

Final Examination
Mathematics 167, Game Theory

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1. (a) Consider a game of nim with 3 piles of sizes 9, 17 and 21. Is this a P-position or an N-position? If an N-position, what is a winning first move?

(b) Consider the take or break game where a player may remove one or two chips from any pile, or he may break any pile into two non-empty piles (but not both). Find the Sprague-Grundy function for piles of size less than or equal to 11.

2. (a) Solve the following two-person zero-sum matrix game.

$$\begin{pmatrix} 5 & 3 & 0 & 2 \\ 0 & 2 & 6 & 4 \end{pmatrix}$$

(b) Player II chooses a positive integer and Player I tries to guess it. If I guesses too high, he loses 2 to II. If he guesses too low by exactly 1 he loses 1 to II. Otherwise (if he is correct or if he is too low by more than one), he wins 1 from II. Set up the matrix and solve. (Use domination.)

3. Player II receives one card at random from a deck of three cards consisting of Ace, King and Joker, probability $1/3$ each. If she receives an Ace or King, she must announce that card. If she receives a Joker, she may announce either the Ace or the King, whichever she wants. Then, after hearing the announcement, Player I must either accept or challenge. If he accepts, he wins 6 if II has the Ace and 3 if II has the King. If he challenges, he wins 18 if II has the Joker and has announced the Ace, and he wins 9 if II has the Joker and has announced the King. There is no payoff otherwise.

(a) Draw the Kuhn Tree.

(b) Find the equivalent strategic form of the game.

4. Consider the non-cooperative bimatrix game: $\begin{pmatrix} (-1, 1) & (0, 2) & (0, 2) \\ (2, 1) & (1, -1) & (0, 0) \\ (0, 0) & (1, 1) & (1, 2) \end{pmatrix}$.

(a) Find the safety levels, and the maxmin strategies for both players.

(b) Find as many strategic equilibria as you can.

5. Suppose in the Cournot duopoly model that the two firms have different production costs and different set-up costs. Suppose Player I's cost of producing x is $x + 2$, and II's cost of producing y is $3y + 1$. Suppose also that the price function is $P(x, y) = 17 - x - y$, where x and y are the amounts produced by I and II respectively. What is the equilibrium production, and what are the players' equilibrium profits?

6. Consider the cooperative TU bimatrix game: $\begin{pmatrix} (3, 2) & (4, 1) & (4, 2) \\ (4, 2) & (2, 3) & (4, 1) \\ (1, 3) & (3, 0) & (4, 3) \end{pmatrix}$.

- (a) Find the TU-values.
- (b) Find the associated side payment.
- (c) Find the optimal threat strategies.

7. (a) Define what it means for a vector $(\bar{u}, \bar{v}) \in S$, where S is the NTU-feasible set, to be Pareto optimal in a two-player NTU game.

(b) Consider the cooperative NTU bimatrix game: $\begin{pmatrix} (2, 4) & (6, 0) \\ (9, 1) & (3, 4) \end{pmatrix}$. Let $(u^*, v^*) = (1, 0)$ be the disagreement point, (threat point, status-quo point). Find the NTU-value.

8. Consider the 3-person game in coalitional form with characteristic function, v , satisfying

$$v(\emptyset) = 0 \quad \begin{array}{ll} v(\{1\}) = -2 & v(\{1, 2\}) = 2 \\ v(\{2\}) = -1 & v(\{1, 3\}) = 1 \\ v(\{3\}) = 0 & v(\{2, 3\}) = 1 \end{array} \quad v(\{1, 2, 3\}) = 3$$

Find the imputations and the core for this game. (Either graph the core or be fairly explicit in your description.)

9. (a) Define a Simple Game.

(b) Find the Shapley value (the Shapley-Shubik Index) for the weighted voting game with four players in which player A holds 10 shares, player B holds 9 shares, player C holds 7 shares and player D holds 6 shares, when 18 or more shares are required pass a measure.

10. Consider the three-person game in coalitional form with characteristic function,

$$v(\emptyset) = 0 \quad \begin{array}{ll} v(\{1\}) = 0 & v(\{1, 2\}) = 2 \\ v(\{2\}) = 1 & v(\{1, 3\}) = 3 \\ v(\{3\}) = 2 & v(\{2, 3\}) = 6 \end{array} \quad v(\{1, 2, 3\}) = 10$$

- (a) Find the Shapley value.
- (b) Find the nucleolus.

