Chapter 2. FINITE HORIZON PROBLEMS.

A stopping rule problem has a finite horizon if there is a known upper bound on the number of stages at which one may stop. If stopping is required after observing $X_1,\ldots,X_T$, we say the problem has horizon T. A finite horizon problem may be obtained as a special case of the general problem as presented in Chapter 1 by setting $y_{T+1} = \cdots = y_\infty = -\infty$. In principle, such problems may be solved by the method of **backward induction**. Since we must stop at stage T, we first find the optimal rule at stage $T-1$. Then, knowing the optimal rule at stage $T-1$, we find the optimal rule at stage $T-2$, and so on back to the initial stage (stage 0). We define $V_T^{(T)}(x_1,\ldots,x_T) = y_T(x_1,\ldots,x_T)$ and then inductively for $j = T-1$, backward to $j = 0$,

$$\begin{aligned} &V_j^{(T)}(x_1,\ldots,x_j) = \\ &\qquad \max\left\{y_j(x_1,\ldots,x_j), \mathrm{E}(V_{j+1}^{(T)}(x_1,\ldots,x_j,X_{j+1})|X_1 = x_1,\ldots,X_j = x_j)\right\}. \end{aligned} \tag{1}$$

Inductively, $V_j^{(T)}(x_1,\ldots,x_j)$ represents the maximum return one can obtain starting from stage j having observed $X_1 = x_1,\ldots,X_j = x_j$. At stage j, we compare the return for stopping, namely $y_j(x_1,\ldots,x_j)$, with the return we expect to be able to get by continuing and using the optimal rule for stages $j+1$ through T, which at stage j is $\mathrm{E}(V_{j+1}^{(T)}(x_1,\ldots,x_j,X_{j+1})|X_1 = x_1,\ldots,X_j = x_j)$. Our optimal return is therefore the maximum of these two quantities, and it is optimal to stop at j if $V_j^{(T)}(x_1,\ldots,x_j) = y_j(x_1,\ldots,x_j)$, and to continue otherwise. The value of the stopping rule problem is then $V_0^{(T)}$.

In this chapter, we present a number of problems whose solutions may be effectively evaluated by this method. The most famous of these is the secretary problem. In the first three sections, we discuss this problem and some of its variations. In the fourth section, we treat the Cayley-Moser problem, a finite horizon version of the house selling problem mentioned in Chapter 1. In the last section, we present the parking problem of MacQueen and Miller. Other examples of finite horizon problems include the fishing problem and the one-armed bandit problem of the Exercises of Chapter 1.

§**2.1 The Classical Secretary Problem.** The secretary problem and its offshoots form an important class of finite horizon problems. There is a large literature on this problem, and one book, *Problems of Best Selection* (in Russian) by Berezovskiy and Gnedin

(1984) devoted solely to it. For an entertaining exposition of the secretary problem, see Ferguson (1989). The problem is usually described as that of choosing the best secretary (the secretary problem), but it is sometimes described as the problem of choosing the best spouse (the marriage problem) or the largest of an unknown set of numbers (Googol). First, we describe what is known as the classical secretary problem, or CSP.

1. There is one secretarial position available.

2. There are n applicants for the position; n is known.

3. It is assumed you can rank the applicants linearly from best to worst without ties.

4. The applicants are interviewed sequentially in a random order with each of the $n!$ orderings being equally likely.

5. As each applicant is being interviewed, you must either accept the applicant for the position and end the decision problem, or reject the applicant and interview the next one if any.

6. The decision to accept or reject an applicant must be based only on the relative ranks of the applicants interviewed so far.

7. An applicant once rejected cannot later be recalled.

8. Your objective is to select the best of the applicants; that is, you win 1 if you select the best, and 0 otherwise.

We place this problem into the guise of a stopping rule problem by identifying stopping with acceptance. We may take the observations to be the **relative ranks**, $X_1, X_2, \ldots, X_n$, where X_j is the rank of the jth applicant among the first j applicants, rank 1 being best. By assumption 4, these random variables are independent and X_j has a uniform distribution over the integers from 1 to j. Thus, $X_1 \equiv 1$, $\mathrm{P}(X_2 = 1) = \mathrm{P}(X_2 = 2) = 1/2$, etc.

Note that an applicant should be accepted only if it is relatively best among those already observed. A relatively best applicant is called a **candidate**, so the jth applicant is a candidate if and only if $X_j = 1$. If we accept a candidate at stage j, the probability we win is the same as the probability that the best among the first j applicants is best overall. This is just the probability that the best candidate overall appears among the first j applicants, namely j/n. Thus,

$$y_j(x_1, \ldots, x_j) = \begin{cases} j/n & \text{if applicant } j \text{ is a candidate,} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Note that $y_0 = 0$ and that for $j \geq 1$, y_j depends only on x_j.

This basic problem has a remarkably simple solution which we find directly without the use of (1). Let W_j denote the probability of win using an optimal rule among rules that pass up the first j applicants. Then $W_j \geq W_{j+1}$ since the rule best among those that pass up the first $j+1$ applicants is available among the rules that pass up only the first j applicants. It is optimal to stop with a candidate at stage j if $j/n \geq W_j$. This means that if it is optimal to stop with a candidate at j, then it is optimal to stop with a candidate at $j+1$, since $(j+1)/n > j/n \geq W_j \geq W_{j+1}$. Therefore, an optimal rule may be found among the rules of the following form, N_r for some $r \geq 1$:

N_r: Reject the first $r-1$ applicants and then accept the next relatively best applicant, if any.

