Maximizing the duration of owning a relatively best object[1]

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Abstract: A version of the secretary problem in which the objective is to maximize the time of possession of a relatively best object is treated. The basic no-information model is related to the Presman-Sonin best-choice problem with a random number of objects. A best-choice duration model similar to one suggested by Gaver is also solved. Extensions to models with a discount, to full-information models, and to problems with and without recall are made. Finally, the duration versions of the random arrival models of Cowan, Zabczyk and Bruss in the no-information case and of Sakaguchi and Bojdecki in the full-information case are solved.

1. Introduction and Summary

We consider a sequential observation and selection problem called the duration problem, which is a variation of the classical secretary problem, treated for example by Gilbert and Mosteller (1966). A decision maker must select one object from a set of n rankable objects. The objects are observed sequentially in random order with all $n!$ orderings being equally likely. As each object is observed, the decision must be made whether to select the object or to continue and observe the next object. The distinguishing feature of the problems treated in this paper is that the payoff to the decision maker is the length of time he is in possession of a relatively best object. Thus, he will only select a relatively best object, receiving a payoff of one as he does so and an additional one for each new observation as long as the selected object stays relatively best.

In Section 2, we treat models in which the decision to select an object must be based only on the relative ranks of the objects observed. Such models are called no-information models.

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The classical secretary problem (CSP) is a best-choice problem; the payoff is one if and only if the selected object is best overall. In Section 2.1, we define a class of relatively-best-choice problems that contains the problems treated in Section 2, and in Lemma 1 state a condition under which an optimal threshold rule exists.

Another distinguishing feature of the CSP is that an object, once rejected by viewing the next object, is no longer available for selection. The problems treated this paper are also of interest if this restriction is removed. Thus in Section 2.2, we treat two cases– sampling with recall, in which any of the observed objects may be selected, and sampling without recall, in which only the most recently observed object may be selected. (The terminology is that of DeGroot (1970) for problems of selling an asset.)

A problem of maximizing the duration of owning a relatively best object has also been considered by Gaver (1976). Several features distinguish his model from ours. First, Gaver treats random arrival models, in which the times at which the objects arrive follow a Poisson process, and since the horizon is finite, the actual number of objects observed becomes random. Second, you only receive the payoff if the object selected is best overall. In Section 2.3, we consider the payoff of Section 2.2 modified by this second feature.

In Section 2.4, we treat the problems of Section 2.2 in which the future is discounted by β, a payoff of 1 received at time t being worth β^t. For such problems, we allow an infinite horizon, and investigate the asymptotics as $\beta \to 1$. The problems of Sections 2.2 and 2.4 are related to best-choice problems with a random number of objects as treated by Presman and Sonin (1972).

In contrast to the no-information problem, the full-information problems are those in which the observations are the true values of the objects, assumed to be independent and identically distributed from a known distribution, taken without loss of generality to be the uniform distribution on the interval $[0, 1]$. In Section 3, we treat the full-information versions of the various problems of Section 2 and see that the solutions differ in significant ways. The finite horizon and discount problems are related to the full-information best-choice problems with a random number of objects as treated by Porosiński (1987).

In the final section, we treat the duration problem in two random arrival models. The first is a no-information model with Poisson arrivals solved in the classical best-choice case by Cowan and Zabczyk (1976) for known arrival rate. Here, we extend to the duration problem the more realistic model of Bruss (1987) in which the arrival rate is unknown and has a prior exponential distribution. The second is an extension of the full- information model of Sakaguchi (1976) and Bojdecki (1978).

2. No-Information Models.

We assume that all that can be observed are the relative ranks of the objects as they are presented. Thus, if X_j denotes the relative rank of the jth object among the first j objects (rank 1 being best), the sequentially observed random variables are $X_1, X_2, \ldots, X_n$. It is well known, under the assumption that the objects are put in random order with the $n!$ orderings being equally likely, that

(2.1) the X_j are independent random variables, and

$$(2.2) \qquad P(X_j = i) = 1/j \quad \text{for} \quad i = 1, 2, \ldots, j, \quad \text{for} \quad j = 1, 2, \ldots, n.$$

In the discounted version of the duration problem, we allow a countable number of observations, so in (2.2) we allow n to be ∞.

2.1 Relatively-Best-Choice Problems.

A best-choice problem is one in which the payoff is positive only if the selected object is the best out of all n objects; that is, if the jth object is selected, the payoff is positive only if $X_j = 1$ and $X_{j+1} > 1, \ldots, X_n > 1$. We generalize the notion of a best choice problem slightly, allowing an infinite number of observations.

We say that a secretary problem is a *relatively-best-choice problem* if the payoff for stopping at k (selecting object k) is of the form

$$(2.3) \qquad Y_k = y_k I(X_k = 1),$$

where the y_k are given nonnegative constants and $I(A)$ represents the indicator function of the event A.

In the following lemma, we show that there is an optimal threshold rule for this problem, provided the sequence y_k is unimodal. *A threshold stopping rule with threshold r* is a stopping rule of the form

$$(2.4) \qquad N_r = \min\{k \geq r : X_k = 1\}$$

for some $r = 1, 2, \ldots$. The proof is an extension of the argument used by Gilbert and Mosteller (1966) in the CSP.

Lemma 1. *For the relatively-best-choice problem* (2.3)*, if y_k is unimodal in the sense that for some finite integer K, $y_j \leq y_{j+1}$ for $j < K$ and $y_j \geq y_{j+1}$ for $j \geq K$, then there is an optimal rule among the threshold rules N_r for $r \leq K$.*

Proof. If stage K is reached, it is clearly optimal to select the next relatively best object, since continuing cannot increase the payoff. Let $W_j = \sup_{N>j} E(Y_N | X_1, \ldots, X_j)$; these quantities are constants because of the independence (2.1). For $j \leq K$, the backward induction equations are

$$(2.5) \qquad W_{j-1} = E \max(Y_j, W_j) \qquad \text{for} \quad j = 1, \ldots, K,$$

with boundary condition, $W_K = EY_{N_K}$, and it is optimal to stop at j with $X_j = 1$ provided $y_j \geq W_j$. Note that for all j, $W_j \geq W_{j+1}$.

Suppose it is optimal to stop at $j < K$ with $X_j = 1$; i.e. suppose $y_j \geq W_j$. Then since $y_j \leq y_{j+1}$, we have $y_{j+1} \geq y_j \geq W_j \geq W_{j+1}$. Hence, it is optimal to stop at $j+1$ with $X_{j+1} = 1$, completing the proof.