Solutions, Final Examination
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1.(a) $9 = 1001_2$, $17 = 10001_2$ and $21 = 10101_2$. The nim sum is 1101_2 , so this is an N-position. The unique winning move is to remove 5 from the pile of 9 leaving $4 = 100_2$.

(b)

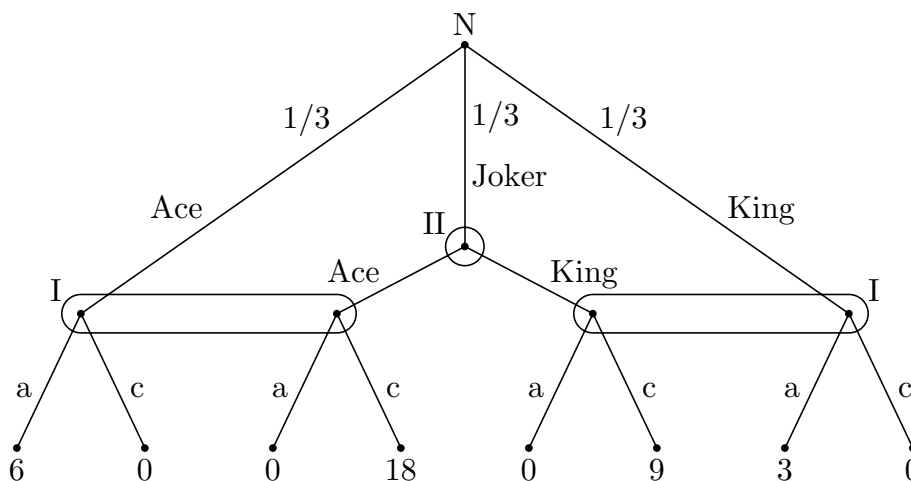
$$\begin{array}{rcccccccccccc} x: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ g(x): & 0 & 1 & 2 & 0 & 3 & 1 & 2 & 0 & 3 & 1 & 2 & 0 \end{array}$$

2.(a) Optimal for I is $\mathbf{p} = (4/7, 3/7)$. Optimal for II is $\mathbf{q} = (0, 6/7, 1/7, 0)$. The value is $V = 18/7$.

(b) The matrix is $\begin{pmatrix} 1 & -1 & 1 & 1 & \dots \\ -2 & 1 & -1 & 1 & \dots \\ -2 & -2 & 1 & -1 & \dots \\ -2 & -2 & -2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$. The first column dominates columns

3, 4, ... and the first row dominates rows 3, 4, ... That leaves the upper left 2 by 2 matrix. Optimal for I is $\mathbf{p} = (3/5, 2/5, 0, 0, \dots)$. Optimal for II is $\mathbf{q} = (2/5, 3/5, 0, 0, \dots)$. The value is $-1/5$.

3.(a)



(b)

$$\begin{array}{l} A \quad B \\ aa \begin{pmatrix} 3 & 3 \\ 2 & 5 \\ 7 & 1 \\ 6 & 3 \end{pmatrix} \\ ac \\ ca \\ cc \end{array}$$

4.(a) In the A matrix, the top row and middle column are dominated. The resulting matrix is $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. I's safety level is $2/3$, and his maxmin strategy is $\mathbf{p} = (0, 1/3, 2/3)$. Similarly, II's safety level is $2/3$, and her maxmin strategy is $\mathbf{q} = (2/3, 0, 1/3)$.

(b) The top row is strictly dominated, and then the middle column is strictly dominated. Removing them does not lose any equilibria (and leads to the Battle of the Sexes). There are two PSE's, one at (second row, first column), and the other at (third row, third column). There is therefore a third SE given by the equalizing strategies in the Battle of the Sexes, namely, (\mathbf{p}, \mathbf{q}) , where $\mathbf{p} = (0, 2/3, 1/3)$ and $\mathbf{q} = (1/3, 0, 2/3)$.

5. The players' profits are $u_1(x, y) = x(17 - x - y) - (x + 2)$ and $u_2(x, y) = y(17 - x - y) - (3y + 1)$. To find the equilibrium production, we set the partial derivatives to zero:

$$\begin{aligned}\frac{\partial u_1}{\partial x} &= 16 - 2x - y = 0 \\ \frac{\partial u_2}{\partial y} &= 14 - x - 2y = 0\end{aligned}$$

which gives $(x, y) = (6, 4)$ as the equilibrium production. The equilibrium profits are $(u_1(6, 4), u_2(6, 4)) = (34, 15)$.

