - S_N) + ES_N - cEN \\ &= \gamma + \mu_1 - (c - \mu)EN \end{aligned} \tag{4.2}$$

where $\mu = \pi\mu_2 - (1-\pi)\mu_1 < c$. We must find EN for the process $Z_n = M_n - S_n$ which stops as soon as $Z_n > \gamma$. To do this, we first find ES_N.

Let K denote the number of times a new record is set; that is, let K denote the cardinality of the set $\{k : 0 < k < N, S_k = M_k\}$. Then K has a geometric distribution,

$$P(K = k) = (1-p)p^k \quad \text{for} \quad k = 0, 1, 2, \ldots$$

where p is the probability that the random walk S_n becomes positive before it becomes less than $-\gamma$. This may be found from (2.15) with $\gamma_1 = \gamma$ and $\gamma_2 = 0$ and is given by

$$\begin{aligned} p &= \frac{\pi(1-\pi)(\mu_1 + \mu_2) - \pi\mu_2 \exp\{-t\gamma\}}{(1-\pi)\mu_1 - \pi\mu_2 \exp\{-t\gamma\}} & \text{for} \quad \mu \neq 0 \\ p &= \frac{\mu_1 + \gamma}{\mu_1 + \mu_2 + \gamma} = \frac{z_1}{z_1 + \mu_2} & \text{for} \quad \mu = 0 \end{aligned} \tag{4.3}$$

where $t = -\mu/(\mu_1\mu_2)$, and where $z_1 = \gamma + \mu_1$. Now note that S_N can be written as $S_N = (U_1 + \ldots + U_K) - (\gamma + W)$, where the U_j are the K jumps above the maximum each having an exponential distribution with mean μ_2, and W is the jump below $-\gamma$, which is exponential with mean μ_1, the variables K, U_j and W being independent. Hence,

$$ES_N = \mu_2 EK - \gamma - \mu_1 = \mu_2 p/(1-p) - z_1.$$

From this, we may obtain EN when $\mu \neq 0$, by the formula $EN = ES_N/\mu$. The value of EN when $\mu = 0$ may be found using $EN = ES_N^2/\sigma^2$. When $\mu = 0$, we have

$$\begin{aligned} ES_N^2 &= \mathrm{Var}\, S_N = \sigma_2^2 EK + \mu_2^2 \mathrm{Var}(K) + \sigma_1^2 \\ &= \sigma_2^2 p/(1-p) + \mu_2^2 p/(1-p)^2 + \sigma_1^2 \\ &= \sigma_2^2 z_1/\mu_2 + z_1(z_1 + \mu_2) + \sigma_1^2. \end{aligned}$$

Hence,

$$\begin{aligned} EN &= [\mu_2 p/(1-p) - z_1]/\mu && \text{for} \quad \mu \neq 0 \\ EN &= [z_1(z_1 + \mu_2 + \sigma_2^2/\mu_2) + \sigma_1^2]/\sigma^2 && \text{for} \quad \mu = 0. \end{aligned} \tag{4.4}$$

We illustrate for the case $\mu = 0$. The optimal rule is of the form (4.1) where γ is chosen to minimize EY_N, which is a quadratic function of γ. The optimizing value of γ is

$$\gamma = \frac{1}{2}\left[\frac{\sigma^2}{c} - \mu_2 - \frac{\sigma_2^2}{\mu_2}\right] - \mu_1 \tag{4.5}$$

if this quantity is positive, and $\gamma = 0$ otherwise. Using the values of these moments for the double exponential case, we find

$$\begin{aligned} \gamma &= 0 && \text{if} \quad c \geq \mu_1\mu_2/(\mu_1 + \mu_2) \\ \gamma &= \mu_1\mu_2/c - \mu_1 - \mu_2 && \text{otherwise.} \end{aligned}$$

It is interesting that this value is symmetric in μ_1 and μ_2 even though the problem is not.
(3) *The double geometric.* For the double geometric distribution of (2.18), we are to find that integer γ that maximizes (4.2) where EN and p are given by the same formulas (4.4) and (4.3) but where the moments are given by (2.21). When $\mu = 0$, (4.2) is a quadratic function of γ, and the optimal integer γ is the nearest nonnegative integer to the expression (4.5). This time, because we do not have $\sigma_2^2 = \mu_2$, the resulting optimal value of γ is not symmetric in p_1 and p_2.

References.

[1] Chow, Y. S. and Robbins, H. (1961) "A martingale system theorem and applications" *Proc. Fourth Berkeley Symp. on Math. Statist. and Prob.* vol. **1**, 93-104.

[2] Chow, Y. S., Robbins, H. and Siegmund, D. (1971) *Great Expectations: The Theory of Optimal Stopping*, Houghton Mifflin, Boston.

[3] Darling, D. A., Liggett, T. and Taylor, H. M. (1972) "Optimal stopping for partial sums" *Ann. Math. Statist.* **43**, 1363-1368.

[4] C. Derman and J. Sacks (1960) Replacement of periodically inspected equipment NRLQ 7 597-607.

[5] Dubins, L. E. and Savage, L. J. (1976) *Inequalities for Stochastic Processes: How to Gamble if You Must*, Dover, New York.

[6] Dubins, L. E. and Schwartz, G. (1987) "A sharp inequality for martingales and stopping-times" *Colloque Paul Lévy sur les Processes stochastiques, Astérisque* **157-158**, *Société Mathématique de France*, 129-145.

[7] Dubins, L. E. and Teicher, H. (1967) "Optimal stopping when the future is discounted" *Ann. Math. Statist.* **38**, 601-605.

[8] Feller, W. (1968) *An Introduction to Probability Theory and Its Applications*, vol. **1**, 3rd Edition, Wiley, New York.

[9] Ferguson, T. S. (1976) "Stopping a sum during a success run" *Ann. Statist.* **4**, 252-264.

[10] Haggstrom, G. (1967) "Optimal stopping when more than one stop is required" *Ann. Math. Stat.* **38**, 1618-1626.

[11] Karlin, S. (1962) "Stochastic models and optimal policy for selling an asset" *Studies in Applied Probability and Management Science*, K. J. Arrow, S. Karlin and H. Scarf, Eds., Stanford University Press, 148-158.

[12] Kemperman, J. H. B. (1961) *The Passage Problem for a Stationary Markov Chain*, Statistical Research Monographs, Vol. 1, University of Chicago Press, Chicago.

[13] J. Kiefer and H. Wolfowitz (1956) "On the characteristics of the general queueing process, with application to random walk" *Ann. Math. Statist.* **27**, 147-161.

[14] Klass, M. J. (1973) "Properties of optimal extended-valued stopping rules for S_n/n" *Ann. Prob.* **1**, 719-757.

[15] Lehoczky, J. P. (1977) "Formulas for stopped diffusion processes with stopping times based on the maximum" *Ann. Prob.* **5**, 601-607.

[16] MacQueen, J. B. and Miller Jr., R. G. (1961) "Optimal Persistence Policies" *Oper. Res.* **8**, 362-380.

[17] Starr, N. (1972) "How to win a war if you must: Optimal stopping based on success runs" *Ann. Math. Statist.* **43**, 1884-1893.

[18] Taylor, H. M. (1975) "A stopped Brownian motion formula" *Ann. Prob.* **3**, 234-246.