HIGH RISK AND COMPETITIVE INVESTMENT MODELS

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Summary. How should we invest capital into a sequence of investment opportunities, if, for reasons of external competition, our interest focuses on trying to invest in the very best opportunity? We introduce new models to answer such questions. Our objective is to formulate them in a way that makes results high-risk specific in order to present true alternatives to other models. At the same time we try to keep them applicable in quite some generality, also for different utility functions. Viewing high risk situations we assume that an investment on the very best opportunity yields a lucrative, possibly time-dependent, rate of return, that uninvested capital keeps its risk-free value, whereas "wrong" investments lose their value. Several models are presented, mainly for the so-called rank-based case. Optimal strategies and values are found, also for different utility functions, and several examples are explicitly solved. We also include results for the so-called full-information case, where, in addition, the quality distribution of investment opportunities is supposed to be known. In addition we present tractable models for an unknown number of opportunities in terms of Pascal arrival processes. Effort is made throghout the paper to justify assumptions in the view of applicability.

Short title: HIGH RISK INVESTMENT MODELS

1. Introduction.

In the classical secretary problem, the selection of a secretary is a yes-or-no choice. We cannot hedge our selection by selecting half of a secretary. In an attempt to make

the selection process smoother, we may consider the following scheme which we view as a suitable model for investment.

Each occasion for an investment will be called *opportunity*. Unless stated otherwise we make no assumptions about the distribution of the value (or measurable quality) of opportunities; indeed, we only suppose that we can rank opportunities with respect to those observed before. In this case we speak of a *rank-based* model, and we will first confine our interest to such models.

1.1 The model. The basic model is as follows: Our initial fortune is x_0. We are going to observe a known number, $n > 0$, of rankable opportunities sequentially in a completely random order. At the first stage after observing the first opportunity, we may invest any amount b_1, $0 \leq b_1 \leq x_0$, in that opportunity, leaving fortune $x_1 = x_0 - b_1$ for future investments. If, after all n opportunities have been observed, this opportunity is best overall, then the return on our investment is $y_1 = \beta_1 b_1$ where $\beta_1 \geq 1$ is a known rate of return available at stage 1. If it is not best overall, we suppose we lose our investment.

Similarly, at stage $k = 2, 3, \ldots, n$, if the kth opportunity is relatively best and our remaining (uninvested) fortune is x_{k-1}, we may invest any amount b_k, $0 \leq b_k \leq x_{k-1}$, in the kth opportunity and the return on the investment will be $y_k = \beta_k b_k$ if the kth opportunity is best overall (=0 otherwise), where $\beta_k \geq 1$. Our problem is to choose a sequence of investments to maximize the expected value of our total fortune after the proceedings have concluded. No interest accumulates on uninvested capital or on lost capital.

More generally, we may have a utility function defined on fortunes, and we may wish to maximize the expected value of the utility of our total fortune at the end of the proceedings. Typical utility functions for such investment problems are

$$
\begin{aligned}
u_\alpha(x) &= (x^\alpha - 1)/\alpha \quad \text{for } \alpha \neq 0 \\
u_0(x) &= \log(x) \quad \text{corresponding to } \alpha = 0.
\end{aligned}
\tag{1.1}
$$

This form of the utility functions is chosen because it is continuous in α at $\alpha = 0$. The case $\alpha = 1$ corresponds to the linear utility of the description above. The use of these utility functions for sequential investment problems goes back to Bellman and Kalaba in the late 1950's. (See Bellman and Kalaba (1965).)

Variations of this basic model will be defined and studied in Sections 5. and 6.

1.2 Model characteristics and comparisons. A typical feature of portfolio selection problems is that investments are spread over a set of stocks or assets which jointly contribute to the global objective which is usually the rate of return on all investments. (For the objective of avoiding bancruptcy see e.g. the model by Assaf et al. (2000))

These contrast to the model we consider. Our model becomes an interesting alternative in those cases where the investment has a specific target and where the best opportunity is essentially the only one which is a winner. Here we want to invest in the one which turns out *best*, or, alternatively, *not to invest* at all. In particular, we do not try

to measure risk coherently (see e.g. Artzner et al. (1999)) but we just concentrate on the optimization problem to select the best opportunity among those we consider as belonging to the high-risk class.

The high risk is reflected in our assumption that wrong investments are lost (or, in reality, essentially lost). Since an opportunity cannot be best of all unless it is the relative-best of those observed so far, we can limit our interest to investment decisions for relative-best opportunities. A relative-best opportunity will be called record opportunity, or simply *record*, throughout the paper.

We think here mainly of that kind of risk in investments which is linked to competition, be it among those who provide investment opportunities to the investor, or else, be it simply among investors themselves. As an example for the first case, we may think of several ventures which follow one specific goal, as e.g. the development of a new technology, and the best will get the whole market. The high risk for the investor stems here from the competition among those who provide investment opportunities. The second case, i.e. competition among investors, is not uncommon either. Some investors, as e.g. investment funds, like to attract attention of the public by a high return (compared with competitors) in order to attract more foreign capital in the future. Hence they typically specialize in high risk investments, though not necessarily in targeted venture investments in order to outperform competitors. We also mention that if, unlike in our present model, uninvested capital is supposed to be lost, then some such problems become equivalent to the problem of stopping (investing everything) on the last record. This problem is conceptually easier. In the case of a record arrival process with independent increments, for instance, the *odds-algorithm* (Bruss (2000), see Theorem 1 and 2.1), gives at once the form of the optimal strategy and its value.

Since our present model is different from the above models, the methods we apply are also quite different. Honoring the relationship in some of the assumptions with those in the classical secretary problem we may name these models "secretarial investment models". However, the chosen title indicates the goal of these models and is thus more informative.

Despite some relationship with the secretary problem (best-choice problem) it should also be pointed out that this relationship has its limits. In particular, investment strategies in our model *cannot* be seen as versions of randomized strategies for secretary problems. So, for example, if investing, at a certain stage, 50 percent say, of the available capital in an opportunity is an optimal decision (that is , this decision maximizes the expected reward), then flipping a fair coin to decide whether to invest all or nothing, and to go on optimally thereafter, is in general not optimal. Thus "hedging" by a partial investment has a real meaning in this context.

1.3 The choice of the payoff structure. The payoff structure we propose raises some questions and we should justify our choice.

We do realize that this choice may have disadvantages. In particular, it may seem somewhat naive that we do not try to link the payoffs β_k to the probability of the kth opportunity being a record. Indeed, one feels that an investor would evaluate an opportunity as being a record more likely if its rate β is itself a record compared with preceding

observed rates. This argument seems then to imply that it would be better to confine our interest to full information models and to try to link the rate β with the observed value (that is, quality of rating or with another measure) under the hypothesis of a known quality distribution of opportunities.

However, there are several major reasons why this is not an adequate idea. The first important reason is that for (targeted) venture capital investments in new technologies, a technological break-through of the chosen company is frequently the main ingredient of success. History shows that small companies are often more flexible and inventive and compete well with large companies, even without adjustment for size. But their rate of return would normally still be relatively modest by comparison since their marketing and sales capacities are usually less developed than those of larger companies. Thus the rate of return β_k is linked with factors others than just the degree of a technological advantage. Time is clearly an important such factor so that we should and do allow for the dependence on k

The second reason for our preference for the proposed model is that full-information models may not only be mathematically more difficult than rank-based models but, as we think, hard to implement. It is clearly not obvious to estimate in practice the function which determines the relationship between the return rate and the probability of success of an opportunity. Moreover, this difficulty is enhanced by barriers of secrecy which are typically present in a competitive environment. Consequently, given that full-information models are, in general, more sensitive to errors in the hypotheses than rank-based models, this is a strong support for the latter.

The β_k's are, as we pointed out already, usually supposed to depend on time. In some continuous time models we denote the corresponding function accordingly $\beta(t)$. Of course it is understood, that the choice of β_k or $\beta(t)$ depends also intrinsically on the specific problem. If the β_k are chosen to be constant then we see them as average return rate per monetary unit. Often the choice of monotone β_k's is the most natural one, and we will treat several such examples. But for the sake of generality we keep results in a form which is independent of monotonicity assumptions.

Before summarizing our results we first derive the basic recursion equations for the fixed-n problem, which is our basic model.