2.2. The Finite-Horizon Duration Problem.

For the CSP, we have

$$(2.6) \qquad \begin{array}{lll} y_k = P(X_{k+1} > 1, \ldots, X_n > 1) = k/n & \text{for} \quad k \le n, & \text{and} \\ y_k = 0 & \text{for} \quad k > n. & \end{array}$$

Lemma 1 applies since the y_j are unimodal with mode $K = n$. As is well known, the optimal rule is the threshold rule, N_{r_n}, with

$$(2.7) \qquad r_n = \min\{k \ge 1 : \sum_{j=k}^{n-1} 1/j \le 1\}.$$

For the duration problem, we define T_k as the time of the first relatively best object after k if there is one, and as $n+1$ if there is none. The duration of posessing a relatively best object selected at time k is $T_k - k$. Instead of maximizing the expected duration of a selection, we choose to maximize the expected proportion of time one is in the possession of a relatively best object, in order to make the solution more easily comparable to the CSP and other problems. Then, this becomes a relatively-best-choice problem by definition (2.3) with $y_k = E(T_k - k)/n$.

Theorem 1. (*a*) *In the finite horizon duration problem, the expected return for stopping at k,*

$$(2.8) \qquad y_k = (k/n) \sum_{j=k}^{n} 1/j,$$

is unimodal with mode at the value

$$(2.9) \qquad K_n = \min\{k \ge 1 : \sum_{j=k+1}^{n} 1/j \le 1\}.$$

(*b*) *The expected payoff using the threshold rule* (2.4) *is*

$$(2.10) \qquad \phi_r = E(Y_{N_r}) = \frac{(r-1)}{n} \sum_{k=r}^{n} \frac{1}{(k-1)} \sum_{j=k}^{n} \frac{1}{j}.$$

Proof. (*a*) We have $P(T_k = n+1) = k/n$ and for $j = k+1, \ldots, n$, $P(T_k = j) = k/((j-1)j)$. Hence,

$$y_k = E(\frac{Y_k - k}{n}) = \sum_{j=k+1}^{n} \frac{k(j-k)}{(j-1)jn} + \frac{k(n+1-k)}{n^2} = \frac{1}{n} \sum_{j=k}^{n} \frac{k}{j}.$$

To check unimodality, compute $y_{k+1} - y_k = \sum_{j=k+1}^{n} 1/j - 1$ and note that this is decreasing in k. The mode is the first k for which $y_{k+1} - y_k \le 0$.

$$(b) \qquad \begin{aligned} \phi_r &= \textstyle\sum_{k=r}^{n} P(T_{r-1} = k) y_k = \sum_{k=r}^{n} (r-1)/((k-1)kn) \sum_{j=k}^{n} k/j \\ &= (1/n) \textstyle\sum_{k=r}^{n} (r-1)/(k-1) \sum_{j=k}^{n} 1/j. \end{aligned}$$

We now apply this theorem to the finite horizon duration problem both with and without recall, and see that the solutions are already in the literature.

First consider the problem of sampling without recall. Since the y_k are unimodal, Lemma 1 implies that one of the threshold rules N_r is optimal. So the optimal rule is N_r for that r that maximizes ϕ_r. However, from the papers of Presman and Sonin (1972) and Rasmussen and Robbins (1976), we see that ϕ_r is exactly the expected payoff for the threshold rule N_r for the best-choice secretary problem with an unknown random number of objects having a uniform distribution on $\{1, \ldots, n\}$. (In this problem, if you pass up all the objects without making a selection, you lose.) In these papers, it is shown that the optimal value of r is

$$r_n^* = \min\{r \geq 1 : \sum_{k=r+2}^{n} \frac{1}{k-1} \sum_{j=k}^{n} \frac{1}{j} \leq \sum_{j=r+1}^{n} \frac{1}{j}\}, \tag{2.11}$$

and that $r_n^*/n \to 1/e^2 = .1353\ldots$ with bounds $(n+1)/e^2 < r_n^* < (n+1)/e^2 + 1.5$. The optimal payoff converges to $2/e^2 = .2707$.

Corollary 1. *In the finite horizon duration problem without recall, the optimal rule is the threshold rule N_r with $r = r^*$ of* (2.11).

The fact that the problem treated here and the problem treated by Presman and Sonin (1972) have the same payoff (2.10) for threshold rules is an example of a general method of Samuels (1990) for relating problems with random horizon to problems with a cost.

Now consider the problem of sampling with recall. We may choose to stop at any time and when we do we naturally select the best object we have seen so far, receiving an expected payoff of y_k if we stop at stage k, independent of the observations, $X_1, \ldots, X_k$. Thus, we stop at that k for which y_k is largest, namely $k = K_n$.

Corollary 2. *In the finite horizon duration problem with recall, the optimal rule is the fixed sample size rule, $N = K_n$.*

There is a simple relation between the cutoff point, r_n of (2.7), for the CSP, and the cutoff point, K_n of (2.10), for the duration problem with recall; namely

$$K_n = r_{n+1} - 1 \qquad \text{for} \quad n \geq 2. \tag{2.12}$$

In particular, $K_n/n \to 1/e$ as $n \to \infty$, and the optimal expected payoff converges to $1/e$.

One may show that this is also the optimal rule for the Presman-Sonin problem with a random uniform number of objects when recall is allowed. This is not immediate since the expected return for stopping at k is not (2.8) but rather $y_k = (k/(n-k)) \sum_{j=k}^{n} 1/j$. Nevertheless, the fixed sample size rule $N = K_n$ is the one-stage look-ahead rule and the problem is monotone, so N is optimal. It is interesting that although sampling with recall makes sense for most secretary problems with a random number of objects, such problems do not seem to have been studied as much, even though they are in general easier.

2.3 The Best-Choice Duration Problem.

We modify the payoff structure of the previous section by requiring that the selected object be best overall; that is, we win an amount equal to the proportion of time we own the object provided it is best overall, and we win nothing if it is not best. If we select a relatively-best object at time k, we win $(n-k+1)/n$ provided the object is best overall. This is a relatively-best-choice problem by definition (2.3) where the y_k are given as

$$y_k = ((n-k+1)/n)P(T_k = n+1) = (n-k+1)k/n^2 \tag{2.13}$$

Since $n^2(y_{k+1}-y_k) = n-2k$, we see that y_k is unimodal with mode at the first k for which this is nonpositive, namely

$$L_n = \lfloor \frac{n+1}{2} \rfloor. \tag{2.14}$$

Thus, there is an optimal rule among the threshold rules N_r for some $r \le L_n$.

