Solutions to the Exercises of Chapter 1.

1. Let $\mathbf{X}_j = (X_{1j}, X_{2j})$ denote the j^{th} observation, where

$$X_{1j} = \quad \text{return from the } j^{th} \text{ burglary, and}$$

$$X_{2j} = \begin{cases} 0 & \text{if he is caught at the } j^{th} \text{ burglary, and} \\ 1 & \text{if he is not caught at the } j^{th} \text{ burglary.} \end{cases}$$

It is assumed that the $\mathbf{X}_j$ are i.i.d. and that X_{1j} and X_{2j} are independent of each other. The distribution of the X_{1j} is assumed known, as is the probability that X_{2j} is 1. The burglar's total fortune, if he stops at n, is

$$Y_n = (\prod_{j=1}^{n} X_{2j})(Y_0 + \sum_{j=1}^{n} X_{1j}) \quad \text{for } n = 1, 2, 3, \ldots,$$

where Y_0 is his initial fortune, and $Y_\infty = 0$. The problem is to find a stopping rule N to maximize $\mathrm{E}(Y_N)$.

2. Let T_j be the time required to catch fish number j. The $T_1, \ldots, T_n$ are i.i.d. with known distribution, F, assumed to have non-decreasing hazard rate. The observations are $X_j = T_{(j)}$, where $T_{(1)} \le \ldots \le T_{(n)}$ are the order statistics. If we stop after catching j fish, we receive a reward, j, and pay $cT_{(j)}$; thus, the reward sequence is given by $Y_0 = 0$, and

$$Y_j = j - cT_{(j)} = j - cX_j \quad \text{for} \quad j = 1, 2, \ldots, n,$$

where $c > 0$. This is a finite horizon problem. We seek a stopping rule $N \le n$ to maximize $\mathrm{E}(Y_N)$.

3. The observations $X_1, X_2, \ldots$ are i.i.d. with $\mathrm{P}(X_i = j) = p_j$ for $j = 1, 2, \ldots$ assumed known. Let $D(n)$ denote the number of distinct observations among $X_1, \ldots, X_n$, $D(n) = 1 + \sum_2^n \mathrm{I}(X_j \ne X_1, \ldots, X_j \ne X_{j-1})$. If we stop after observing X_n, we receive

$$Y_n = D(n) - nc \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

where $c > 0$, and $Y_\infty = -\infty$. The problem is to find a stopping rule, N, to maximize $\mathrm{E}(Y_N)$.

4. Let X_n denote the number of misprints found on the n^{th} proofreading. The observations are $X_1, X_2, \ldots,$ and their joint distribution may be described as follows. Given the number M of misprints, X_1 has a binomial distribution with sample size M and probability of success, p_1, denoted by $\mathcal{B}(M, p_1)$. Similarly, given M and $X_1, \ldots, X_{n-1}$, the distribution of X_n is $\mathcal{B}(M_n, p_n)$, where $M_n = M - X_1 - \cdots - X_{n-1}$, the number of misprints remaining after the first $n-1$ proofreadings. The prior distribution of M is

assumed known. Since each proofreading costs $c_1 > 0$ and each undetected misprint costs $c_2 > 0$, stopping after proofreading number n costs

$$Y_n = nc_1 + (M - \sum_1^n X_i)c_2 \quad \text{for} \quad n = 0, 1, \ldots, \quad \text{and} \quad Y_\infty = \infty.$$

We seek a stopping rule N to minimize $\mathrm{E}Y_N$. Note that the payoff depends on M which is not known exactly when the decision to stop is made. We may, as pointed out in Remark 2, replace the payoff by the known function,

$$y_n(X_1, \ldots, X_n) = nc_1 + (\mathrm{E}\{M|X_1, \ldots, X_n\} - \sum_1^n X_i)c_2 \quad \text{for} \quad n = 0, 1, \ldots,$$

and $Y_\infty = \infty$.

5. The observations, $X_1, X_2, \ldots$, are i.i.d. with $\mathrm{P}(X_j = 1) = p = 1 - \mathrm{P}(X_j = 0)$. The net return for stopping at n is

$$Y_n = K \sum_{j=1}^n (\prod_{i=n-j+1}^n X_i) - nc \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

and $Y_\infty = -\infty$, where K and c are positive constants. We seek a stopping rule, N, to maximize $\mathrm{E}(Y_N)$.

Solutions to the Exercises of Chapter 2.