1.4 The Recursion Equations. Let $V_k(x, y)$ represent the expected utility of the final fortune using an optimal strategy, when at stage k, $1 \le k \le n$, before we observe the kth opportunity, we have an amount x available for future investments, and a preceding investment that will return y if the present record is best overall. We first note that we would invest in an opportunity only if it is a re cord because otherwise it cannot be best of all. We now show that the initial (backward) condition and recursion equations are

$$
\begin{aligned}
V_n(x,y) &= \frac{n-1}{n} u_\alpha(x+y) + \frac{1}{n} u_\alpha(\beta_n x) \quad \text{and} \\
V_k(x,y) &= \frac{k-1}{k} V_{k+1}(x,y) + \frac{1}{k} \max_{0 \le b \le x} V_{k+1}(x-b, \beta_k b) \quad \text{for } k = 1, \dots, n-1.
\end{aligned}
\tag{1.2}
$$

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Indeed, if at the kth stage, the kth opportunity is not a record, which happens with probability $(k-1)/k$, we proceed to the next stage with the same x and y; if the kth opportunity is a record (with probability $1/k$), the investment y is lost, but we now choose to invest b in the kth opportunity to maximize our expected utility. This explains the second line equations in (1.2). The first line ($k = n$) is just a special case: At the nth stage, if the nth opportunity is a record, it is certain to be best overall, so we must invest all our remaining fortune.

1.5 Summary of results. In this paper, we find the solution of the recursion equations (1.2) for arbitrary $\alpha \leq 1$. Two cases, $\alpha = 1$ and $\alpha = 0$, are somewhat special so we treat them first separately. The log utility is a central case. The resulting optimal rule, which may be seen as a Kelly betting system, has been studied extensively and shown to have various optimality properties in standard investment problems for fixed n, independent of the choice of a utility function. See, among others, Breiman (1961), Bell and Cover (1980), Algoet and Cover (1988), and Browne and Whitt (1996)

We find that, in all cases, the optimal investment policy is a proportional investment system, that is to say that if a record appears at stage k, it is optimal to invest some proportion $0 \leq a_k \leq 1$, of the remaining fortune in this opportunity. Proportional investment systems in standard investment problems have also been studied extensively. See, for example, Ethier and Tavaré (1983), and also the surprising universal portfolio of Cover (1991).

We find the asymptotic form of the optimal rules and payoff, for large horizon, n. We also discuss briefly so-called *full-information* risk investment problem. Borrowing again from the terminology of secretary problems, (see Samuels (1991) for a survey), we use this name for those models where, in contrast to the rank-based case, the distribution of the qualities of opportunities is known. In our final section we drop the fixed-n hypothesis and study the case where the number of opportunities may be unknown with some hypotheses about the process of arriving opportunities.

2. Linear Utility

Let us first consider the case, $\alpha = 1$, of linear utility. For simplicity, we take $u_1(x) = x$ instead of $u_1(x) = x - 1$ as defined in (1.1), because the optimization problem is clearly equivalent. Then, according to the first equation in (1.2),

$$V_n(x, y) = \frac{n-1}{n}(x + y) + \beta_n \frac{x}{n}.$$

To compute V_{n-1}, we need to find $\max_{0 \leq b \leq x} V_n(x - b, \beta_{n-1}b)$. But according to (1.2) with $u_\alpha(x) = x$, this is linear in b and so its maximum occurs at either $b = 0$ or at $b = x$. Hence

$$\max_{0 \leq b \leq x} V_n(x - b, \beta_{n-1}b) = \max\{V_n(x, 0), V_n(0, \beta_{n-1}x)\}.$$

Therefore, from the second equation of (1.2),

$$V_{n-1}(x,y) = \frac{n-2}{n-1}\left[\frac{n-1}{n}(x+y) + \beta_n \frac{x}{n}\right] + \frac{1}{n-1}\frac{n-1}{n}\max\{1 + \frac{\beta_n}{n-1}, \beta_{n-1}\}x$$

$$= \frac{n-2}{n}(x+y) + \left[\frac{n-2}{n-1}\beta_n + \max\{1 + \frac{\beta_n}{n-1}, \beta_{n-1}\}\right]\frac{x}{n}$$

and the optimal investment at stage $n-1$ is to invest everything on a record opportunity if $n-1 \geq \beta_n/(\beta_{n-1}-1)$ and nothing otherwise. We see that the optimal return at stage $n-1$ has the same form as it does at stage n.

Using backward induction we arrive at the following theorem:

Theorem 1. *In the case $\alpha = 1$, we have*

$$V_k(x,y) = \frac{k-1}{n}[x+y+c_k x] \tag{2.1}$$

for $k = 2, \ldots, n$, and

$$V_1(x,y) = \frac{1}{n}\max\{1 + c_2, \beta_1\}x,$$

where $c_n = \beta_n/(n-1)$ and for $k = n-1, \ldots, 2$,

$$c_k = c_{k+1} + \frac{1}{k-1}\max\{1 + c_{k+1}, \beta_k\}. \tag{2.2}$$

An optimal investment policy is to invest everything in the first record, with arrival number k, for which $\beta_k > c_{k+1} + 1$.

Proof. We have seen that (2.1) holds for $k = n$. Suppose that (2.1) holds for $k+1$. Then from (1.2)

$$V_k(x,y) = \frac{k-1}{k}\left[\frac{k}{n}(x+y+c_{k+1}x)\right] + \frac{1}{k}\max_{0 \leq b \leq x}\frac{k}{n}[x - b + \beta_k b + c_{k+1}(x-b)]$$

$$= \frac{k-1}{n}[x+y+c_{k+1}x] + \frac{x}{n}\max\{1 + c_{k+1}, \beta_k\}$$

$$= \frac{k-1}{n}[x+y+(c_{k+1} + \frac{1}{k-1}\max\{1 + c_{k+1}, \beta_k\})x]$$

completing the induction. ∎

Note that it is impossible to have anything invested when entering the first stage. This is reflected in the fact that $V_1(x,y)$ does not depend on y anyway.

2.1 Asymptotic form. We allow the β_k and the c_k to depend on n, say $\beta_{k,n}$ and $c_{k,n}$. We assume there is a continuous function, $\beta(t) \geq 1$ on $(0,1]$, such that $\beta_k = \beta(k/n)$. We write the recursion (2.2) as

$$\frac{c_{k+1,n} - c_{k,n}}{1/n} = -\frac{n}{k-1}\max\{1 + c_{k,n}, \beta(k/n)\}. \tag{2.3}$$

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This shows that the $c_{k,n}$ are monotone decreasing. If we define monotone functions $f_n(t)$ on $[0,1]$ that interpolate the points $(k/n, c_{k,n})$, then (2.3) becomes

$$\frac{f_n((k+1)/n) - f_n(k/n)}{1/n} = -\frac{n}{k-1} \max\{1 + f_n(k/n), \beta(k/n)\}. \tag{2.4}$$

We will use (2.4) to derive a differential equation which will yield a useful *limiting optimal* solution. To make precise what we understand here by "limiting optimal" we need the following definition.

Definition. 2.1 Let β be a real-valued function defined on $[0,1]$, and let S_β be the set of all return sequences $((\beta_{1,n}, \beta_{2,n}, \cdots, \beta_{n,n}))_{n=1,2,\dots}$ satisfying $\beta_{k(n),n} \to \beta(t)$ whenever $k(n)/n \to t$ as $n \to \infty$. For $s \in S_\beta$ let $\rho_n(s)$ be the optimal expected utility for the corresponding reward vector $(\beta_{1,n}^s, \beta_{2,n}^s, \cdots, \beta_{n,n}^s)$. We say that an investment policy is limiting optimal on S_β with reward $\rho(\beta)$, if $\lim_{n\to\infty} \rho_n(s)$ exists for all $s \in S^\beta$ and satisfies

$$\lim_{n\to\infty} \rho_n(s) = \rho(\beta).$$

Theorem 2. *If β is continuous on $[0,1]$ and $\beta(t) > 1$ on $[0,1[$ then a limiting optimal policy exists and is of the following form: Invest all available capital in the first record opportunity whose arrival time t satisfies*

$$\beta(t) \geq 1 + f(t), \tag{2.5}$$

where $f(t)$ is the solution of the differential equation

$$f'(t) = -\frac{1}{t} \max\{1 + f(t), \beta(t)\} \tag{2.6}$$

with boundary condition $\lim_{t\to 0+} f(t) = \infty$.