6.(a) The maximum total payoff is $\sigma = 7$, with payoff $(4, 3)$. The difference matrix is $\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ -2 & 3 & 1 \end{pmatrix}$. The last two and column are dominated, so $\delta = \text{Val} \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} = 7/5$. Therefore, $\varphi = ((\sigma + \delta)/2, (\sigma - \delta)/2) = (21/5, 14/5)$.

(b) To get to this from $(4, 3)$ requires Player II to pay $1/5$ to Player I.

(c) The threat strategies are $\mathbf{p} = (3/5, 2/5, 0)$ and $\mathbf{q} = (4/5, 1/5, 0)$.

7.(a) A vector $(\bar{u}, \bar{v}) \in S$ is Pareto optimal if the only point $(u, v) \in S$ such that $u \geq \bar{u}$ and $v \geq \bar{v}$ is $(u, v) = (\bar{u}, \bar{v})$ itself.

(b) The Pareto optimal boundary is the line segment from $(3, 4)$ to $(9, 1)$. The equation of this line is $v - 4 = (-1/2)(u - 3)$ or $v = (11 - u)/2$. We seek the point on this line that minimizes $(u - 1)v = (u - 1)(11 - u)/2$. Setting the derivative to zero gives $u = 6$, which gives $v = 5/2$. Thus $(6, 5/2)$ is the NTU solution since it is on the line segment.

8. The set of imputations is the set on points $\{\mathbf{x} : x_1 + x_2 + x_3 = 3, x_1 \geq -2, x_2 \geq -1, x_3 \geq 0\}$. This is the equilateral triangle with vertices $(-2, -1, 6)$, $(-2, 5, 0)$ and $(4, -1, 0)$. The core is the subset of this set that satisfies $x_1 + x_2 \geq 2$, $x_1 + x_3 \geq 1$, and $x_2 + x_3 \geq 1$. This is the four-sided figure with vertices $(1, 2, 0)$, $(2, 1, 0)$, $(2, 0, 1)$ and $(0, 2, 1)$.

9.(a) A coalitional game (N, v) is said to be simple if for all $S \subset N$, $v(S)$ is either zero or one.

(b) Player A can change a coalition from losing to winning if and only if that coalition is one of $\{B\}$, $\{B, C\}$, $\{B, D\}$ or $\{C, B\}$. Therefore,

$$\phi_A = \frac{1!2!}{4!} + 3\frac{2!1!}{4!} = 1/3.$$

Similarly, $\phi_B = 1/3$, $\phi_C = 1/6$ and $\phi_D = 1/6$.

$$\phi_{\{1\}} = (1/3)0 + (1/6)1 + (1/6)1 + (1/3)4 = 5/3$$

$$10.(a) \phi_{\{2\}} = (1/3)1 + (1/6)2 + (1/6)4 + (1/3)7 = 11/3$$

$$\phi_{\{3\}} = (1/3)2 + (1/6)3 + (1/6)5 + (1/3)8 = 14/3$$

(b) The Shapley value was found to be $(5/3, 11/3, 14/3)$ so we might try $(2, 4, 4)$ as an initial guess at the nucleolus. The largest excess occurs at either of the coalitions $\{1\}$, $\{3\}$ and $\{2, 3\}$. The first and last cannot be made smaller without making the other larger. So $x_1 = 2$ in the nucleolus. The excess at $\{3\}$ can be made smaller by making x_3 larger. This increases the excess of $\{2\}$. These are equal at $x_3 = 4.5$ and $x_2 = 3.5$. The nucleolus is $(2, 3.5, 4.5)$.

Coalition	Excess	$(2, 4, 4)$	$(2, 3.5, 4.5)$
$\{1\}$	$-x_1$	-2	-2
$\{2\}$	$1 - x_2$	-3	-2.5
$\{3\}$	$2 - x_3$	-2	-2.5
$\{1, 2\}$	$2 - x_1 - x_2 = x_3 - 8$	-4	-3.5
$\{1, 3\}$	$3 - x_1 - x_3 = x_2 - 7$	-3	-3.5
$\{2, 3\}$	$6 - x_2 - x_3 = x_1 - 4$	-2	-2