Such a rule is called a **threshold rule** with threshold r. The probability of win using N_r is

$$\begin{aligned} P_r &= \sum_{k=r}^{n} \mathrm{P}(k^{th} \text{ applicant is best and is selected }) \\ &= \sum_{k=r}^{n} \mathrm{P}(k^{th} \text{ applicant is best })\mathrm{P}(k^{th} \text{ applicant is selected } \mid \text{ it is best }) \\ &= \sum_{k=r}^{n} \frac{1}{n}\mathrm{P}(\text{best of first } k-1 \text{ appears before stage } r) \\ &= \sum_{k=r}^{n} \frac{1}{n}\,\frac{r-1}{k-1} = \frac{r-1}{n}\sum_{k=r}^{n}\frac{1}{k-1}, \end{aligned} \tag{3}$$

where $(r-1)/(r-1)$ represents 1 if $r=1$. The optimal r_1 is the value of r that maximizes P_r. Since

$$\begin{aligned} &P_{r+1} \leq P_r \qquad \text{if and only if} \\ &\frac{r}{n}\sum_{r+1}^{n}\frac{1}{k-1} \leq \frac{r-1}{n}\sum_{r}^{n}\frac{1}{k-1} \qquad \text{if and only if} \\ &\sum_{r+1}^{n}\frac{1}{k-1} \leq 1, \end{aligned}$$

we see that the optimal rule is to select the first candidate that appears among applicants from stage r_1 on, where

$$r_1 = \min\{r \geq 1 : \sum_{r+1}^{n}\frac{1}{k-1} \leq 1\}. \tag{4}$$

The following table is easily constructed.

$n =$	1	2	3	4	5	6	7	8
$r_1 =$	1	1	2	2	3	3	3	4
$P_{r_1} =$	1.0	.500	.500	.458	.433	.428	.414	.410

It is of interest to compute the approximate values of the optimal r_1 and the optimal P_{r_1} for large n. Since $\sum_{r+1}^{n} 1/(k-1) \sim \log(n/r)$, we have approximately $\log(n/r_1) = 1$, or $r_1/n = e^{-1}$. Hence, for large n it is approximately optimal to pass up a proportion

$e^{-1} = 36.8\%$ of the applicants and then select the next candidate. The probability of obtaining the best applicant is then approximately e^{-1}.

§**2.2 Arbitrary Monotonic Utility**. Problems in which the objective is to select the best of the applicants, as in assumption 8, are called **best-choice problems**. These are problems of the very particular person who will be satisfied with nothing but the very best. We extend the above result to more arbitrary payoff functions of the rank of the selected applicant. For example, we might be interested in getting one of the best two applicants; or we might be interested in minimizing the expected rank of the applicant selected, rank 1 being best. Let $U(j)$ be your payoff if the applicant you select has rank j among all applicants. It is assumed that $U(1) \geq U(2) \geq \cdots \geq U(n)$. The lower the ranking the more valuable. If no selection is made at all, the payoff is a fixed number denoted by Z_∞ (allowed to be greater than $U(n)$). In the best-choice problem, $U(1) = 1$, $U(2) = \cdots = U(n) = 0$, and $Z_\infty = 0$.

Let X_j be the relative rank of the jth applicant among the first j applicants observed. Then the X_j are independent, and X_j is uniformly distributed on the integers, $\{1, 2, ..., j\}$. The reward function $y_j(x_1, \ldots, x_j)$ is the expected payoff given that you have selected the jth applicant and it has relative rank x_j. The probability that an applicant of rank x among the first j applicants has eventual rank b among all applicants is the same as the probability that the applicant of rank b will be found in a sample of size j and have rank x there, namely

$$f(b|j, x) = \frac{\binom{b-1}{x-1}\binom{n-b}{j-x}}{\binom{n}{j}} \qquad \text{for} \quad b = x, \ldots, n-j+x,$$

(the negative hypergeometric distribution). Hence for $1 \leq j \leq n$,

$$y_j(x_1, ..., x_j) = \sum_{b=x}^{n-j+x} U(b) f(b|j, x)$$

where $x = x_j$. To force you to take at least one observation, we take $y_0 = -\infty$. We note that y_j depends only on x_j and may be written $y_j(x_j)$. As a practical matter, computation may be carried out conveniently using a backward recursion. The recursion for the probabilities,

$$f(b|j-1, x) = \frac{x}{j} f(b|j, x+1) + \frac{j-x}{j} f(b|j, x),$$

implies the backward recursion for the expected values,

$$y_{j-1}(x) = \frac{x}{j}\, y_j(x+1) + \frac{j-x}{j}\, y_j(x) \tag{5}$$

for $j > 1$, with initial conditions, $y_n(x) = U(x)$ for $1 \leq x \leq n$.

The horizon for the secretary problem is n. If you go beyond the horizon, you receive Z_∞, so the initial condition on the $V^{(n)}$ is: $V_n^{(n)}(x_n) = \max(U(x_n), Z_\infty)$. Since the X_i are independent, the conditional expectation in the right side of (1) reduces to an unconditional expectation. Since y_j depends on $x_1, \ldots, x_j$ only through the values of x_j, the same is true of $V_j^{(n)}$. Hence, for $j = n-1, \ldots, 1$,

$$V_j^{(n)}(x_j) = \max\{y_j(x_j), \frac{1}{j+1}\sum_{x=1}^{j+1} V_{j+1}^{(n)}(x)\}.$$

It is optimal to stop at j if

$$y_j(x_j) \geq \frac{1}{j+1}\sum_{x=1}^{j+1} V_{j+1}^{(n)}(x)$$

and to continue otherwise. The generalization of the result for the CSP that the optimal rule is a threshold rule, is contained in the following lemma.