Proof. Let $t \in]0,1[$ and $(k(n))$ be a sequence such that $k(n)/n \to t$ as $n \to \infty$. For n fixed, we define $k(n)/n$ as the arrival time of the $k(n)$-th opportunity on the interval $[0,1]$. Fix t with $0 < t < 1$. Then the policy of investing in the first record after time t, satisfying a certain condition, is defined as the policy of investing at the earliest record time k/n with $k/n \geq t$ (and satisfying the same condition), if such a k exists, and not to invest at all after t, otherwise. Let us call this policy 'n-policy (for t)'. Each reward vector of length n may have a different n-policy.

Our attack of embedding a t-policy in a sequence of n-policies is as follows. We first show that, as $n \to \infty$, n-policies for t allow, for each $s \in S_\beta$, for a well-defined limiting object ("t-policy") in the sense that the instruction of investing in the first record after t stays is well-defined. This requires only to show, that the first record (if any) does not appear (a.s) in a point of accumulation of records. We then show that, for some t^*,

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the limiting optimal reward for n-policies does exist and that it is also achieved by the t^*-policy, which we then can determine via the differential equation given in the Theorem.

For fixed n, let I_k be the indicator of the kth opportunity being a record. Let $R_{k,n} = I_{k+1} + I_{k+2} + \cdots + I_n$, that is, $R_{k,n}$ denotes the number of records among the $n - k$ last opportunities. It is well-known that the I_j are independent with $E(I_j) = 1/j$. It is easy to see from the generating function of $R_{k,n}$ (and well-known) that, as $k(n)/n \to t$, $R_{k,n}$ converges in law to a Poisson random variable, $R[t, 1]$, say, with parameter $\int_t^1 (1/s)ds = -\log(t)$. Hence, for $t > 0$, the variable $R[t, 1]$ is finite with probability one, so that, for $n = \infty$, there is a.s. no accumulation point of records after time t. Hence, in particular, there is no accumulation point $t > 0$ satisfying the investment condition $\beta(t) > 1 + f(t)$.

To see that we can ignore the point $t = 0$, we show now that the investment condition $\beta(t) > 1 + f(t)$ cannot hold unless $t > 0$. According to (2.3), we have for all $k \geq 2$ the inequality $-c_{k+1,n} + c_{k,n} \geq 1/(k-1)$, since all $c_{k,n}$ are nonnegative. Thus, as $n \to \infty$,

$$c_{2,n} - c_{n,n} = \sum_{k=2}^{n-1}(c_{k,n} - c_{k+1,n}) \to \infty,$$

and so, since $c_{n,n} \to 0$ by definition,

$$\lim_{n \to \infty} c_{2,n} = \lim_{n \to \infty} f_n\left(\frac{2}{n}\right) = \lim_{t \to 0+} f(t) = \infty.$$

Further, since the function β is continuous on $[0, 1]$ it is also bounded, and hence $\beta(t) > 1 + f(t)$ implies that $t > 0$.

Now for $t > 0$ the recursion equations (2.4) are just Euler-Cauchy approximations to the differential equation (2.5). Since β is continuous, this holds for any $s \in \beta$. Hence $f_n(t) \to f(t)$ for $t \in (0, 1]$ by standard arguments. See Henrici (1962) for example. Again, since β is continuous, $\beta(k(n)/n) \to \beta(t)$ as $k(n)/n \to t$, and hence from (2.1)

$$V_t(x, y) = \lim_{k(n)/n \to t} V_{k(n)}(x, y) = t(x + y + f(t)x).$$

Consequently

$$\rho = \lim_{t \to 0+} V_t(x, y) = \lim_{t \to 0+} tf(t)x.$$

This completes the proof. ∎

2.2 Computing the value. It is of interest to find the overall value of the investment model, $\lim_{t \to 0} V_t(x, y)$, for reasonable rate functions, $\beta(t)$. Note that $f(t) \to \infty$ as $t \to 0$. If $\beta(t)$ is bounded on the interval $(0, 1)$, then there is an interval $0 < t < t_0$ in which $\beta(t) \leq 1 + f(t)$, and equation (2.5), $f'(t) = -(1 + f(t))/t$, has the solution, $f(t) = c/t - 1$ where the constant, c, of integration satisfies $f(t_0) = c/t_0 - 1$. Thus $f(t) = (1 + f(t_0))t_0/t - 1$ for $0 < t < t_0$, and so

$$V_0(x, y) = \lim_{t \to 0} V_t(x, y) = x(1 + f(t_0))t_0. \tag{2.7}$$

Any t_0 such that $\beta(t) \leq 1 + f(t)$ for $0 < t < t_0$ may be used in this equation. Some examples will make this procedure clear.

2.3 Examples.

2.3.1 Constant return. As a first example, consider the constant return function, $\beta(t) \equiv \beta > 1$. The equation (2.5) becomes for $0 < t \leq 1$,

$$f'(t) = -\frac{1}{t} \begin{cases} \beta & \text{if } 1 + f(t) \leq \beta \\ 1 + f(t) & \text{if } 1 + f(t) > \beta \end{cases}$$

with boundary condition $f(1) = 0$. Since $f(t)$ is decreasing, there is a unique point t_0 such that $1 + f(t_0) = \beta$. We then have

$$f(t) = \begin{cases} -\beta \ln(t) & \text{if } t_0 < t \leq 1 \\ (t_0\beta/t) - 1 & \text{if } 0 < t \leq t_0 \end{cases}$$

where

$$t_0 = e^{-(\beta-1)/\beta}. \tag{2.8}$$

The optimal final expected fortune is the limit of $V_t(x, y)$ as t tends to zero as in (2.7), namely,

$$V_t(x, y) = t[x + y + ((t_0\beta/t) - 1)x] \to t_0\beta x. \tag{2.9}$$

For $\beta = 2$, this is essentially the three-value secretary problem of Sakaguchi (1984), where our repective rewards $0, 1, 2$ are replaced by $-1, 0, 1$. According to (2.8) we obtain $t_0 = e^{-1/2}$ and hence, from (2.9), a total optimal expected reward of $2e^{-1/2}x$ for an initial capital x.

2.3.2 Time proportional total return. As another example, consider $\beta(t) = c(1 - t)$ where c is a positive constant. Clearly, we would not invest if $c(1 - t) < 1$, i.e. we can limit our interest, without further mentioning, to $c > 1$ and $t < 1 - 1/c$. Equation (2.5) becomes

$$f'(t) = -\frac{1}{t} \begin{cases} 1 + f(t) & \text{if } f(t) \geq c(1 - t) - 1 \\ c(1 - t) & \text{if } f(t) < c(1 - t), \end{cases}$$

with boundary condition $f(1) = 0$. For t close to 1, the upper inequality is satisfied and the differential equation $f'(t) = -(1 + f(t))/t$ has solution, $\log(1 + f(t)) = -\log(t) + a_0$ for some constant of integration, a_0. By the boundary condition, the constant must be 0, and we have

$$f(t) = \frac{1}{t} - 1 \qquad \text{for } t_1 < t \leq 1.$$

This holds for t from 1 down until $1 + f(t) = c(1 - t)$. This quadratic equation has two roots $t = (1 \pm \sqrt{1 - (4/c)})/2$. We see that if $c \leq 4$, then $t_1 = 0$. Assume then that $c > 4$. The two roots are in the interval $(0, 1)$ and straddle $1/2$. Let $t_1 = (1 + \sqrt{1 - (4/c)})/2$ denote the larger root. Immediately to the left of t_1, the differential equation becomes $f'(t) = -c(1 - t)/t$, with solution

$$f(t) = -c\log(t) + ct + a_1 \qquad \text{for } t_0 < t \leq t_1,$$

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where $a_1 = (1/t_1) - 1 + c\log(t_1) - ct_1$. This holds for t down to the root t_0 of the equation, $1 + f(t) = c(1-t)$.