Theorem 2. *In the best-choice duration problem, the optimal rule is N_{r^*} where*

$$r^*(n) = \min\{r \ge 1 : n \sum_{k=r+1}^{n} \frac{1}{k-1} \le 2n-2r+1\}. \tag{2.15}$$

Proof. The expected payoff using threshold rule N_r is, for $r = 1, \ldots, n+1$,

$$\phi_r = E(Y_{N_r}) = \sum_{k=r}^{n} P(T_{r-1} = k)y_k = \frac{(r-1)}{n} \sum_{k=r}^{n} \frac{n-k+1}{(k-1)}. \tag{2.16}$$

To see which of the ϕ_r is largest, we look at the differences for $r = 1, \ldots, n$,

$$n(\phi_{r+1} - \phi_r) = \sum_{k=r+1}^{n} \frac{n-k+1}{k-1} - (n-r+1). \tag{2.17}$$

By taking a second difference, we find that (2.17) is decreasing for $r < L_n$ and increasing thereafter, taking a negative value at $r = n$. Therefore, the largest value of ϕ_r occurs at the first r at which (2.17) is negative. This reduces to (2.15), completing the proof.

It is of interest to find an approximation to $r^*(n)$ for large n. If we divide both sides of the inequality on the right side of (2.15) by n and let $n \to \infty$, we find that $r^*(n)/n \to \alpha$ where $\alpha = .20388\ldots$ is the root of the equation,

$$f(\alpha) = -\log(\alpha) - 2 + 2\alpha = 0, \tag{2.18}$$

between zero and one. This gives the approximation, $r^*(n) = n\alpha$. A more accurate approximation to $r^*(n)$ may be found using the Euler summation formula approximation,

$\sum_1^n 1/k = \log(n) + \gamma - \frac{1}{2n} + \epsilon_n$, where γ is Euler's constant and where $0 < \epsilon_n < \frac{1}{12n^2}$. Write (2.15) in the form,

$$\begin{aligned} r^*(n) &= \min\{r \geq 0 : \sum_{k=r+1}^{n} \frac{1}{k} \leq 2 - \frac{2r}{n}\} + 1 \\ &= \min\{r \geq 0 : f(r/n) \leq \frac{n-r}{2rn} + \epsilon_r - \epsilon_n\} + 1, \end{aligned}$$

where $|\epsilon_r - \epsilon_n| < 1/(12r^2)$ is negligible in the limit. Use the expansion, $f(r/n) = ((r/n) - \alpha)f'(\alpha) + \cdots$, where $f'(\alpha) = -((1/\alpha) - 2)$, and replace the term, $(n-r)/(2nr)$, by its approximation, $(1-\alpha)/(2\alpha(1-2\alpha))$, to find

$$r^*(n) = \min\{r \geq 0 : r \geq n\alpha - \frac{1-\alpha}{2\alpha(1-2\alpha)}\} + 1.$$

Replacing α by its value leads to the approximation,

$$r^*(n) = \lfloor 0.20388n + 1.3279 \rfloor. \tag{2.19}$$

The limiting value of the expected payoff as $n \to \infty$ and $r/n \to x$ is easily found to be

$$\phi_r \to -x\log(x) - x(1-x). \tag{2.20}$$

Evaluating this at $x = \alpha$ gives the optimal return, $.1618\ldots$.

These results agree with the asymptotic results found by Gaver (1976). In the model of Gaver, the objects arrive at random times given by the points of a Poisson process on $[0,1]$ at rate λ. As $\lambda \to \infty$, the number of arrivals tends to infinity, and the fact that there are a random number of arrivals does not seem to matter. Gaver derives formula (2.20) as the limiting payoff for $\lambda \to \infty$ of the fixed sample size rule that waits until time x and then selects the next relatively best object, and finds $x = \alpha$ as the optimal rule in this class. Gaver also treats the random arrival best-choice problem. For finite λ, his formulas (6.6) and (6.7) for the expected payoff are inaccurate, but may easily be corrected by adding the term, $e^{-\lambda T}\phi(U, -\infty)$, a term that becomes negligible as $\lambda \to \infty$.

The best-choice duration problem with recall is easily treated. If we select the best we have seen so far at time k, we receive y_k. So the optimal rule is the fixed sample size rule that stops at L_n. The optimal return converges to $1/4$.

2.4 The Discount Model.

We now allow an infinite horizon but discount future payoffs by an amount β per period, $0 < \beta < 1$. To make expected payoffs comparable to those of the previous sections, we multiply the return by $(1-\beta)$. Thus, an amount of 1 received at time k is worth $(1-\beta)\beta^k$ at time 0.

If the kth object is relatively best, and if we select it, our conditional expected return is for $k = 1, 2, \ldots$,

$$\begin{aligned} y_k &= \sum_{j=k}^{\infty} (1-\beta)\beta^j P(T_k > j) \\ &= (1-\beta)k \sum_{j=k}^{\infty} \beta^j / j. \end{aligned}$$

To check unimodality, compute the differences

$$\begin{aligned} (y_{k+1} - y_k)/(1-\beta) &= (k+1) \sum_{j=k+1}^{\infty} \beta^j/j - k \sum_{j=k}^{\infty} \beta^j/j \\ &= \beta^k \big[\sum_{j=1}^{\infty} \beta^j/(k+j) - 1 \big]. \end{aligned}$$

So the y_k are unimodal with mode at

$$K(\beta) = \min\{k \geq 1 : \sum_{j=1}^{\infty} \beta^j \frac{1}{k+j} \leq 1\}. \tag{2.23}$$

When $k = 1$, the sum in (2.23) is equal to $-\frac{1}{\beta}\log(1-\beta) - 1$, so $K(\beta) = 1$ if $-\log(1-\beta) < 2\beta$. This becomes $\beta < 1 - \alpha$, where α is given in (2.18). This implies that for $\beta < .79612\ldots$, the y_k are decreasing, so it is optimal to select the first object. The general result is as follows.