Lemma. *If it is optimal to select an applicant of relative rank x at stage k, then*
(a) it is optimal to select an applicant of relative rank $x-1$ at stage k, and
(b) it is optimal to select an applicant of relative rank x at stage $k+1$.

Proof. Let $A(j) = (1/j)\sum_{i=1}^{j} V_j^{(n)}(i)$. The hypothesis is that $y_k(x) \geq A(k+1)$. We are to show that (a) $y_k(x-1) \geq A(k+1)$, and (b) $y_{k+1}(x) \geq A(k+2)$. (a) follows since $y_k(x-1) \geq y_k(x)$. To see (b), note first that $A(k+1) \geq A(k+2)$ since $A(k+1)$ is an average of quantities $V_{k+1}^{(n)}(i)$ each at least as large as $A(k+2)$. Thus, (b) will follow if we show $y_{k+1}(x) \geq y_k(x)$. To see this, use the recursion (5) to obtain,

$$\begin{aligned} y_k(x) &= (x/(k+1))y_{k+1}(x+1) + ((k+1-x)/(k+1))y_{k+1}(x) \\ &\leq (x/(k+1))y_{k+1}(x) + ((k+1-x)/(k+1))y_{k+1}(x) \\ &= y_{k+1}(x). \ \blacksquare \end{aligned}$$

This lemma implies that an optimal rule has the following form. Let r_x denote the first stage at which it is optimal to select an applicant of relative rank x. Then, $1 \leq r_1 \leq r_2 \leq \ldots \leq r_n \leq n$ and if at stage j you see an applicant of relative rank x, you stop if $j \geq r_x$. For example, when $U(3) = U(4) = \ldots = U(n) = Z_\infty = 0$, the threshold rule takes the form $N_{r,s}$: continue through stage $r-1$; from stages r through $s-1$, select an applicant of relative rank 1; from stages s through n, select an applicant of relative rank 1 or 2. (See Exercise 3.)

As another application consider the problem of **minimizing the expected rank** of the applicant selected. In this case, $U(j) = j$, $y_j(x_j) = (n+1)x_j/(j+1)$ as may be found using (5), and we are trying to *minimize* $\mathrm{E}y_N(X_N)$. D. V. Lindley (1961) introduced this problem. For small values of n the optimal values of the r_x, call them $r_x(n)$, may be computed without difficulty. The problem of approximating these values when n is large has been solved by Chow, Moriguti, Robbins, and Samuels (1964). Their results may be summarized as follows. Let $V(n)$ denote the expected rank of the applicant selected by an optimal rule when there are n applicants. Then as $n \to \infty$,

$$V(n) \to V(\infty) = \prod_1^\infty (1 + 2/j)^{1/(j+1)} = 3.8695\cdots \quad \text{and}$$

$$r_x(n)/n \to (1/V(\infty)) \prod_1^{x-1} (1 + 2/j)^{1/(j+1)} = b(x).$$

Thus, $b(1) = .2584\cdots$, $b(2) = .4476\cdots$, $b(3) = .5640\cdots$, and for large x, $b(x) = 1 - 2/(x+1) + O(1/x^3)$. Therefore the optimal rule has the following description. Wait until 25.8% of the applicants have passed; then select any applicant of relative rank 1. After 44.8% of the applicants have passed, select any applicant of relative rank 1 or 2. After 56.4% have passed, select any candidate of relative rank 1, 2 or 3, etc.

The paper of Mucci (1973) contains an extension of these results to general non-decreasing payoff based on the ranks. An interesting possibility is that for some reward functions, y_j, the function $b(x) = \lim_{n\to\infty} r_x(n)/n$ may not tend to 1 as $x \to \infty$; that is, there may be an upper bound on the proportion of applicants it is optimal to view.

§**2.3 Variations.** There is a large literature dealing with many variations of the CSP. For example, there may be a probability q that an applicant will not accept an offered job (Exercise 2). Or, in the viewpoint of the marriage problem, it may be a worse outcome to marry the wrong person than not to get married at all (Exercise 1).

In Gilbert and Mosteller (1966), the intrinsic values of the objects are revealed, and you are allowed to use the information in your search for the best object. To be specific, there is a sequence of independent identically distributed continuous random variables, $X_1, \ldots, X_n$, that are shown to you one at a time, and you may stop the proceedings by selecting the number being shown to you. As before, the payoff is a function of the true rank of the number you select. For the best-choice problem, you win 1 if you select the largest X_i and you win 0 otherwise. Gilbert and Mosteller solve the best-choice problem when the distribution of the X_i's is fully known, and show that the limiting probability of choosing the best converges to $v* = .58016\cdots$ as $n \to \infty$. This is to be compared to the limiting value e^{-1} when the intrinsic values are not revealed.

The first description of the secretary problem in print occurs in Martin Gardner's Scientific American column of February 1960, in which he describes a game theoretic version called "Googol". In this version, player I chooses n numbers, $X_1, \ldots, X_n$, writes them on slips of paper and puts them in a hat. Player II, not knowing the numbers, pulls them out at random one at a time. Player II may stop at any time, and if he does he wins one if the last number he has drawn is the largest of all the n numbers. Clearly, player II can achieve at least probability P_{r_1} of winning by using only the relative ranks of the numbers drawn and the optimal stopping rule for the CSP. Can he do better using the actual values of the X's? Can player I choose the X's without giving away any extra information? (See Exercise 4.)