This equation has no simple solution so we consider a specific value of c as an example. Suppose $c = 16/3$. Then $t_1 = 3/4$, $a_1 = -5.20097$, and $t_0 = .31289$. For $t < t_0$, $f'(t) = -(1 + f(t))/t$ again, and

$$f(t) = 1/t - a_2 \qquad \text{for } 0 < t \le t_0,$$

for some constant a_2. In conclusion, the optimal strategy is to invest everything in the first record that appears between $t_0 = .31289$ and $t_1 = .75$. The overall value of the investment model is $V_0(x,0) = xc(1-t_0)t_0 = 1.1466x$.

2.3.3 Principal and interest for the best. Take $\beta(t) = 1 + (1-t)p$ where $p > 0$ represents an interest rate. Here interest means simple interest.

Equation (2.5) becomes

$$f'(t) = -\frac{1}{t} \begin{cases} 1 + (1-t)p & \text{if } (1-t)p \ge f(t) \\ 1 + f(t) & \text{if } (1-t)p < f(t), \end{cases}$$

with boundary condition $f(1) = 0$. If $p \le 1$, then $f(t)$, being convex with slope 1 at $t = 1$, is greater than $(1-t)p$ for all $t \in (0,1)$; so $f(t) = (1/t) - 1$ and it is optimal never to invest. Assume $p > 1$. Then to the immediate left of 1, we have $f'(t) = -(1/t)(1 + (1-t)p)$, and

$$f(t) = -(1+p)\log(t) - p(1-t) \qquad \text{for } t_0 \le t \le 1,$$

where t_0 is the root of the equation $f(t) = (1-t)p$ below 1. Below t_0, we have $f(t) > (1-t)p$ so that

$$f(t) = \frac{1}{t} - a_1 \qquad \text{for } 0 < t < t_0$$

where a_1 is a constant that makes $f(t)$ continuous at $t = t_0$. The optimal strategy for $p > 1$, is to invest in the first record, if any, to appear after t_0, where t_0 satisfies

$$-(1+p)\log(t_0) = 2(1-t_0)p, \quad 0 < t_0 < 1.$$

For example, if $p = e/(e-2) = 3.7844\cdots$, then $t_0 = 1/e$. We note that if p does not exceed 1 (that is 100 percent interest rate for the horizon $[0,1]$) it is optimal to never invest. Indeed, if we see a record at time t then this record is best overall with probability t. In this case we obtain $1 + (1-t)p$ per unit of investment (and zero otherwise) whereas we can keep the unit uninvested until the end and hence keep 1. But $t(1 + (1-t)p) > 1$ implies $p > 1/t \ge 1$.

Compound interest. Similarly, if we want to model *compound* interest with interest rate p percent per capitalization time unit on some horizon T, we simply adapt $\beta(t)$

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on the horizon. For continuous capitalization for instance we define $\beta(t)$ as the limit of $\beta_{k(n),n} = (1+p'/n)^{(n-k)T}$ as $n \to \infty$ with $k(n)/n \to t$ and $p' = \log(1+p)$. We then obtain

$$\beta_{k(n),n} = (1+\frac{p'}{n})^{(n-k)T} = (1+\frac{p'}{n})^{(n(n-k)/n)T} \to e^{p'(1-t)T} =: \beta(t).$$

Hence $\beta(t) = (1+p)^{(1-t)T}$, as desired, and the procedure to compute the optimal strategy and value is similar to above. Other capitalization periods can be handled similarly.

3. Log Utility

Now we look at the case $\alpha = 0$ of log utility linear, which may be considered as the most important utility function after linear utility. The recursion equations (1.2) become now

$$V_n(x,y) = \frac{n-1}{n} \log(x+y) + \frac{1}{n} \log(\beta_n x) \qquad \text{and}$$

$$V_k(x,y) = \frac{k-1}{k} V_{k+1}(x,y) + \frac{1}{k} \max_{0 \le b \le x} V_{k+1}(x-b, \beta_k b) \quad \text{for } k = n-1, \ldots, 1.$$

(3.1)

Solving the recursion we see that, in contrast to Theorem 1, we do *not* invest all or nothing in a record opportunity. Rather the optimal investment is a proportion, a_k, of our fortune, where

$$a_k = \left(\frac{k\beta_k - n}{n(\beta_k - 1)} \right)^+ . \tag{3.2}$$

It is to be understood that $a_k = 0$ if $\beta_k \le 1$.

Interestingly, this is just the Kelly betting system applied to the sequence of record opportunities. See Kelly (1956). For log utility, with an investment opportunity affording a return of $\beta > 1$ per unit invested with probability p and loss of the investment with probability $1-p$, the optimal proportion of fortune to invest is $a = (p\beta - 1)^+/(\beta - 1)$. Here, if the kth opportunity is a record, its probability of being absolutely best, and so returning the reward β_k, is $p = k/n$, leading to (3.2).

The remarkable feature of (3.2) is that, in contrast to the $\alpha \ne 0$ case, no backward computation needs to be done to find the optimal investment policy.

Theorem 3. *For $k = 1, 2, \ldots, n$,*

$$V_k(x,y) = \frac{k-1}{n} \log(x+y) + \frac{n-k+1}{n} \log(x) + c_k, \tag{3.3}$$

where $c_n = (1/n) \log \beta_n$, and for $k = n-1, n-2, \ldots, 1$,

$$c_k = c_{k+1} + \frac{1}{k} \left[\frac{k}{n} \log(1 + (\beta_k - 1)a_k) + \frac{n-k}{n} \log(1-a_k) \right], \tag{3.4}$$

where the a_k are given by (3.2).

It is optimal at a record stage k to invest the proportion a_k of the remaining fortune into the present record opportunity.

Proof. The proof is by backward induction.

From (3.1), the result is true for $k = n$. Suppose it is true for $k + 1$, that is,

$$V_{k+1}(x, y) = \frac{k}{n} \log(x + y) + \frac{n - k}{n} \log(x) + c_{k+1}. \tag{3.5}$$

Then

$$\max_{0 \le b \le x} V_{k+1}(x - b, \beta_k b) = \max_{0 \le b \le x} \left[\frac{k}{n} \log(x + (\beta_k - 1)b) + \frac{n - k}{n} \log(x - b) + c_{k+1} \right] \tag{3.6}$$

It is easy to check that the maximum occurs at $b = a_k x$. Thus the optimal investment in a record at stage k is $a_k x$. Substituting this in (3.6), we find

$$\max_{0 \le b \le x} V_{k+1}(x - b, \beta_k b) = \frac{k}{n} \log(x(1 + (\beta_k - 1)a_k)) + \frac{n - k}{n} \log(x(1 - a_k)) + c_{k+1}$$

$$= \log(x) + \frac{k}{n} \log(1 + (\beta_k - 1)a_k) + \frac{n - k}{n} \log(1 - a_k) + c_{k+1}.$$

Substituting this and (3.5) into (3.1) completes the proof. ∎

3.1 Asymptotic form. The asymptotic form of the optimal investment policy is easy to find. We assume there is a continuous function, $\beta(t) \ge 1$ on $(0, 1]$, such that $\beta_k = \beta(k/n)$ and pass to the limit in (10) as $n \to \infty$ and $k/n \to t$. Writing $a_{k,n}$ for a_k in (3.2), we find

$$a_{k,n} \to a(t) := \begin{cases} (t\beta(t) - 1)/(\beta(t) - 1) & \text{if } t\beta(t) > 1 \\ 0 & \text{if } t\beta(t) \le 1 \end{cases} \tag{3.7}$$

We then have the following theorem analogous to Theorem 2 for linear utility.

Theorem 4. *The limiting optimal expected fortune is*

$$V_t(x, y) = \lim_{\substack{n \to \infty \\ k/n \to t}} V_k(x, y) = t \log(x + y) + (1 - t) \log(x) + f(t). \tag{3.8}$$

where $f(t)$ satisfies the differential equation

$$f'(t) = \begin{cases} 0 & \text{for } t\beta(t) \le 1 \\ -\frac{1}{t} \left[\log \beta(t) - (1 - t) \log(\beta(t) - 1) + t \log(t) + (1 - t) \log(1 - t) \right] & \text{for } t\beta(t) > 1, \end{cases} \tag{3.9}$$

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for $t \in (0,1]$, with boundary condition $f(1) = 0$. The limiting optimal policy is to invest proportion $a(t)$ of the remaining fortune in a record that appears at time t.