Theorem 3. *In the discounted duration problem, the optimal rule is N_{r^*} where*

$$r^*(\beta) = \min\{r \geq 1 : \sum_{j=r+1}^{\infty} \beta^j/j \sum_{k=r+1}^{j} 1/(k-1) \leq \sum_{j=r}^{\infty} \beta^j/j\}. \tag{2.24}$$

Proof. Since the y_k are unimodal with mode at $K(\beta)$, we know that there exists an optimal threshold rule, N_r, for some $r \leq K(\beta)$. The expected payoff using N_r is

$$\begin{aligned} \phi_r = E(Y_{N_r}) &= \sum_{k=r}^{\infty} P(T_{r-1} = k) y_k \\ &= (1-\beta)(r-1) \sum_{k=r}^{\infty} \frac{1}{k-1} \sum_{j=k}^{\infty} \beta^j/j \qquad \text{for} \quad r > 1 \\ &= y_1 \qquad \text{for} \quad r = 1. \end{aligned} \tag{2.25}$$

Again, we consider the differences

$$(\phi_{r+1} - \phi_r)/(1-\beta) = \sum_{k=r+1}^{\infty} \frac{1}{k-1} \sum_{j=k}^{\infty} \beta^j/j - \sum_{j=r}^{\infty} \beta^j/j. \tag{2.26}$$

The first r at which (2.26) is less than or equal to zero reduces to (2.24). To show this is optimal, it is sufficient to show that if (2.26) is nonpositive for some $r \leq K(\beta)$, then it is also nonpositive for $r+1$. Hence, assume that (2.26) is nonpositive and that $r \leq K(\beta)$, that is, that $\sum_{r+1}^{\infty} \beta^j/j \geq \beta^r$. Then,

$$
\begin{aligned}
(\phi_{r+2} - \phi_{r+1})/(1-\beta) &= \sum_{k=r+1}^{\infty} \frac{1}{k-1} \sum_{j=k}^{\infty} \beta^j/j - \frac{r+1}{r} \sum_{r+1}^{\infty} \beta^j/j \\
&\leq \sum_{j=r}^{\infty} \beta^j/j - \frac{r+1}{r} \sum_{r+1}^{\infty} \beta^j/j \qquad \text{(using (2.26)} \leq 0) \qquad (2.27) \\
&= \beta^r/r - \frac{1}{r} \sum_{r+1}^{\infty} \beta^j/j \leq 0,
\end{aligned}
$$

completing the proof.

It may be shown that the expected return (2.25) is exactly the same as the expected return using the same rule, N_r, in the Presman-Sonin model with an unknown random number of objects having a geometric distribution with probability β, ($P(n \text{ objects}) = (1-\beta)\beta^{n-1}$). Since Presman and Sonin show that there is an optimal rule of the form N_r for some r, the rule N_{r^*} is optimal for this problem as well.

The infinite sums that appear above for y_k, ϕ_r, etc. converge slowly, but they may be reduced to finite sums by applying the following lemma.

Lemma 2. Let $\theta = -\log(1-\beta)$. For $r \geq 1$,

$$
\sum_{j=r}^{\infty} \frac{\beta^j}{j} = \theta - \sum_{j=1}^{r-1} \frac{\beta^j}{j} \qquad (2.28)
$$

$$
\sum_{k=r+1}^{\infty} \frac{1}{k-1} \sum_{j=k}^{\infty} \frac{\beta^j}{j} = \frac{\theta^2}{2} - \sum_{i=1}^{r-1} \frac{1}{i} \left[\theta - \sum_{j=1}^{i} \frac{\beta^j}{j} \right]. \qquad (2.29)
$$

Proof. Equation (2.28) follows from the power series expansion for $-\log(1-\beta)$. To check (2.29), first let $f(\theta)$ denote the left side of (2.29) when $r = 1$, and note that $f(0) = 0$, $f'(0) = 0$ and $f''(\theta) \equiv 1$. Hence, $f(\theta) = \theta^2/2$. The rest of (2.29) for $r > 1$ now follows from this using (2.28).

Using this lemma, we find that $\phi_1 = y_1 = -(1-\beta)\log(1-\beta)$, and $\phi_2 = (1-\beta)(\log(1-\beta))^2/2$. Since $r^*(\beta) = 1$ if and only if $\phi_1 \leq \phi_2$ or, equivalently, $-\log(1-\beta) \geq 2$, (i.e. $\beta \leq 1 - e^{-2}$), this implies that it is optimal to select the first object if and only if $\beta \leq .86466\ldots$.

For values of β close to 1, the threshold points $r^*(\beta)$ are more difficult to compute. We investigate asymptotic approximations as $\beta \to 1$. With β replaced by $1-(1/n)$, the inequality in (2.24) may be expressed as

$$\sum_{j=r+1}^{\infty} \frac{((1-\frac{1}{n})^n)^{j/n}}{j/n} \left[\sum_{k=r+1}^{j} \frac{1}{(k-1)/n} \frac{1}{n} - 1 \right] \frac{1}{n} \le 0.$$

If we let $r/n \to x$ (i.e. $r(1-\beta) \to x$), this is a Riemann sum approximation to the integral,

$$\int_x^\infty \frac{e^{-y}}{y} \left[\int_x^y \frac{1}{z}\, dz - 1 \right] dy = \int_x^\infty \frac{e^{-y}}{y} \left[\log(y/x) - 1 \right] dy \le 0. \tag{2.30}$$

This is a decreasing function of x that has a unique root that may be evaluated numerically to be $x = .17404\ldots$. In a similar manner, the limit of the expected payoff, ϕ_r of (2.25), may be shown to converge as $\beta \to 1$ and $r(1-\beta) \to x$ to

$$x \int_x^\infty \frac{1}{z} \int_z^\infty \frac{e^{-y}}{y}\, dy\, dz = x \int_x^\infty \frac{e^{-y}}{y} \log(\frac{y}{x})\, dy,$$

which evaluated at the root of the function (2.30) becomes $x \int_x^\infty e^{-y}/y\, dy = .17404 \times 1.338 = .23287\ldots$.

Thus, we have the approximations $r^*(\beta) \simeq .17404/(1-\beta)$ and $\phi_{r^*} \simeq .23287$. These approximations were discovered by Presman and Sonin (1972) for the problem with a random, geometric number of objects. The approximations for $r^*(\beta)$ are quite good. Exact calculations give $r^*(.99) = 18$, $r^*(.999) = 174$, and $r^*(.9999) = 1740$.

As in the previous two sections, the corresponding problem for sampling with recall has an optimal fixed sample size rule, namely, stop at time $K(\beta)$. For β close to one, we have the approximation, $K(\beta) \simeq c/(1-\beta)$, where $c = .4348\cdots$ satisfies $\int_0^\infty e^{-y}/(c+y)\, dy = 1$. The asymptotic optimal return is $y_{K(\beta)} \simeq ce^{-c} = .2815\cdots$.

3. Full Information Models.

In full information models, the true worth of each object is observed as it appears. It is assumed that the true worths of the objects are i.i.d. from a known continuous distribution on the real line, taken without loss of generality to be the uniform distribution on the interval [0,1]. An object is relatively best at time k if it possesses the largest true worth seen by time k. We are to select a relatively best object with the view of maximizing the duration it stays relatively best, and we may base our choice of a stopping time on the true values.

Such problems are quite distinct from the problems studied in earlier sections. In particular, an optimal rule at stage k will not depend on the number of objects seen so far, but only on the maximum of the true values seen and on the number of observations yet to see. The reason for this is similar to the argument for basing rules on sufficient statistics: the distribution of future payoffs and observations depends on the past observations only through these two quantities.