Problems such as Googol, in which the distribution of the X's is completely unknown, are called **no-information** problems. The classical secretary problem is included in this group. Problems such as the problem of Gilbert and Mosteller, in which the X's are i.i.d. with a known distribution, are called **full-information** problems. A number of intermediate versions have been proposed featuring some sort of **partial information** about the distribution of the X's. Petruccelli (1980) treats the normal distribution with unknown mean and shows that the best invariant rule achieves $v*$ as a limiting probability of selecting the best. Thus, asymptotically nothing is lost by not knowing the mean of the normal distribution. On the other hand, if the distribution is uniform on the interval (a, b) with a and b unknown, Stewart (1978) and Samuels (1981) show that the minimax stopping rule is based only on relative ranks, and so gives limiting probability e^{-1}, so that learning the distribution does you no good asymptotically. Campbell and Samuels (1981) consider the problem of selecting the best among the last n objects from a sequence of $m+n$ objects. The success probabilities converge as $m/(m+n) \to t$ to a quantity $p^*(t)$, where $p^*(t)$ increases from e^{-1} at $t = 0$ (no-information) to $v*$ at $t = 1$ (full-information).

Another variation lets the total number of objects that are going to be seen be unknown, violating assumption 2 of the CSP. It is assumed instead that the number of objects is random with some known distribution. These problems were introduced by Presman and Sonin (1972). (See Exercise 5.) Abdel-Hamid, Bather, and Trustrum (1982) consider admissibility of stopping rules for this problem, and J. D. Petruccelli (1983) gives conditions under which the optimal rule is a threshold rule.

Still another variation, introduced by Yang (1974) allows us to attempt to select an object we have already passed over. This is called backward solicitation. The probability of success of backward solicitation depends on how far in the past the object was observed. For example, one may be allowed to choose any of the last m objects that have been observed, as in Smith and Deely (1975). This problem has been extended to the full information case by Petruccelli (1982), and to the case where an option may be bought to be able to recall a desirable object subsequently by J. Rose (1984).

§**2.4 The Cayley-Moser Problem.** A finite horizon version of the problem of selling an asset was proposed by Arthur Cayley in 1875. A smooth version of Cayley's problem, due to Moser (1956), is presented in this section. Another important finite horizon problem, the parking problem of MacQueen and Miller (1960), is described in the next section.

In the Cayley problem, a population of m objects with known values, $x_1, x_2, \ldots, x_m$, is given. From this population, a simple random sample of size at most n, where $n \le m$, is drawn sequentially without replacement. You may stop the sampling at any time by selecting the last chosen object and you receive the value of that object as a reward, but you must stop and select the nth sample if you have not stopped before that stage. The problem is to find a stopping rule that maximizes the expected value of the selected object. Cayley discusses this problem and suggests solving it by the method of backward induction. He then applies the method to a simple example of a population of size $m = 4$

with values $\{1,2,3,4\}$, and finds for $n = 1,2,3,4$, the optimal rules and the maximal expected rewards. (Exercise 7.) The Cayley problem (sampling without replacement from a population of size m of known values $1,2,\ldots,m$) has been extended to sampling with recall and an arbitrary payoff function by Chen and Starr (1980).

Moser reformulates Cayley's problem in a way that provides an approximation to the problem when m is large and the population values are $\{1,2,\ldots,m\}$. He assumes that the observations, $X_1, X_2, \ldots, X_n$, are independent and identically distributed according to a uniform distribution on the interval, $(0,1)$. The return for stopping at stage j is $Y_j = X_j$, at least one observation must be taken (so $y_0 = -\infty$), and stopping is required if stage n is reached.

Let us derive the optimal stopping rule by the method of equation (1), assuming an arbitrary common distribution, $F(x)$, for the X_j, with finite first moment. We have $V_n^{(n)} = Y_n = X_n$, and inductively for $j = n-1$ down to 1, $V_j^{(n)} = \max\{X_j, \mathrm{E}(V_{j+1}^{(n)}|\mathcal{F}_j)\}$. The independence of the X_j implies inductively that $V_j^{(n)}$ depends only on X_j and that $A_{n-j} = \mathrm{E}(V_{j+1}^{(n)}|\mathcal{F}_j)$ is a constant that depends only on $n-j$, the number of stages to go. Thus, the optimal stopping rule stops at j if $X_j \geq A_{n-j}$, where the A_j may be computed inductively as

$$\begin{aligned} A_0 &= -\infty \\ A_1 &= \mathrm{E}X_1 \qquad \text{and for } j \geq 1, \\ A_{j+1} &= \mathrm{E}\max\{X, A_j\} = \int_{-\infty}^{A_j} A_j\,dF(x) + \int_{A_j}^{\infty} x\,dF(x). \end{aligned} \tag{6}$$

When the X_j are uniformly distributed on $(0,1)$, we may as well put $A_0 = 0$, and we find $A_1 = 1/2$, and for $j \geq 1$,

$$A_{j+1} = \int_0^{A_j} A_j\,dx + \int_{A_j}^1 x\,dx = (A_j^2 + 1)/2 \tag{7}$$

We have $A_2 = 5/8$, $A_3 = 89/128$, and so on. Moser finds an approximation to the A_n for large n as

$$A_n \simeq 1 - \frac{2}{n + \log(n) + c} \tag{8}$$

for some constant c.