3.2 Example. As an example, consider the constant return function, $\beta(t) \equiv \beta > 1$. The optimal investment policy is to invest proportion

$$a(t) = \begin{cases} (t\beta - 1)/(\beta - 1) & \text{if } t > 1/\beta \\ 0 & \text{if } t \leq 1/\beta \end{cases}$$

of the remaining fortune on a record appearing at time t.

For $\beta(t) = c(1-t)$, the set of t on which investment is made is $\{t : t(1-t) > 1/c\}$. For $c \leq 4$, this is empty, while for $c > 4$, it is an interval symmetric about $1/2$. For $\beta(t) = 1 + (1-t)p$, the investment set is empty if $p < 1$, while if $p > 1$, the investment set is an interval $(1/p, 1)$, and the proportion of fortune invested at an opportunity occurring at time t is $a(t) = t - (1/p)$.

4. Arbitrary α.

Let us consider the case, $\alpha < 1$, $\alpha \neq 0$. For simplicity, we take

$$u_\alpha(x) = x^\alpha / \alpha.$$

This differs from the utility of (1.1) only by a change of location. Then

$$V_n(x, y) = \frac{n-1}{n} u_\alpha(x+y) + \frac{1}{n} u_\alpha(\beta_n x) = \frac{n-1}{n}[u_\alpha(x+y) + \frac{\beta_n^\alpha}{n-1} u_\alpha(x)].$$

Theorem 5. *In the case $\alpha < 1$, $\alpha \neq 0$, we have*

$$V_k(x, y) = \frac{k-1}{n}[u_\alpha(x+y) + c_k u_\alpha(x)] \tag{4.1}$$

for $k = 2, \ldots, n$, and

$$V_1(x, y) = \frac{u_\alpha(x)}{n} \begin{cases} 1 + c_2 & \text{if } \beta_1 - 1 \leq c_2 \\ \beta_1^\alpha (1 + (\beta_1 - 1)\theta_1)^{1-\alpha} & \text{if } \beta_1 - 1 > c_2 \end{cases}, \tag{4.2}$$

where $c_n = \beta_n^\alpha/(n-1)$ and for $k = n-1, \ldots, 2$,

$$c_k = c_{k+1} + \frac{1}{k-1} \begin{cases} 1 + c_{k+1} & \text{if } \beta_k - 1 \leq c_{k+1} \\ \beta_k^\alpha (1 + (\beta_k - 1)\theta_k)^{1-\alpha} & \text{if } \beta_k - 1 > c_{k+1} \end{cases}, \tag{4.3}$$

where $\theta_k = (c_{k+1}/(\beta_k - 1))^{1/(1-\alpha)}$. The optimal investment policy is the proportional investment system, that is to invest a proportion, $(1-\theta_k)/(1+(\beta_k - 1)\theta_k)$ of the remaining

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fortune in the kth opportunity, if it is a record and if $\beta_k > c_{k+1}+1$. Otherwise it is optimal not to invest.

Proof. We first show (4.1) by backward induction. Equation (1.2) shows that (4.1) holds for $k = n$ with $c_n = \beta_n^\alpha/(n-1) > 0$. Suppose (4.1) holds down to $k+1$. To show it holds for k, we need to find

$$\max_{0 \le b \le x} V_{k+1}(x - b, \beta_k b) = \max_{0 \le b \le x} \frac{k}{n}[u_\alpha(x + (\beta_k - 1)) + c_{k+1}u_\alpha(x - b)].$$

Let $\phi(b) = u_\alpha(x + (\beta_k - 1)b) + c_{k+1}u_\alpha(x - b)$. Then

$$\phi'(b) = [\frac{\beta_k - 1}{(x + (\beta_k - 1))^{1-\alpha}} - \frac{c_{k+1}}{(x - b)^{1-\alpha}}]. \tag{4.4}$$

From (4.3) and the induction hypothesis, we see that $c_{k+1} > 0$. We note that $\phi''(b) < 0$ so that $\phi(b)$ is concave on $(0, x)$. If $\beta_k - 1 \le c_{k+1}$, then $\phi'(0) \le 0$ so that $\phi(b)$ is decreasing and takes its maximum value over $0 \le b \le x$ at $b = 0$. Otherwise, $\phi(b)$ has a unique maximum in the interval $(0, x)$ at the root of $\phi'(b)$ in that interval. Thus the optimal investment in a record opportunity is b^* where

$$b^* = \begin{cases} 0 & \text{if } \beta_k - 1 \le c_{k+1} \\ \dfrac{1 - \theta_k}{1 + (\beta_k - 1)\theta_k}x & \text{if } \beta_k - 1 > c_{k+1}, \end{cases} \tag{4.5}$$

where $\theta_k = (c_{k+1}/(\beta_k - 1))^{1/(1-\alpha)}$. Noting that $1 + c_{k+1}\theta_k^\alpha = 1 + (\beta_k - 1)\theta_k$, we find

$$\max_{0 \le b \le x} V_{k+1}(x - b, \beta_k b) = \frac{k}{n}\phi(b^*) = \frac{k}{n}u_\alpha(x) \begin{cases} 1 + c_{k+1} & \text{if } \beta_k - 1 \le c_{k+1} \\ \beta_k^\alpha(1 + (\beta_k - 1)\theta_k)^{1-\alpha} & \text{if } \beta_k - 1 > c_{k+1}. \end{cases}$$

Finally we have

$$V_k(x, y) = \frac{k-1}{k}\frac{k}{n}[u_\alpha(x + y) + c_{k+1}u_\alpha(x)] + \frac{1}{k}\frac{k}{n}\phi(b^*)$$

$$= \frac{k-1}{n}[u_\alpha(x + y) + c_k u_\alpha(x)]$$

for $k = n, \ldots, 2$, while for $k = 1$ the first term disappears and we have (4.2). In both cases the c_k are determined by (4.3) and the proof is complete. ∎

Remark The fact that the optimal investment policy is a proportional investment system follows from a general theorem in an unpublished paper of Ferguson and Gilstein (1985). It depends strongly on the assumed form of the utility functions.

4.1 Asymptotic forms. We allow the β_k and the c_k to depend on n, and we assume there is a continuous function, $\beta(t) \ge 1$ on $[0, 1]$, such that $\beta_k = \beta(k/n)$. We see from (4) that the c_k are monotone decreasing. We rewrite the recursion (4.3) as

$$\frac{c_{k+1} - c_k}{1/n} = -\frac{n}{k+1}\begin{cases} 1 + c_{k+1} & \text{if } c_{k+1} \ge \beta_k - 1 \\ \beta_k^\alpha\left(1 + (\beta_k - 1)(\frac{c_{k+1}}{(\beta_k - 1)})^{1/(1-\alpha)}\right)^{1-\alpha} & \text{if } c_{k+1} < \beta_k - 1. \end{cases} \tag{4.6}$$

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The corresponding limiting result is given by the following theorem:

Theorem 6. *As n tends to ∞ and $k/n \to t$, then $c_{k,n} \to f(t)$ where $f(t)$ satisfies the differential equation,*

$$f'(t) = -\frac{1}{t}\begin{cases} 1 + f(t) & \text{if } f(t) \geq \beta(t) - 1 \\ \beta(t)^\alpha \left(1 + f(t)^{1/(1-\alpha)}(\beta(t) - 1)^{-\alpha/(1-\alpha)}\right)^{1-\alpha} & \text{if } f(t) < \beta(t) - 1 \end{cases} \quad (4.7)$$

on $(0, 1]$ with boundary condition $f(1) = 0$. The optimal investment policy is the proportional investment system, to invest a proportion,

$$(1 - \theta(t))/(1 + (\beta(t) - 1)\theta(t)) \quad (4.8)$$

of the remaining fortune in a record opportunity appearing at time t if $\beta(t) > f(t) + 1$, where

$$\theta(t) = (f(t)/(\beta(t) - 1))^{1/(1-\alpha)}. \quad (4.9)$$

The optimal expected fortune is

$$V_t(x, y) = t[u_\alpha(x + y) + f(t)u_\alpha(x)]. \quad (4.10)$$

4.2 Dependence on α. It is of interest to compare how the optimal investment policy changes with changing α. We take the case of constant $\beta(t) \equiv 2$ as an example. For $\alpha = 1$, the optimal policy is a threshold policy that invests everything in the first record after time $t = e^{-1/2} = 0.6065\ldots$. For $\alpha = 0$, it is the Kelly betting system that invests proportion $2t - 1$ of the fortune on a record opportunity that appears at time $t > 1/2$.