3.1 The Finite Horizon Full Information Model.

We assume that $X_1, X_2, \ldots$ are i.i.d. random variables, uniformly distributed on [0,1], where X_n denotes the value of the object at the nth stage from the end. We let $w(x,n)$ denote the expected payoff given that the nth object from the last is a relatively best object of value $X_n = x$ and we select it.

$$w(x,n) = 1 + x + x^2 + \cdots + x^{n-1} = (1-x^n)/(1-x). \tag{3.1}$$

Since there is a finite horizon, the optimal rule can be found by backward induction. Let $v(x,n)$ denote the optimal expected return when there are n objects yet to be observed and the present maximum of past observations is x. The $v(x,n)$ are defined inductively by

$$v(x,n) = xv(x,n-1) + \int_x^1 \max\{w(t,n), v(t,n-1)\}\,dt \tag{3.2}$$

with initial condition, $v(x,0) \equiv 0$. Thus, for example, $v(x,1) = 1-x$.

Note that the $w(x,n)$ are increasing in x for every n. Moreover, the $v(x,n)$ are nonincreasing in x for every n. The argument for v is quite general and independent of the $w(x,n)$: $v(x,n)$ is the supremum of something over the set of strategies available at state (x,n); this set contains the set of strategies available at (x',n) when $x' > x$ so that $v(x',n)$ cannot be larger than $v(x,n)$.

Hence, there exists unique numbers x_n such that if $x \geq x_n$, one should accept a candidate of value $X_n = x$ at n stages from the end. The values of the x_n are as follows.

Theorem 4. *In the full information duration problem, it is optimal to select a relatively best object of value $X_n = x$ at n stages from the end if $x \geq x_n$, where $x_1 = 0$ and for $n > 1$, x_n is the unique root of the equation,*

$$\sum_{k=1}^{n} x^{k-1} = \sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k} (1-x^j)/j \tag{3.3}$$

Proof. We show in fact that the rule described in the theorem is the 1-stage look-ahead rule and that the problem is monotone. Then, since the horizon is finite, this rule is optimal.

If we stop with a relatively-best object of value $X_n = x$, n stages from the end, we receive w(x,n). If instead, we continue and select the next relatively-best object if any, we expect to receive

$$\begin{aligned} u(x,n) &= \textstyle\sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t,n-k)\,dt \\ &= \textstyle\sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k} (1-x^j)/j \end{aligned} \tag{3.4}$$

The one-stage look-ahead rule calls for stopping if $w(x,n) \geq u(x,n)$; that is, if

$$\sum_{k=1}^{n} x^{k-1}(1 - \sum_{j=1}^{n-k}(1-x^j)/j) \geq 0. \tag{3.5}$$

To show the problem is monotone, we must show that if this inequality holds for n it also holds for $n-1$, or equivalently, if the reverse inequality holds for n, then it also holds for $n+1$. We write the reverse inequality, after changing index from k to $i=n-k$ and cancelling x^{n-1}, as

$$1+\sum_{i=1}^{n-1} x^{-i}[1-\sum_{j=1}^{i}(1-x^j)/j]<0. \tag{3.6}$$

If this is true for some n, then since the term in square brackets is decreasing in i, this term must be negative for $i=n$ and hence for $i=n+1$. Hence, inequality (3.6) also holds for $n+1$, and the proof is complete.

Just as the duration problem of Section 2.2 is related to the random horizon best-choice problem of Presman and Sonin (1972), the corresponding full-information duration problem is related to the full-information random horizon best-choice problem of Porosiński (1987). The optimal rule found in Theorem 4 is the same as the optimal rule found by Porosiński for the full- information best-choice problem when the total number of objects is given a uniform distribution on the integers $\{1,2,...,n\}$. Moreover, Porosiński gives an asymptotic approximation for the x_n that may be obtained as follows. Write x_n as $1-z_n/n$ and write (3.3) in the form

$$\sum_{k=1}^{n}(1-\frac{z_n}{n})^{k-1}[1-\sum_{j=1}^{n-k}(1-(1-\frac{z_n}{n})^j)/j]=0. \tag{3.7}$$

In order for this to stay equal to zero as $n\to\infty$, the z_n must converge to a constant, $z_n\to z$, where z satisfies the equation,

$$\int_0^1 e^{-zv}[1-\int_0^{1-v}(1-e^{-zu})/u\,du]\,dv=0,$$

or, equivalently

$$\int_0^z e^v[1-\int_0^v \frac{1}{u}(1-e^{-u})\,du]\,dv=0. \tag{3.8}$$

The root of this equation may be found by numerical methods to be $z=2.1198\ldots$, giving the approximation, $x_n\simeq 1-2.1198/n$. Tables of x_n and its approximation for $n=2,3,\ldots,61$, may be found in Porosiński's paper. (Note that his b_k is equal to our x_{k+1}.)

Sampling With Recall. If we are allowed to select any of the observed objects, then it is easy to see that the one-stage look-ahead rule is optimal. If we stop n stages from the end with an object of value x, then we receive $w(x,n)$ as the expected payoff. If we continue one stage and then stop, we expect to receive,

$$xw(x,n-1)+\int_x^1 w(y,n-1)\,dy=\sum_{j=1}^{n-1}x^j+\sum_{j=1}^{n-1}\frac{1}{j}(1-x^j).$$

The one-stage look-ahead rule therefore stops n stages from the end if this is less than or equal to $w(x,n)$, that is if

$$\sum_{j=1}^{n-1} \frac{1}{j}(1-x^j) \le 1. \tag{3.9}$$

Since the left side is decreasing in x, the set of x for which the one stage look ahead rule calls for stopping is an interval of the form $[x_n, 1]$.

Theorem 5. *In the full information duration problem with recall, it is optimal to stop n stages from the end if the largest of the values observed is at least x_n, where x_n is the root in $[0,1]$ of the equation*

$$\sum_{j=1}^{n-1} \frac{1}{j}(1-x^j) = 1. \tag{3.10}$$

Proof. Since the horizon is finite, it is sufficient to show that the problem is monotone. Suppose then that x satisfies (3.9) for some n. We are to show that it satisfies (3.9) for $n-1$. But this is obvious since the left side gets smaller as n gets smaller.

One may show that this is also the solution to the full- information best-choice problem with a uniform random number of objects when recall is allowed.