This may be demonstrated as follows. If we let $B_n = 2/(1-A_n) - n$, then $B_0 = 2$ and (7) reduces to

$$B_{n+1} - B_n = \frac{1}{B_n + n - 1},$$

and we are to show that $B_n - \log(n) \to c$. It is clear that the B_n are increasing. Then, $B_n \geq 2$ for all n so that the differences $B_{n+1} - B_n$ are bounded above by $1/(n+1)$ and

$$B_n = B_0 + \sum_{j=0}^{n-1}(B_{j+1} - B_j) \leq 2 + \sum_{j=0}^{n-1} \frac{1}{j+1}. \tag{9}$$

This upper bound on B_n gives us a lower bound on the differences,

$$B_{n+1} - B_n \geq \frac{1}{n + 1 + \sum_{j=0}^{n-1} \frac{1}{j+1}}$$

from which we may obtain a similar lower bound on the B_n:

$$B_n \geq 2 + \sum_{j=0}^{n-1} \frac{1}{j + 1 + \sum_{i=0}^{j-1} \frac{1}{i+1}}. \tag{10}$$

The difference of the bounds (9) and (10) is a sum of the form $\sum \log(n)/n^2$ and so is bounded. Thus we have $B_n = \log(n) + O(1)$. Finally, let $C_n = B_n - \sum_{j=1}^{n} 1/j$. Then,

$$C_{n+1} - C_n = \frac{1}{n - 1 + C_n + \sum_1^n 1/j} - \frac{1}{n+1} \sim -\frac{\log(n)}{(n+1)^2}$$

shows that C_n is a Cauchy sequence and hence converges. Thus, $B_n - \log(n)$ also converges to some constant c and (8) follows. Gilbert and Mosteller (1966) have approximated the constant c. To six figures, it is 1.76799.

Equations for the A_j are discussed in Guttman (1960) for the normal distribution, in Karlin (1962) for the exponential distribution (see Exercise 8), in Gilbert and Mosteller (1966) for the Pareto distribution and in Engen and Seim (1987) for the gamma distribution.

Various finite horizon extensions of the Moser problem have been suggested. Karlin (1962) and Saario (1986) treat the problem in which there are several objects to be selected. Hayes (1969) allows future returns to be discounted. Karlin (1962), Mastran and Thomas (1973) and Sakaguchi (1976) treat problems with random arrivals. But the main extension of Moser's problem is to allow an unbounded horizon, in which case a cost of observation or a discount factor must be included so that one is not inclined to continue forever. These problems are referred to as house-selling or selling an asset or search for the best offer and are treated in Chapter 4.

Another important generalization of the Moser problem may be found in the work of Derman, Lieberman and Ross (1972). In this problem, there are n workers of known values, $p_1 \leq \ldots \leq p_n$ and n jobs that appear sequentially with random values, $X_1, \ldots, X_n$, assumed to be independent and identically distributed according to a known distribution. The return for assigning worker i to job j is the product $p_i X_j$. The problem is to assign workers to jobs, immediately as the job values become known, in such a way to maximize the expectation of the sum of the returns. This contains the Moser problem when $p_1 = \cdots = p_{n-1} = 0$ and $p_n = 1$. It is a more complex decision problem than the stopping rule problems treated here since at each stage we must decide among a number of actions, not just two. Derman et al. show that there is an optimal policy of the following form: For each $n \geq 1$, there are numbers $-\infty = a_{0n} \leq a_{1n} \leq a_{2n} \leq \cdots \leq a_{nn} = +\infty$ such that if there are n workers remaining to be assigned and a job of value X_1 appears, the job is assigned to worker i if X_1 is contained in the interval $[a_{i-1,n}, a_{in}]$. A remarkable feature of this result is that the a_{ij}'s do not depend on the values of the p_i's, so long as they are arranged in increasing order. This result has been extended to random arrival infinite horizon problems with a discount by Albright (1974). See R. Righter (1990) for further extensions and recent developments.

§**2.5 The Parking Problem.** (MacQueen and Miller (1960)) You are driving along an infinite street toward your destination, the theater. There are parking places along the street but most of them are taken. You want to park as close to the theater as possible without turning around. If you see an empty parking place at a distance d before the theater, should you take it?

Here, we model this problem in a discrete setting. We assume that we start at the origin and that there are parking places at all integer points of the real line. Let $X_0, X_1, X_2, \ldots$ be independent Bernoulli random variables with common probability p of success, where $X_j = 1$ means that parking place j is filled and $X_j = 0$ means that it is available. Let $T > 0$ denote your target parking place. You may stop at parking place j if $X_j = 0$, and if you do you lose $|T - j|$. You cannot see parking place $j+1$ when you are at j, and if you once pass up a parking place you cannot return to it. If you ever reach T, you should choose the next available parking place. If T is filled when you reach it, your expected loss is then $(1-p) + 2p(1-p) + 3p^2(1-p) + \ldots = 1/(1-p)$, so that we may consider this as a stopping rule problem with finite horizon T and with loss

$$y_T = 0 \quad \text{if} \quad X_T = 0 \qquad \text{and} \qquad y_T = 1/(1-p) \quad \text{if} \quad X_T = 1$$

and for $j = 0, \ldots, T-1$,

$$y_j = T - j \quad \text{if} \quad X_j = 0 \qquad \text{and} \qquad y_j = \infty \quad \text{if} \quad X_j = 1.$$

The value $y_j = \infty$ forces you to continue if you reach a parking place j and it is filled. We seek a stopping rule, $N \le T$, to minimize $\mathrm{E}Y_N$.

First we show that if it is optimal to stop at stage j when $X_j = 0$, then it is optimal to stop at stage $j+1$ when $X_{j+1} = 0$. As in Moser's problem, $V_j^{(T)}$ depends only on X_j, and $A_{n-j} = \mathrm{E}(V_{j+1}^{(T)} | \mathcal{F}_j)$ is a constant that depends only on $n-j$. It is optimal to stop at stage $n-j$ if $y_{n-j} \le A_j$. We are to show that if $n - j \le A_j$, then $n - j - 1 \le A_{j-1}$. This follows from the inequalities, $n - j - 1 < n - j \le A_j \le A_{j-1}$.