For $\beta(t) \equiv 2$, the differential equation (4.7) becomes, for t in a neighborhood of 1,

$$f'(t) = -\frac{2^\alpha}{t}(1 + f(t)^{1/(1-\alpha)})^{1-\alpha}.$$

This is a variables-separable equation, easily solvable by numerical methods. We plot in Figure 1 the optimal investment proportions for $\alpha = 1, .5, 0$ and -1. As α decreases from 1, the investment proportion is continuous in α, but the investor becomes more and more conservative, so much so that when $\alpha = -1$, the investor will even hedge by investing (very small amounts) at unfavorable odds ($t < .5$).
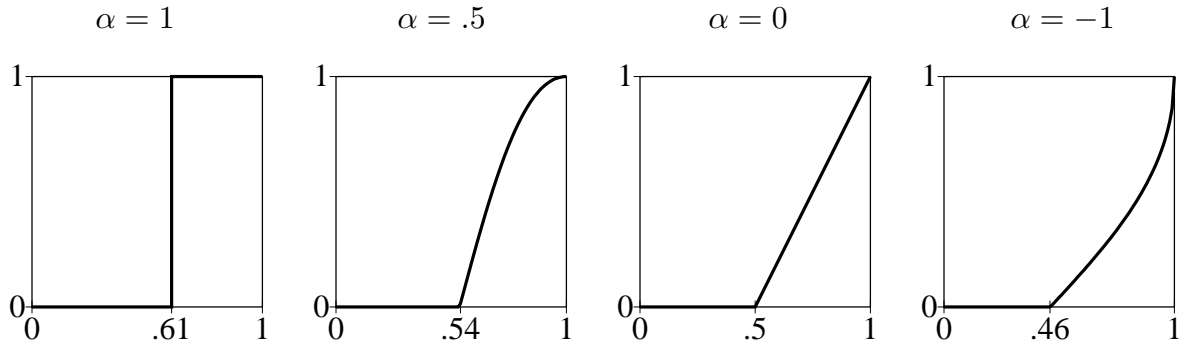

$\alpha = 1$ $\alpha = .5$ $\alpha = 0$ $\alpha = -1$

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Figure 1.

5. Full Information.

In the full information version of the problem, each opportunity has an observable value that determines the rank, the larger the better. The values are assumed to be i.i.d. from a known continuous distribution, which, since we are interested only in the ranks, is assumed without loss of generality to be uniform on $(0, 1)$.

When dealing with full information problems, it is more convenient to let k denote the number of stages to go rather than the number of stages from the beginning. Let the values of the opportunities be $\ldots, U_2, U_1$ i.i.d. from a uniform distribution on $(0, 1)$. Let $V_k(x, y, z)$ represent the expected utility of the final fortune using an optimal strategy, when there are k stages to go, $k \geq 1$, before we observe U_k, the kth from last opportunity, and we have an amount x available for future investments, and a current investment that will return y if the current record of value z, is best overall. Then,

$$V_0(x, y, z) = u_\alpha(x + y)$$

$$V_k(x, y, z) = zV_{k-1}(x, y, z) + \int_z^1 \max_{0 \leq b \leq x} V_{k-1}(x - b, \beta_k b, u) \, du \qquad (5.1)$$

$$\text{for } k = 1, 2, 3, \ldots,$$

where $\beta_k \geq 1$ is the return on a successful investment on a record opportunity with value U_k if it turns out to be absolutely best (largest). We can find useful formulas for the optimal strategies for linear and log utiltites.

5.1 Linear Utility Let us first look at linear utility, $\alpha = 1$, in which case the initial equation is $V_0(x, y, z) = x + y$. If U_1 is a record, it is certain to be best overall, and it is optimal to invest the whole amount x in it. This leads to

$$V_1(x, y, z) = z(x + y) + (1 - z)\beta_1 x. \qquad (5.2)$$

Continuing, we find

$$V_2(x, y, z) = z[z(x + y) + (1 - z)\beta_1 x]$$

$$+ \int_z^1 \max_{0 \leq b \leq x} [u(x - b + \beta_2 b) + (1 - u)\beta_1(x - b)] \, du$$

$$= z^2(x + y) + z(1 - z)\beta_1 x + x \int_z^1 \max\{u + (1 - u)\beta_1, u\beta_2\} \, du$$

$$= z^2(x + y) + c_2(z)x$$

where

$$c_2(z) = [z(1 - z)\beta_1 + \int_z^1 \max\{u + (1 - u)\beta_1, u\beta_2\} \, du].$$

An optimal strategy with 2 stages to go is to invest everything on a record of value $U_2 = u$ if $u + (1 - u)\beta_1 < u\beta_2$.

Continuing the backward induction, we can prove the following theorem.

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Theorem 7. *In the full information case with $\alpha = 1$, we have*

$$V_k(x, y, z) = z^k(x + y) + xc_k(z) \qquad \text{for } k = 1, 2, \ldots \tag{5.3}$$

where $c_0(z) = 0$, and for $k = 1, 2, 3, \ldots$,

$$c_k(z) = zc_{k-1}(z) + \int_z^1 \max\{u^{k-1} + c_{k-1}(u), u^{k-1}\beta_k\}\, du. \tag{5.4}$$

An optimal strategy with k stages to go is to invest everything in a record of value $U_k = u$, if and only if $c_{k-1}(u) < u^{k-1}(\beta_k - 1)$.

5.2 The Full Information Three-Value Secretary Problem. Consider the case with constant return, $\beta_k = \beta > 1$ for all k. This may be considered as an extension to the full information case of the three-value secretary problem of Sakaguchi (1984). On the last stage, it is optimal to invest in any record. On the next to last stage, it is optimal to invest in a record if its value, $U_2 = u$, satisfies $(1 - u)\beta/u < \beta - 1$, or explicitly, $u > z_2 := \beta/(2\beta - 1)$. To go further, we need to find $c_k(z)$ from (5.4). But since the $c_k(z)/z^k$ are increasing in k, each cutoff point can be found from the previous one by replacing the maximum inside the integral sign by $u^{k-1}\beta$. Thus, with k stages to go, we invest in a record opportunity with value $U_k = z$ provided $c_{k-1}(z) < z^{k-1}(\beta - 1)$, where

$$c_k(z) = zc_{k-1}(z) + \int_z^1 u^{k-1}\beta\, du = zc_{k-1}(z) + \frac{\beta(1 - z^k)}{k}.$$

For the purposes of finding the optimal strategy, we have

$$c_k(z) = \beta z^k \sum_{j=1}^k \frac{z^{-j} - 1}{j}.$$

Let $z_1 = 0$, and for $k > 1$ let z_k denote the root of $c_{k-1}(z) = z^{k-1}(\beta - 1)$ in $(0, 1)$. Then it is optimal, with k stages to go, to invest in a record of value U_k provided $U_k > z_k$, where $z_1 = 0$ and for $k > 1$, z_k satisfies

$$\sum_{j=1}^{k-1} \frac{z_k^{-j} - 1}{j} = \frac{\beta - 1}{\beta}.$$

To get an idea of how much the information is worth to the investor, we may compare return per unit available for investment of the full-information problem to that of the rank-based problem when $\beta = 2$. In the rank-based problem with large horizon, the optimal expected return per unit available for investment for large horizon is approximately $\beta e^{-(\beta-1)/\beta} = 2e^{-1/2} = 1.21306$, as found in the first example of Section 2.2. In the full information problem above, it is approximately 1.4276, essentially twice the rate of return.