We may find asymptotic approximations to the x_n. Replace x in (3.10) by $(1-z/n)$ and take the limit as $n \to \infty$ to find

$$\int_0^1 \frac{1}{y}(1-e^{-zy})\,dy = 1.$$

Numerical methods give $z = 1.3450\cdots$ as the solution of this equation, leading to the approximation, $x_n \simeq 1 - 1.3450/n$. The accuracy may be judged by the following table.

n	True	Approx
3	.4142	.5516
10	.8549	.8655
50	.9727	.9731

3.2 The Best-Choice Duration Problem.

Here we treat the full information version of the problem treated in Section 2.3 where we win the duration of owning the best object only if it is best overall. If $w(x,n)$ represents the expected payoff given that we select the nth object from the end and $M_n = X_n = x$, then

$$w(x,n) = nx^{n-1}.$$

The values, $v(x,n)$, when there are n objects to be observed and the present maximum is x, again satisfy the recursion (3.2) with initial condition, $v(x,0) \equiv 0$. The same argument shows that there are unique numbers x_n such that one should accept a candidate of value $X_n = x \geq x_n$, n stages from the end.

Theorem 6. *In the full information best-choice duration poblem, it is optimal to select a relatively best object of value $X_n = x$ at n stages from the end if $x \geq x_n$, where $x_1 = 0$ and for $n > 1$, x_n is the unique root of the equation,*

$$\sum_{k=0}^{n-2} x^k - nx^{n-1} = 0. \tag{3.11}$$

Proof. x_n is the unique root of the equation, $v(x, n-1) = w(x, n)$. If the x_n so found are increasing, then for $x > x_n$, with n stages to go and the present maximum value equal to x, it is optimal to select the first object of value greater than x. Hence, for $x > x_n$,

$$\begin{aligned} v(x,n) &= \textstyle\sum_{k=1}^{n} x^{k-1} \int_x^1 w(t, n+1-k)\,dt \\ &= \textstyle\sum_{k=1}^{n} x^{k-1}(1 - x^{n-k+1}) \\ &= \textstyle\sum_{k=0}^{n-1} x^k - nx^n \end{aligned}$$

Then, x_{n+1} may be found by equating this to $w(x, n+1)$ and solving. For x_n, this becomes (3.11). To show that the x_n are increasing, let $f_n(x)$ denote the left side of (3.11) multiplied by $1 - x$ so that

$$f_n(x) = (2n-1)x^n - 2nx^{n-1} + 1.$$

This polynomial is one at $x = 0$ and zero at $x = 1$, and is decreasing for $x < 2(n-1)/(2n-1)$ and increasing thereafter. Hence, $f_n(x)$ has a unique root $x_n < 2(n-1)/(2n-1)$. We will show that $x_{n+1} > x_n$ by showing that $f_{n+1}(x_n) > 0$. Write $f_{n+1}(x) - f_n(x) = x^{n-1}(1-x)(2n-(2n+1)x)$. Then, $f_{n+1}(x_n) = x_n^{n-1}(1-x_n)(2n-(2n+1)x_n) > 0$ since $x_n < (2n-2)/(2n-1) < 2n/(2n+1)$, completing the proof.

Asymptotically as $n \to \infty$, we find $x_n \simeq 1 - \frac{\alpha}{n}$, where α satisfies $e^\alpha = 1 + 2\alpha$, or $\alpha = 1.25643\cdots$.

n	True	Approx
2	.3333	.3718
10	.8719	.8744
50	.9748	.9749

With Recall. Assume we are allowed to recall past observations and that we are n stages from the end with $M_n = x$. If we stop we receive $w(x, n) = nx^{n-1}$; if we continue one stage and stop, we expect to receive

$$xw(x, n-1) + \int_x^1 w(t, n-1)\,dt = (n-2)x^{n-1} + 1.$$

The one-stage look-ahead rule stops if the former is greater than the latter, namely if $x^{n-1} \geq 1/2$. Clearly the problem is monotone and hence this rule is optimal.

Theorem 7. *In the full information best-choice duration problem with recall, it is optimal to stop n stages from the end if the largest of the values observed is at least* $x_n = 2^{-1/(n-1)}$.

Asymptotically, we have $x_n \simeq 1 - \log(2)/n$.

3.3 The Discounted Problem.

Consider the full-information duration problem with recall and with infinite horizon in which future payoffs are discounted by an amount β per period, where $0 < \beta < 1$. The payoff now depends on time from the beginning rather than on time to go. The objects, in order of observation, have values $X_1, X_2, \cdots$, assumed to be independent uniform (0,1) random variables. Let $M_n = \max\{X_1, \cdots, X_n\}$ for $n = 1, 2, \cdots$. The conditional expected return for selecting the best object at stage n, given that $M_n = x$ is

$$y_n(x) = (1-\beta)\beta^n[1 + x\beta[1 + x\beta[1 + \cdots]]] = (1-\beta)\beta^n/(1 - x\beta).$$

This is a stopping rule problem with observations $\{X_n\}$ and with payoff for stopping at n equal to $Y_n = y_n(M_n)$. Since $E\sup_n Y_n$ is finite (in fact, $y_n(x) \le 1$), the rule given by the principle of optimality is optimal. This rule is

$$N^* = \min\{n \ge 1 : Y_n \ge \operatorname{ess\,sup}_{N>0} E\{Y_N | X_1, \cdots, X_n\}\}.$$

There is a time invariance for this problem that makes it easy to solve. If V(x) denotes the optimal expected payoff for the problem modified so that stopping at a value less than x is prohibited, then for $n \ge 1$, $\operatorname{ess\,sup}_{N \ge n} E\{Y_N | X_1, \cdots, X_n\} = \beta^n V(M_n)$ a.s.. With this and the formula for y_n substituted into the inequality in the expression for N^*, we deduce the inequality $(1-\beta)/(1-\beta X_n) \ge V(X_n)$. Since the left side is increasing in X_n and the right side is nonincreasing in X_n, there is a unique root, y, so the rule N^* may be found among rules of the form

$$N_y = \min\{n \ge 1 : X_n \ge y\}.$$

We have only to evaluate the expected return for such rules, and find the value of y that gives the maximum expected return. Given $N_y = n$, the conditional distribution of X_n is uniform on (y,1). Hence,

$$EY_{N_y} = E\beta^{N_y}(1-\beta)/(1 - X_{N_y}) = E\frac{\beta^{N_y}}{1-y}\int_y^1 \frac{1-\beta}{1-x\beta}\,dx = \frac{(1-\beta)}{(1-y)\beta}\log(\frac{1-y\beta}{1-y})E\beta^{N_y}.$$

Since N_y has a geometric distribution, $E\beta^{N_y}$ is easily computed to be $(1-y)\beta/(1-y\beta)$. Hence,

$$EY_{N_y} = \frac{1-\beta}{1-y\beta}\log(\frac{1-y\beta}{1-\beta}). \tag{3.15}$$

The optimal choice of y is the value of $y \ge 0$ that maximizes this. Taking a derivative with respect to y and setting to zero gives $\log((1-y\beta)/(1-\beta)) = 1$, or equivalently,

$$y = \frac{1 - e(1-\beta)}{\beta}.$$

If this is negative, then (3.15) is a decreasing function of y and the optimal value of y is zero.