Thus, there is an optimal rule of the threshold form, N_r for some $r \ge 0$: continue until r places from the destination and park at the first available place from then on. Let P_r denote the expected cost using this rule. Then, $P_0 = p/(1-p)$, and for $r \ge 1$, $P_r = (1-p)r + pP_{r-1}$. We will show by induction that

$$P_r = r + 1 + \frac{2p^{r+1} - 1}{1 - p}. \tag{11}$$

$P_0 = p/(1-p) = 1 + (2p-1)/(1-p)$, so it is true for $r = 0$. Suppose it is true for $r-1$; then $P_r = (1-p)r + pP_{r-1} = (1-p)r + pr + p(2p^r - 1)/(1-p) = (r+1) + (2p^{r+1} - 1)/(1-p)$, as was to be shown.

To find the value of r that minimizes (11), look at the differences, $P_{r+1} - P_r = 1 + (2p^{r+2} - 2p^{r+1})/(1-p) = 1 - 2p^{r+1}$. Since this is increasing in r, the optimal value is the first r for which this difference is nonnegative, namely, $\min\{r \ge 0 : p^{r+1} \le 1/2\}$. For example, if $p \le 1/2$, you should reach the destination before looking for a parking place. But if $p = .9$, say, we should start looking for a parking place $r = 6$ places before the destination.

There are various extensions to this problem. There may be a cost of time or gas. (See Exercise 10.) MacQueen and Miller (1960) allow you to drive around the block in search for a parking place, and Tamaki (1988) allows you to make a U-turn, at a cost, to return to a previously observed parking space. In Tamaki (1985), the problem is extended to allow the probability of finding parking space j free to depend on j. (See Exercises 11 and 12.) Other extensions are treated in Chapter 5.

§2.6 Exercises.

1. *The win-lose-or-draw marriage problem.* (Sakaguchi (1984)) Consider the marriage problem in which the payoff is 1 if you select the best, -1 if you select someone who is not the best, and 0 if you stay single.
(a) Find the optimal rule.
(b) Show that for large n the rule is approximately to pass up the first $1/\sqrt{e} = 60.6\cdots\%$ of the possibilities and to select the next relatively best, if any.

2. *Uncertain employment.* (Smith (1975)) Consider the classical secretary problem with the added possibility that an observed candidate may be unavailable (the applicant may refuse the offer). If this is the case, you are not allowed to select the applicant and the search goes on. Let ϵ_k represent the indicator of the event that applicant k is available; it is assumed that the ϵ_k are independent and independent of the X_j with $\mathrm{P}(\epsilon_k = 1) = p$ for all k. The outcome of availability or not is made known to you at the time of observation.
(a) Show that $y_k = (k/n)I(X_k = 1, \epsilon_k = 1)$.
(b) Show that there is a threshold rule that is optimal.
(c) Show that the probability of win using the threshold rule N_r is

$$P_r = \frac{p}{n}\sum_{k=r}^{n} \frac{\Gamma(r)\Gamma(k-p)}{\Gamma(k)\Gamma(r-p)}.$$

(d) Use $(n-p)^p\Gamma(n-p)/\Gamma(n) \to 1$ as $n \to \infty$ to show that the optimal threshold is approximately $r = np^{1/(1-p)}$.

3. *One of the best two.* (Gilbert and Mosteller (1966), Gusein-Zade (1966)) Consider the secretary problem with $U(3) = \ldots = U(n) = Z_\infty = 0$ and $U(1) = a \ge U(2) = b \ge 0$. An optimal rule is of the form, $N_{r,s}$: do not stop in stages 1 through $r-1$; in stages r through $s-1$, select an object if it has relative rank 1; in stages s through n, select an

object if it has relative rank 1 or 2.
(a) Show P(select the best$|N_{r,s}) = ((r-1)/n)[\sum_r^{s-1} 1/(k-1) + (s-2)\sum_s^n 1/((k-1)(k-2))]$.
(b) Show P(select second best$|N_{r,s}) = ((r-1)/n)[(1/(n-1))\sum_r^{s-1}(n-k)/(k-1) + (s-2)\sum_s^n 1/((k-1)(k-2))]$.
(c) Let $V(r,s) = \mathrm{E}(\mathrm{return}|N_{r,s})$. Let $n \to \infty$, $r/n \to x$, and $s/n \to y$. Show that $V(r,s) \to (a+b)x(\log(y/x) + 1 - y) - bx(y-x)$.
(d) Show that this limit is maximized if $y = (a+b)/(a+2b)$ and x satisfies $\log(x) = (2b/(a+b))x - 1 + \log(y)$.
(e) Specialize to the case $a = b = 1$. (Ans. $x = .3475\cdots$, $y = 2/3$ and limiting probability of selecting one of the best two $= .574\cdots$.

4. *Googol.* (Berezovskiy and Gnedin (1984)) Let $X_1, \ldots, X_n$ be i.i.d. uniform on $(0, \theta)$ where θ has the Pareto distribution, $\mathcal{P}a(\alpha, 1)$. The Pareto distribution, $\mathcal{P}a(\alpha, x)$ is defined to be the distribution with density

$$f(\theta|\alpha, x) = \alpha x^{\alpha} \theta^{-(1+\alpha)} I(\theta > x),$$

where $\alpha > 0$ and x is an arbitrary real number. Let $M_0 = X_0 = 1$, and for $j = 1, \ldots, n$ let $M_j = \max\{X_0, X_1, \ldots, X_j\} = \max\{M_{j-1}, X_j\}$. You observe the X's one at a time and you may stop at any time. If the most recently observed X_j when you stop is the largest of all the X's, including X_0, you win.
(a) Show the posterior distribution of θ given $X_1, \ldots, X_j$ is $\mathcal{P}a(j + \alpha, M_j)$.
(b) Show $y_j = \mathrm{P}\{X_j = M_n | X_1, \ldots, X_j\} = ((j+\alpha)/(n+\alpha)) I(X_j = M_j)$.
(c) Show that if it is optimal to stop at stage j with $X_j = M_j$, then it is optimal to stop at stage $j+1$ if $X_{j+1} = M_{j+1}$.
(d) Show that the optimal stopping rule is to pass up $r-1$ numbers and then stop at the next j such that $X_j = M_j$, where r is that integer that maximizes $P_r = ((r - 1 + \alpha)/(n+\alpha))\sum_{j=r}^n 1/(j - 1 + \alpha)$.