5.3 Log Utility. Consider now the case $\alpha = 0$ of log utility. The initial equation is $V_0(x, y, z) = \log(x + y)$. If U_1 is a record, it is certain to be best overall, and it is optimal to invest the whole amount x in it. This leads to

$$V_1(x, y, z) = z\log(x + y) + (1 - z)\log(\beta_1 x) = z\log(x + y) + (1 - z)\log(x) + c_1(z),$$

where $c_1(z) = (1 - z)\log(\beta_1)$. We can prove

Theorem 8. *In the full information case with log utility, it is optimal at stage $k \geq 1$ from the end to invest proportion $a_k(u)$ in a record opportunity of value $U_k = u$, where*

$$a_k(u) = \begin{cases} (u^{k-1}\beta_k - 1)/(\beta_k - 1) & \text{if } u^{k-1}\beta_k > 1 \\ 0 & \text{if } u^{k-1}\beta_k \leq 1. \end{cases}$$

The value functions satisfy

$$V_k(x, y, z) = z^k \log(x + y) + (1 - z^k)\log(x) + c_k(z),$$

where $c_0(z) = 0$ and for $k = 1, 2, \ldots,$

$$c_k(z) = zc_{k-1}(z) + \int_z^1 [u^{k-1}\log(1 + (\beta_k - 1)a_k(u)) + (1 - u^{k-1})\log(1 - a_k(u))]\, du.$$

6. Unknown Number of Opportunities

We now turn to the problem of an unknown number of opportunities. This case is harder than the fixed-n case or the corresponding asymptotic case. Our interest will be confined to the rank-based model with linear utility. We have given several reasons in the Introduction why we think of rank-based models as usually being more adequate. Our preference for linear utility is based on the facts that this utility function is commonly seen as a resonable utility function and that it presents, at the same time, the easiest case.

6.1 The all-or-nothing-rule for linear utility. Similar to what we have seen in Section 2, linear utility implies again that we need only consider those investment strategies which invest either nothing or, alternatively, all available capital on a record opportunity. To see this, suppose that it is optimal to invest the fraction a with $0 < a < 1$ in a present record opportunity. The principle of optimality requires then that the contribution of each unit of money invested now is expected to yield an at least as high contribution as using it for optimal investments later on. But the expected contribution from the present amount of investment increases proportional to a whereas the contribution of investments under an optimal behavior in the future is proportional to $1 - a$. Thus, since the model imposes no constraint on the fraction of invested capital, we must, by the principle of optimality, choose $a = 1$, if the expected present contribution is strictly higher, and $a = 0$ if it is strictly lower. If the expected contribution from a unit of the present investment is equal to the one reserved for future optimal investment then we are indifferent to the choice of a, that is, $a = 1$ or $a = 0$ are also optimal.

6.2 Aspects of statistical inference in modelization. Suppose now that we would like to invest in one from an unknown number of opportunities arriving on some finite horizon $[0, T]$, say. Let $N(t)$ denote the number of opportunities up to time t, $0 \leq t \leq T$ and let $N = N(T)$. We suppose that the value of N is unknown and must be inferred from sequential observation.

There are several possibilities to model this situation. One is to suppose that all opportunities have i.i.d. arrival times according to a known distribution F on $[0, T]$. As time progresses, one may update information about the value of N. For example, we may estimate N by the maximum likelihood estimate, $\hat{N}(t) = \lfloor N(t)/F(t) \rfloor$ (the greatest integer less than or equal to $N(t)/F(t)$). We may then replace the fixed number n (in Subsection 1.1) at each arrival time, t_k, by $\hat{N}(t_k) = \lfloor k/F(t_k) \rfloor$ and use the corresponding optimal rule. However, this procedure is not simple to use because the optimal thresholds must be recomputed each time a new record appears. Also, though we expect such procedures to be reasonably good, we have no idea how close they are to optimal procedures.

Another way to model this situation is to assume that the counting process, $N(t)$, $0 \leq t \leq T$, belongs to a certain class and to update the knowledge of its parameters by sequential observation. Clearly, one would first try relatively simple Poisson process. The fact that the posterior law of N must be recomputed in general with the arrival of each new record leads in general to the same difficulty in computation.

6.3 Pascal Processes. For both classes of models however, there exist some special cases for which these computational problems dissolve. A particularly nice case is the one where the record process stays unaffected by updating the arrival process, $N(t)$, or, in the first model, unaffected by updating the law of N. This is the case of the Pascal processes of Bruss and Rogers (1991). These are processes for which the distribution of $N(t)$ given $N(s)$ for fixed $s < t$ is Pascal (negative binomial). They can be obtained by either a geometric prior for N in the first model, or by an exponential prior on the intensity rate of the Poisson process in the second, or by limiting priors of these. Also, any strictly monotone time-scale transformation of a Pascal process yieldss again a Pascal process.

A Pascal process has the remarkable property that the process of records forms a Poisson process (see Bruss and Rogers (1991) p. 333). Note that the record process is the only relevant process for decision making in our model. Thus, from the independent increments property, statistical inference from the past of the process is redundant and the optimal policy (if it can be computed at time 0) will be invariant on the whole investment interval.

If $\lambda(t)$ denotes the intensity function of records and if $\varphi(t)$ represents the probability of no records in the interval (t, T), then $\lambda(t)$ and $\varphi(t)$ are related by

$$\int_t^T \lambda(s)\, ds = -\log \varphi(t). \tag{6.1}$$

We note that $\varphi(t)$ is, by definition, nondecreasing in t. We also note that the class of Pascal processes covers the arrival process which defines the infinite secretary problem of Gianini and Samuels (1976) where the process of record arrivals is Poisson with intensity rate $1/t$. This is the case $T = 1$ and $\varphi(t) = t$, because in the Gianini-Samuels model, arrival times of different ranks are uniform on $[0, 1]$ and there is no record after time t iff rank 1 arrives in $[0, t]$. Thus, from (6.1), $\lambda(t) = 1/t$ for $0 \leq t \leq 1$. In general we have $\varphi(0) > 0$, however, because a Pascal process may show no arrivals at all, and thus no records.

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The optimal rule and optimal reward for a Pascal process is given in the following theorem.

Theorem 9. *Let $(\Pi_t)_{0 \le t \le T}$ be a Pascal process with parameter function $\varphi(t)$ on some horizon $[0, T]$, and let $\beta(t)$ be a continuous reward function on this horizon. Then, for linear utility, it is optimal to invest all capital in a record opportunity that appears at time t if*

$$\beta(t)\varphi(t) \ge r(t), \tag{6.2}$$

where $r(t)$ satisfies the differential equation

$$r'(t) = -\lambda(t)\Big(\beta(t)\varphi(t) - r(t)\Big)^+ \tag{6.3}$$

subject to the boundary condition $r(T) = 1$. If no such time exists it is optimal not to invest at all.

The optimal reward is given by $r(0)$ per unit of intial capital.

Proof. Let $r(t)$ be the expected reward for one unit of available capital by investing optimally after time t when t is not a record time, and let $\tilde{r}(t)$ be the corresponding optimal expected reward at time t if t is a record time. Note that if t is not a record time, then we must pass over t so that $r(t)$ is the same as the expected reward for optimal investments after time t. If a record arriving at time t is selected for investment, then the expected return is $\beta(t)\varphi(t)$. Hence, by the principle of optimality, for each record time t,

$$\tilde{r}(t) = \max\{\beta(t)\varphi(t), r(t)\}.$$

By the same principle, we must invest the given unit of capital if $\tilde{r}(t) > r(t)$, that is, if $\beta(t)\varphi(t) > r(t)$. According to the all-or-nothing rule for linear utility, we must then invest all capital (see subsection 6.1), and we may do so if $\beta(t)\varphi(t) \ge r(t)$. This implies the first statement of the Theorem with inequality (6.2).

We now derive the differential equation for $r(t)$. If we have for some $t' \in (t, t + \delta t)$ a record time, then we obtain $\tilde{r}(t')$ under optimal continuation; otherwise we obtain $r(t+\delta t)$. Using the Poisson property of the process of records, and its intensity rate at time t, $\lambda(t)$, given through (6.1), this argument yields

$$r(t) = (\lambda(t)\delta t + o(\delta t))\tilde{r}(t') + (1 - \lambda(t)\delta t + o(\delta t))r(t + \delta t). \tag{6.4}$$

We note that $t' \to t$ as $\delta t \to 0$ and recall that $\beta(t)$ is continuous. Hence, subtracting in (6.4) $r(t + \delta t)$ from both sides, dividing by δt and taking the limit as $\delta t \to 0$ yields the differential equation

$$r'(t) = -\lambda(t)\Big(\beta(t)\varphi(t) - r(t)\Big)^+.$$

This proves (6.3).