Theorem 8. *The optimal rule for the discounted full-information duration problem with recall is N_y, where*

$$y = \left(\frac{1 - e(1-\beta)}{\beta}\right)^{+}.$$

In particular, if $\beta \leq 1 - 1/e = .63212\cdots$, it is optimal to select the first object. The optimal payoff is equal to $1/e$ for all $\beta \geq 1 - 1/e$.

This is exactly the rule discovered by Porosiński to be optimal in the full-information best-choice problem with a random number of objects having a geometric distribution with probability β. As he points out, this rule is even simpler than the Gilbert-Mosteller rule when the number of objects is known.

Without Recall. Theorem 8 holds for the full-information duration problem without recall as well. The optimal rule never recalls a previously discarded observation. So if it is best when one is allowed to recall, it is best also within the smaller class of rules for which recall is not allowed.

4. Random Arrival Models.

We assume that objects appear at random times given by a Poisson process on $[0, T]$ at rate $\lambda > 0$. We look briefly at the problem of selecting a relatively best object with the view of maximizing the duration that it remains relatively best. Here, duration is measured in "real" time, not just in the number of arrivals.

4.1 The No-Information Case.

The no-information best-choice random arrival problem has been studied by Cowan and Zabczyk (1978). They have shown that it is optimal to select the mth object to arrive provided it is relatively best and its arrival time t satisfies $\lambda(T - t) \leq x_m$, where x_m is the unique solution of the equation,

$$\sum_{j=0}^{\infty} \frac{x^j}{j!(m+j)} \left(1 - \sum_{k=1}^{j} \frac{1}{k+m-1}\right) = 0.$$

The direct extension of their result to the duration problem is rather difficult. Instead, we treat the model of Bruss (1987) where λ is allowed to be unknown and is assigned a prior exponential distribution, $a \exp\{-a\lambda\}\, I(\lambda > 0)$ where $a > 0$ is a known parameter. As is shown in Bruss, the conditional distribution of the number of arrivals in $[t, s]$ for $s > t$ given the history, $\mathcal{F}_t$, to time t is negative binomial:

$$P\{j \text{ arrivals in } [t,s] | \mathcal{F}_t\} = P(j|t,s,k) = \frac{(j+k)!}{j!\,k!} \left(\frac{s-t}{s+a}\right)^j \left(\frac{a+t}{a+s}\right)^{k+1} \quad \text{for} \quad j = 0, 1, \ldots$$

where k represents the number of arrivals in $[0,t]$. From this, one can deduce the probability of no new *records* in $[t,s]$ given k arrivals in $[0,t]$ to be

$$P\{\text{no records in } [t,s] | \mathcal{F}_t\} = \sum_{j=0}^{\infty} P(j|t,s,k) \frac{k}{k+j} = \frac{a+t}{a+s}. \tag{4.1}$$

The remarkable feature of this probability is that it is independent of k. An investigation of other arrival processes that share this feature with the Poisson-exponential model has been carried out by Bruss and Rogers (1990). Using this, we may derive the optimal rule.

Theorem 9. *In the no-information duration poblem with random arrivals on $[0,T]$ following a Poisson process at rate $\lambda > 0$ having an exponential distribution with rate parameter a, it is optimal to select a relatively best object that appears at time t if*

$$\frac{t+a}{T+a} \geq e^{-2}. \tag{4.2}$$

Proof. Let S denote the time of the appearance of the first record after time t, and let $F(s|a,t)$ denote its distribution function, $F(s|a,t) = (s-t)/(a+s)$ from (4.1). If a relatively best object appears at time t and we select it, we expect to receive

$$\begin{aligned} E(S-t) &= \int_t^T (s-t)\, dF(s|a,t) + (T-t)(1-F(T|a,t)) \\ &= (a+t)\log\big(\frac{a+T}{a+t}\big). \end{aligned} \tag{4.3}$$

If, instead, we reject it and then select the next relatively best object to appear, we expect to receive

$$\int_t^T (a+s)\log\big(\frac{a+T}{a+s}\big)\, dF(s|a,t) = \frac{a+t}{2}[\log\big(\frac{a+T}{a+t}\big)]^2.$$

The one-stage look-ahead rule has us stop if the former is at least as great as the latter, which reduces to the condition in the statement of the theorem, From this one sees that the problem is monotone with bounded payoff, so this rule is optimal.

It is interesting to observe the behavior of the expected payoff of the optimal rule. Let $t(a)$ denote the time at which it is optimal to select the next relatively best object, and let $v(a)$ denote the corresponding optimal return. Then from the theorem we find that:
If $a \leq 1/(e^2+1) = .1565\cdots$, then $t(a) = T\,e^{-2} - a(1-e^{-2})$ and $v(a) = 2e^{-2}(T+a)$.
If $a \geq 1/(e^2+1)$, then $t(a) = 0$ and $v(a) = (a/2)\log((a+T)/a)$.
For a close to zero (large arrival rate), the optimal return is close to $2e^{-2}T$, the limiting optimal value for the finite horizon duration problem of section 3. As a increases, the optimal return increases to a maximum of about $.406T$ at $a/T = 0.255\cdots$, (the root of $\log((a+T)/a) = T/(a+T)$), and then decreases to zero as $a \to \infty$. At the point of maximum return, it is optimal to select the first arrival.

With Recall. The no information random arrival model treated above has an even simpler solution if recall is permitted. We assume that one is allowed to stop at any time and select any of the past observations. The expected payoff for stopping at t is just (4.3). The value of t that minimizes (4.3) is easily found to satisfy

$$\frac{a+t}{a+T} = e^{-1}.$$

That this is the optimal rule follows as in the proof of Theorem 9 by discretizing time, finding the 1-stage look-ahead rule and passing to the limit. Alternatively, one may show that this is the infinitesimal look-ahead rule and apply the theory of Ross (1971). Thus the optimal rule for this problem is a fixed-time stopping rule.

4.2 The Full Information Case.

We assume that the objects arrive at the times of a Poisson process with known rate λ. Associated with each object is its value; the values are assumed to be i.i.d. uniform $[0,1]$, independent of the arrival times. Otherwise, the problem is as before.