5. *Random Number of Applicants.* (Presman and Sonin (1972), Rasmussen and Robbins (1975)) The number, K, of applicants is unknown, but it is assumed to be random with a uniform distribution on $\{1, \ldots, n\}$. The assumptions are as in the CSP, but if you pass up an applicant and it turns out that there are no more, you lose. You want to select the best of the K applicants.
(a) Find y_j for $j = 1, \ldots, n$.
(b) Show there is a threshold rule that is optimal.
(c) Find the probability of win using an arbitrary threshold rule.
(d) Find approximately, for large n, the optimal threshold rule and the optimal reward.

6. *The Duration Problem.* (Ferguson, Hardwick and Tamaki (1992)) The first 7 assumptions of the CSP are in force, but the payoff now is the proportion of time you are in possession of the relatively best applicant. Thus, if the jth applicant is relatively best and you select him, you win $(k-j)/n$ if the next relatively best applicant is the kth, and you win $(n+1-j)/n$ if the jth applicant is best overall.
(a) Find y_j for $j = 1, \ldots, n$.
(b) Show there is a threshold rule that is optimal.

(c) Find the expected winnings using an arbitrary threshold rule. Note that the answer is exactly the same as for Problem 5(c) so the asymptotic values of 5(d) hold for this problem as well.

7. *The Cayley Example.* Suppose in Cayley's problem that the population consists of $m = 4$ objects of values $1, 2, 3$ and 4. For $n = 1, 2, 3$ and 4, find the optimal stopping rule and its expected payoff if one may sample at most n times.

8. *Moser's problem with an exponential distribution.* (Karlin (1962)) In Moser's problem, suppose the common distribution of the X_i is exponential on $(0, \infty)$.
(a) Find the recurrence relation for the A_j.
(b) Show that $A_n = \log(n) + o(1)$; that is, show $A_n - \log(n) \to 0$.
(c) *Moser's problem with two stops.* (Karlin (1959) vol. 2, Chap. 9, Exercise 13.) In Moser's problem with $\mathcal{U}(0, 1)$ variables and finite horizon n, suppose you are allowed to choose two numbers from the sequence and are given the sum as a payoff. How do you proceed?

9. *Moser's problem with nonidentical distributions.* (Moser (1956)) Suppose, in Moser's problem, that $X_1, \ldots, X_n$ are independent and X_i has a uniform distribution on the interval $(0, n + 1 - i)$.
(a) Find the recurrence for the sequence of cutoff points A_n.
(b) Show that $A_n \simeq n - \sqrt{2n}$ in the sense that $(n - A_n)/\sqrt{2n} \to 1$.

10. *The parking problem with cost.* Solve the parking problem if the loss for stopping at parking place j is changed to $|T - j| + cj$, where $c > 0$ represents a cost of time or gas.

11. *The parking problem with arbitrary probabilities and distances between parking places.* Let $X_0, X_1, \ldots, X_T$ be independent Bernoulli random variables where $P(X_i = 1) = p_i$, $0 \le p_i \le 1$ for all i. Let $s_0 \le s_1 \le \ldots \le s_T$ be a nondecreasing sequence of numbers with $s_T > 0$. Consider the finite horizon stopping rule problem with observations $\{X_i\}$ and rewards Y_i for stopping at i, where $Y_i = s_i$ if $X_i = 1$, and $Y_i = 0$ if $X_i = 0$, for $i = 0, 1, \ldots, T$. (This contains the parking problem with non-constant distances between parking spaces, and probability of a free space dependent on its position.) Let W_j denote the optimal expected reward if it is decided to continue from stage j for $j = 0, 1, \ldots, T-1$, and $W_T = 0$. Note that $W_{j-1} = \mathrm{E}\{V_j^{(T)} | \mathcal{F}_{j-1}\}$ is a constant.
(a) Find the backward recursion equations for the W_j.
(b) Show there is an optimal rule of the form N_r: Stop at the first $j \ge r$ at which $X_j = 1$.

12. *The parking problem with random distances.* Generalize the parking problem to allow random nonnegative distances, Z_j, between parking places. (This extension also contains Exercise 11 when the ϵ_j below are identically one.) One can think of the Z_j as the travel time between parking places which is random due to traffic fluctuations. To be specific: Assume that the observations, $X_j = (Z_j, \epsilon_j)$ for $j = 1, \ldots, T$, are independent with the Z_j nonnegative, the ϵ_j Bernoulli with probability p_j of success, and with Z_j and ϵ_j independent for all j. Let $S_n = \sum_1^n Z_j$, and let the reward for stopping at n be $Y_n = S_n I(\epsilon_n = 1)$. Let $W_j(S_j) = \mathrm{E}\{V_{j+1}^{(T)} | \mathcal{F}_j\}$ denote the optimal expected reward if it is decided to continue from stage j.
(a) Find the backward recursion equations for the $W_j(s)$.