Finally, since we do not invest (a.s.) at time 0, $r(0)$ is the optimal reward per investment unit by definition of $r(t)$, completing the proof. ∎

6.4 Example. Suppose $\beta(t)$ is a constant, $\beta > 1$. From (6.1) we have $\lambda(t) = \varphi'(t)/\varphi(t)$. Then (6.3) becomes, at least for t in a neighborhood of T,

$$r'(t) = -\varphi'(t)\beta + (\varphi'(t)/\varphi(t))r(t),$$

with solution, satisfying $r(T) = 1$ (since $\varphi(T) = 1$),

$$r(t) = (1 - \beta \log \varphi(t))\varphi(t) \tag{6.5}$$

as is easily checked. This holds for t down until, if ever, $r(t) > \beta\varphi(t)$, or $\varphi(t) < e^{-(\beta-1)/\beta}$. From this point down to 0, $r(t)$ stays constant. The optimal rule is to invest all capital in the first record appearing after time

$$t^* = \begin{cases} \varphi^{-1}(e^{-(\beta-1)/\beta}) & \text{if } \varphi(0) < e^{-(\beta-1)/\beta} \\ 0 & \text{otherwise.} \end{cases} \tag{6.6}$$

If $\varphi(0) < e^{-(\beta-1)/\beta}$, the optimal expected reward equals $r(0) = r(t^*) = \beta e^{-(\beta-1)/\beta}$, independent of $\varphi(t)$.

6.5 Modeling investment problems with Pascal processes. The case of an unknown number of opportunities is the most realistic one for applications, and, to our knowledge, Pascal processes provide the easiest access with explicit solutions. Therefore, we briefly show how to model problems with Pascal processes.

6.5.1 Very Weak Information. Suppose first that we have little information about the dependence of arrival times of opportunities. We may have no idea about the location of subintervals of $[0, T]$ where opportunities may be more likely, and neither about the distribution of the total number of opportunities. Due to the lack of information it is then natural to model the arrival process by a homogeneous Poisson process. Still, the question remains how to choose the rate, λ. If $[0, T]$ represents one year say, then a rate of 1 per two months, or, in contrast, 6 per month, say, makes a huge difference for the distribution of $N = N(T)$. But we would like to have a robust selection rule, one whose behavior is rather independent of the choice of λ, if this is possible.

The central point here is that we frame our (weak) knowledge in a compatible but suitable way. It is known (see Bruss (1987 p. 924)) that if the rate λ is random with exponential density with parameter a (i.e. mean $1/a$), then the arrival process $N(t)$ becomes a Pascal process with parameter function

$$\varphi(t) = (t + a)/(T + a).$$

(Here the Gianini-Samuels infinite secretary problem model (1976) corresponds to the limiting parameter function $\varphi(t) = t$ as $a \to 0$ for $T = 1$).

As an example, choose the horizon T to be one year, and the reward rate (per unit of invested capital) $\beta = 2$ for the best opportunity. If we estimate λ to be somewhere between $1/2$ and 6 per month, say, then the choice $\varphi(t) = (t+a)/(12+a)$, with a somewhere between $1/6$ and 2, would seem reasonable. (We may be slightly more sophisticated by maximizing, as a function of a, the probability of the rate λ falling between $1/2$ and 6 per month, but, as we shall see, it does not really matter.) We note that $\varphi(t)$ is strictly increasing so that the inverse function φ^{-1} exists. Now, with $\beta = 2$,

$$\varphi(t^*) = \varphi(\varphi^{-1}(e^{-1/2}) = e^{-1/2} \text{ if } \varphi(0) < e^{-1/2}.$$

The optimal reward $r(t^*) = r(0)$ is thus independent of a for

$$0 \le a \le 12e^{-1/2}/(1 - e^{-1/2}) = 18.4979\cdots,$$

since $\varphi(0) = a/(12 + a)$. The optimal rule is not independent of a, however, since t^* of (6.6) depends on a in this range. Nevertheless, $r(t^*)$ of (6.5) is not very sensitive to a wrong choice of $t^* = t^*(a)$ as Figure 2 illustrates.

This graph shows for $\beta = 2$ and for various values of a, the reward $r_{\text{inv}}(t)$ defined as the expected return if we invest in the first record which appears after time t. The lowest curve on the LHS is the case $a = 0$, really the limit as $a \to 0$, and is practically indistinguishable form $a = 1/6$. (Recall that the smaller a, the more opportunities we expect). Also plotted are the cases $a = 1$, $a = 2$, and $a = 6$ (highest curve on LHS). The optimal reward $r(t)$ is the maximum of the straight line on top and the corresponding curve, each reaching its maximum value of $2e^{(1-\beta)/\beta} = 1.213\ldots$. Note that the optimal waiting times $t^* = t^*(a)$ are very close to each other, ranging from 7.278 at $a = 0$ to 6.491 at $a = 2$. Similarly the corresponding reward curves r_{inv} are very flat in this region.
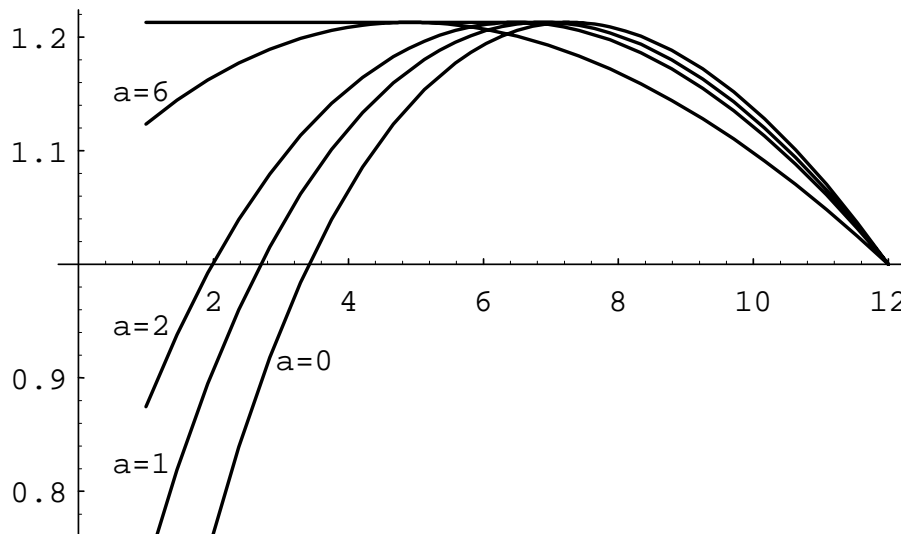

Figure 2.

Only when a is large compared to T does $t^*(a)$ become sensitive to the choice of a. This contrasts a choice of large a compared to medium and smaller a (the curve for $a = 6$ is the flattest curve). $t^*(a)$ moves to the left as a increases (very few expected arrivals).

To summarize the solution of this example: Observing the market for about $6\frac{1}{2}$ months and then investing all available capital in the next record opportunity (if any), is an excellent strategy, for few as well as many expected opportutinies, with an expected return of over 20 percent (moreover, the money could be placed on a fixed return contract for the first six months, of course.)

6.5.2 Information on Arrival Time Densities. Now suppose that we have some prior information which enables us to say *when* opportunities are more likely to arrive without necessarily knowing the distribution of the total number of arrivals For instance, we may know patterns of seasonal variation in arrival times, but we do not know yet whether it will be, generally, a good year. Here we need an "inhomogeneous" version of a Pascal process.

A convenient way to model this situation is as follows. We draw a graph what we think of as being the arrival time density f for opportunities on $[0, T]$. Let $F(t) = \int_0^t f(s)\,ds$, and let N be geometric with law $P(N = n+1) = q^n p$ for $n = 1, 2, \ldots$. Here we may choose p according to a prior belief on the size of $E(N) = 1 + 1/p$. The arrival process $N(t)$ is then Pascal with $\varphi(t) = p + qF(t)$ (Bruss and Rogers p. 332), and the solution can again be derived conveniently in the same way as in Example *6.6.1*.

Again we can see that the solution is robust, both with respect to errors in p and with respect to F. With this robustness in p and F it is an excellent model indeed, because the choice of F allows for a great deal of flexibility. However, we need that the assumption of geometric prior is not unreasonable, because it is exactly there that the flexibility stops. The geometric prior can be shown to be the only prior with finite expectation, which generates in this model a Pascal arrival Process.

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