The best-choice version of this problem has been solved independently by Sakaguchi (1976) and Bodjecki (1978). In these papers it is shown that it is optimal to select a relatively best object of value x with time t remaining if $x \geq 1 - \frac{z}{\lambda t}$, where $z = .80435\cdots$ satisfies

$$\int_0^z \frac{1}{t}(e^t - 1)\, dt = 1.$$

This is the one-stage look-ahead rule and the problem is monotone. This result extends easily to the duration problem.

Theorem 10. *In the full information duration poblem with random arrivals following a Poisson process at rate $\lambda > 0$, it is optimal to select a relatively best object of value x that appears with time t to go, if*

$$x \geq 1 - \frac{z}{\lambda t},$$

where $z = 2.11982\cdots$ satisfies (3.8).

Proof. We show that this rule is the 1-stage look- ahead rule and the problem is monotone, and since the loss is bounded this rule is optimal. As Sakaguchi shows, the probability of no records in time t given that the present maximum value is x is

$$\sum_{j=0}^{\infty} x^j e^{-\lambda t}(\lambda t)^j/j! = e^{-\lambda t(1-x)},$$

so that the distribution of the arrival time of the next record, given the present maximum is x, is exponential at rate $\lambda(1-x)$. If we select a relatively best object of value x with time t remaining, our expected return is

$$\begin{aligned} w(x,t) &= \int_0^t s\lambda(1-x)e^{-\lambda(1-x)s}\, ds + te^{-\lambda(1-x)t} \\ &= \int_0^t e^{-\lambda(1-x)s}\, ds \end{aligned} \tag{4.4}$$

If, instead, we continue and select the next relatively best object to appear, we expect to win

$$\begin{aligned}
v(x,t) &= \int_0^t \lambda(1-x)e^{-\lambda(1-x)s}\int_x^1 \frac{1}{1-x}w(y,t-s)\,dy\,ds \\
&= \lambda\int_0^t e^{-\lambda(1-x)s}\int_0^{t-s}\int_x^1 e^{-\lambda(1-y)r}\,dr\,dy\,ds \\
&= \int_0^t e^{-\lambda(1-x)s}\int_0^{\lambda(1-x)(t-s)}\frac{1}{u}(1-e^{-u})\,du\,ds \\
&= \frac{e^{-z(x,t)}}{\lambda(1-x)}\int_0^{z(x,t)} e^v\int_0^v \frac{1}{u}(1-e^{-u})\,du\,dv,
\end{aligned}$$

where $z(x,t) = \lambda(1-x)t$. The one-stage look-ahead rule requires stopping if $w(x,t) \geq v(x,t)$, or equivalently, if

$$\int_0^{z(x,t)} e^v[1 - \int_0^v \frac{1}{u}(1-e^{-u})\,du]\,dv \geq 0. \tag{4.5}$$

This reduces to the rule stated in the theorem. To see that the problem is monotone, note that the left side of (4.5) is increasing in z if $\int_0^z (1-e^{-u})/u\,du \leq 1$, that is if $z > 1.345\cdots$, and decreasing in z thereafter. Since $z(x,t)$ is decreasing as time to go decreases or as x increases, once (4.5) becomes satisfied, it stays satisfied. Thus the problem is monotone and the proof is complete.

With Recall. Suppose that recall of past observations is allowed. We may stop at any time and when we stop we select the best of the objects observed. If we stop at time t with an object of value x, we receive w(x,t) of (4.4). If we continue for an additional length of time Δt and then stop, we expect to receive, to first order terms,

$$\exp\{-\lambda(1-x)\Delta t\}w(x,t+\Delta t) + (1-\exp\{-\lambda(1-x)\Delta t\})\int_x^1 w(y,t+\Delta t)\,dy/(1-x)$$

The Δt-look-ahead rule stops at t if this quantity is less than $w(x,t)$. After dividing the resulting inequality by Δt and passing to the limit, we obtain the differential equation,

$$\lambda(1-x)w(x,t) - \frac{\partial}{\partial t}w(x,t) \geq \lambda\int_x^1 w(y,t)\,dy.$$

Replacing w(x,t) by its value from (4.4) and changing the variable y in the integral to $u = \lambda t(1-y)$, we obtain the infinitesimal look-ahead rule in the form: Stop as soon as the time to go, t, and the present observed maximum, x satisfy the inequality

$$\int_0^{\lambda t(1-x)} \frac{1-e^{-u}}{u}\,du \leq 1. \tag{4.6}$$

Since the left side decreases as x increases and as t decreases, the problem is monotone and this is the optimal rule. It stops as soon as $\lambda t(1-x) \leq .80435\cdots$.

This is exactly the same inequality that describes the solution to the Sakaguchi-Bodjecki problem. However, the actual stopping rules differ. In the Sakaguchi-Bodjecki problem, stopping only occurs at the arrival of a relatively-best object. Here stopping can occur as t decreases to satisfy (4.6) with equality.

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References.

T. Bodjecki (1978) "On optimal stopping of a sequence of independent random variables – Probability maximizing approach" *Stoch. Proc. & Their Appl.* **6** 153-163.

F. Thomas Bruss (1987) "On an optimal selection problem of Cowan and Zabczyk" *J. Appl. Prob.* **24** 918-28.

F. Thomas Bruss and L. C. G. Rogers (1990) "Pascal processes and their characterization" *Stoch. Proc. & Their Appl.* To appear.

R. Cowan and J. Zabczyk (1978) "An optimal selection problem associated with the Poisson process" *Theory Prob. Appl.* **23** 584-592.

M. H. DeGroot (1970) *Optimal Statistical Decisions* McGraw-Hill, New York.

D. P. Gaver (1976) "Random record models" *J. Appl. Prob.* **13** 538- 547.

J. Gilbert and F. Mosteller (1966) "Recognizing the maximum of a sequence" *JASA* **61** 35-73.

Z. Porosiński (1987) "The full-information best-choice problem with a random number of observations" *Stoch. Proc & Appl.* **24** 293-307.

E. L. Presman and I. M. Sonin (1972) "The best-choice problem for a random number of objects" *Theory Prob. & Their Appl.* **17** 657-668.

W. T. Rasmussen and H. Robbins (1975) "The candidate problem with unknown population size" *J. Appl. Prob.* **12** 692-701.

S. M. Ross (1971) "Infinitesimal look-ahead stopping rules" *Ann. Math. Statist.* **42** 297-303.

M. Sakaguchi (1976) "Optimal stopping problems for randomly arriving offers" *Mathematicae Japonicae* **21** 201-217.

S. M. Samuels (1990) "Secretary problems" Chapter 16 in *Handbook of Sequential Analysis* P. K. Sen and B. K. Ghosh eds., Marcel-Dekker.

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