(b) Show that there exist numbers, $r_1 \ge r_2 \ge \cdots \ge r_T = 0$, such that it is optimal to stop at stage j if $S_j \ge r_j$ and $\epsilon_j = 1$.

13. *Fishing.* (Starr (1974)) Suppose that a lake contains n fish whose catch times, $T_1, \ldots, T_n$, are i.i.d. exponential, with density $f(t) = \exp\{-t\}I(t > 0)$. Let $X_1, \ldots, X_n$ denote the order statistics of the T_j. If you stop after catching j fish, you receive $Y_j = j - cX_j$, for $j = 0, 1, \ldots, n$ where $X_0 = 0$ and $c > 0$. Find an optimal stopping rule. Hint: Recall that $X_1, X_2 - X_1, X_3 - X_2, \ldots, X_n - X_{n-1}$ are independent and that $Z_j = X_j - X_{j-1}$ has density $f(z) = (n + 1 - j)\exp\{-(n + 1 - j)z\}I(z > 0)$.

14. *A Symmetric Random Walk Secretary Problem.* (Blackwell and MacQueen, (1996), personal communication.) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables. Let $S_k = \sum_1^k X_i$ denote the partial sums with $S_0 = 0$, and let $M_k = \max\{S_0, S_1, \ldots, S_k\}$ be the maxima of the partial sums. If we stop at stage k, $0 \le k \le n$, we win if and only if $S_k = M_n$. This is a full-information secretary problem with the worths of the secretaries following a random walk. Note that ties among the S_j may occur, but you win if you tie for best. If you stop at a stage k with $S_k < M_k$ you can't possibly win. If you stop at stage k with $S_k = M_k$, then your probability of winning is $y_k = p_{n-k}$, where $p_j = \mathrm{P}(S_1 \le 0, \ldots, S_j \le 0)$. Thus this is a finite horizon stopping rule problem with payoff for stopping at k equal to $y_k \mathrm{I}(S_k = M_k)$.

The problem seems hard in general, but make the assumption that *the distribution of the X's is symmetric about* 0, and do the following.

(a) Suppose you are at a stage k with $S_k = M_k$ and you decide to continue until stage n in the hope that $S_n = M_n$. The probability you win is then q_{n-k}, where $q_j = \mathrm{P}(S_0 \le S_j, \ldots, S_{j-1} \le S_j)$. Show that $p_j = q_j$ for all j.

(b) Show that any stopping rule is optimal provided it does not stop at any $k < n$ for which $S_k < M_k$. In particular, the rule that continues to the last stage and stops is optimal.

(c) Suppose that the distribution of the X's is Bernoulli with $\mathrm{P}(X = 1) = \mathrm{P}(X = -1) = 1/2$. Show $p_n = \binom{n}{\lfloor n/2 \rfloor} 2^{-n}$.

(d) Suppose that the distribution of the X's is double exponential with density $\frac{1}{2}e^{-|x|}$. Show that

$$p_n = \frac{1}{2^{2n-1}}\binom{2n-1}{n} \qquad \text{for } n = 1, 2, \ldots$$

One way is to use the fundamental identity of Wald ($\mathrm{E}e^{tS_N}M(t)^{-N} \equiv 1$, valid for t for which $M(t)$ is finite, where $M(t) = \mathrm{E}e^{tX}$ is the moment generating function of X) to find the probability generating function of $N = \min\{n > 0 : S_n > 0\}$ is $\mathrm{E}u^N = 1 - \sqrt{1 - u}$.

(e) Show that the answer to (d) holds for *any* continuous symmetric distribution of the X's. (Use Theorem 8.4.3 of Chung, (1974).)

15. *The Cayley-Moser problem with independent non-identically distributed distributions.* (Assaf, Goldstein and Samuel-Cahn (2000)) If, in the Cayley-Moser problem, the X_i have different distributions with different means etc., the decision maker should be able to improve his expected return by choosing the order in which he observes the X_i. Let $F_j(x)$ denote the distribution function of X_j and assume that $X_j \ge 0$ for all j. Suppose

we observe the X_i sequentially in reverse order, $X_n, X_{n-1}, \ldots, X_1$. Then corresponding to (6), the optimal rule stops at X_{j+1} if $X_{j+1} > A_j$, where $A_0 = 0$, $A_1 = \mathrm{E}X_1$ and for $j \geq 1$

$$A_{j+1} = \mathrm{E}\max\{X_{j+1}, A_j\} = \int_0^{A_j} A_j\, dF_{j+1}(x) + \int_{A_j}^{\infty} x\, dF_{j+1}(x).$$

Define the function $g_j(\alpha) = \mathrm{E}\max\{X_j, \alpha\}$; then $A_1 = g_1(0)$, $A_2 = g_2(g_1(0))$, and so forth.

Suppose the random variable X_j has distribution function $F(x|\theta_j)$, where

$$F(x|\theta) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{(1-\theta)^2}{(1-x\theta)^2} & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

for $0 < \theta < 1$. Note the $F(x|\theta)$ has a mass at 0 of size $(1-\theta)^2$. The θ_j are known and between 0 and 1.
(a) Show $\mathrm{E}X_j = \theta_j$.
(b) Show

$$g_j(\alpha) = 1 - \frac{(1-\theta_j)(1-\alpha)}{1-\theta_j\alpha} \qquad \text{for } 0 < \alpha < 1.$$

(c) Show $g_1(g_2(\alpha)) = g_2(g_1(\alpha))$ for all $0 \leq \alpha < 1$.
(d) Conclude that A_n is a symmetric function of the θ_i so that the decision maker is indifferent as to the order in which he observes the X_i!