1.(a) By the Lemma, there is an optimal threshold rule R_r. Let E_r denote the expected payoff using R_r. Then,

$$\begin{aligned} E_r &= \mathrm{P}(\text{select best}|R_r) - \mathrm{P}(\text{select not best}|R_r) \\ &= 2\mathrm{P}(\text{select best}|R_r) - \mathrm{P}(\text{select something}|R_r). \end{aligned}$$

Now, $\mathrm{P}(\text{select something}|R_r) = \mathrm{P}(\text{the best occurs after } r-1) = (n-r+1)/n$, and $\mathrm{P}(\text{select the best}|R_r)$ is the same as it is for the CSP. Hence,

$$E_r = (2(r-1)/n)\sum_{k=r}^{n} 1/(k-1) - (n-r+1)/n.$$

The optimal r is the first r such that $E_{r+1} \le E_r$, which reduces to the first r such that $\sum_{k=r+1}^{n} 1/(k-1) \le 1/2$.
(b) Using $\sum_{k=r+1}^{n} 1/(k-1) \sim \log(n/r)$, we find for the optimal r that $\log(n/r) \sim 1/2$, or $r \sim n/\sqrt{e}$.

2.(a) If applicant k is a candidate and is available, then the probability of win is the same as for the CSP.
(b) Same argument as for the CSP.

(c)
$$\begin{aligned} P_r &= \sum_{k=r}^{n} \mathrm{P}(\text{select } k \text{ and it wins}|N_r) \\ &= \sum_{k=r}^{n} \mathrm{P}(\text{reach } k|N_r)\mathrm{P}(X_k = 1, \epsilon_k = 1, \text{ and it wins}). \end{aligned}$$

$$\mathrm{P}(\text{reach } k|N_r) = \frac{r-p}{r}\,\frac{r+1-p}{r+1}\cdots\frac{k-1-p}{k-1} = \frac{\Gamma(r)\Gamma(k-p)}{\Gamma(k)\Gamma(r-p)}$$

$$\mathrm{P}(X_k = 1, \epsilon_k = 1, \text{ and it wins}) = (1/k)p(k/n) = p/n.$$

(d) Computation reveals that

$$P_{r+1} - P_r = \left[\frac{p}{r-p}\sum_{r+1}^{n}\frac{\Gamma(r)\Gamma(k-p)}{\Gamma(k)\Gamma(r-p)} - 1\right]\frac{p}{n}.$$

Hence, the optimal rule is $N_{r'}$, where r' is

$$r' = \min\{r \ge 1 : \frac{p\Gamma(r)}{\Gamma(r+1-p)}\sum_{r+1}^{n}\frac{\Gamma(k-p)}{\Gamma(k)} \le 1\}.$$

Using the approximation $k^p\Gamma(k-p)/\Gamma(k) \to 1$ as $k \to \infty$, we find

$$\sum_{r+1}^{n} \frac{\Gamma(k-p)}{\Gamma(k)} \sim \int_r^n x^{-p}\,dx = \frac{n^q - r^q}{q},$$

where $q = 1-p$. Combined with $p\Gamma(r)/\Gamma(r+1-p) \sim p/r^q$, this gives the result.

3. Let $A(k,r)$ be the event $\{$best of first $k-1$ occurs before $r\}$, and let $B(k,s)$ be the event $\{$2nd best of first $k-1$ occurs before $s\}$. Let $P_k = \mathrm{P}(A(k,r)) = (r-1)/(k-1)$ and $Q_k = \mathrm{P}(B(k,s)|A(k,r)) = (s-2)/(k-2)$. Then,

(a)
$$\begin{aligned} P_A = \mathrm{P}(\text{select best}) &= \sum_{k=r}^{n} \mathrm{P}(\text{best at } k \text{ and is selected}) \\ &= \sum_{k=r}^{n} (1/n)\mathrm{P}(\text{reach } k) \\ &= \sum_{k=r}^{s-1} (1/n)P_k + \sum_{k=s}^{n} (1/n)P_kQ_k, \end{aligned}$$

which gives the formula in (a).

(b) Let $Z_k = \mathrm{P}(\text{best occurs after } k|\text{2nd best at } k) = (n-k)/(n-1)$. Then,

$$\begin{aligned} P_B = \mathrm{P}(\text{select 2nd best}) &= \sum_{k=r}^{n} \mathrm{P}(\text{2nd best at } k \text{ and is selected}) \\ &= \sum_{r}^{s-1} (1/n)P_kZ_k + \sum_{s}^{n} (1/n)P_kQ_k, \end{aligned}$$

which gives the formula in (b).

(c) The expected return using rule $R_{r,s}$ is

$$V(r,s) = aP_A + bP_B \to W(x,y)$$

where

$$W(x,y) = (ax)\log(y/x) + bx(\log(y/x) - (y-x)) + (a+b)x(1-y).$$

(d) The derivative of $W(x,y)$ with respect to y yields $(a+b)x/y - ax - 2bx = 0$ which gives $y = (a+b)/(a+2b)$. The derivative of $W(x,y)$ with respect to x yields $(a+b)(\log(y/x) - y) - b(y-x) + bx = 0$ which reduces to the formula in (d).

(e) If $a = b = 1$, then $y = 2/3$ and $\log(x) = \log(2/3) - 1 + x$. Newton's method gives $x = .3475\cdots$.

4. (a) The joint density of $\theta, X_1, \ldots, X_j$ is

$$\begin{aligned} g(\theta, x_1, \ldots, x_j) &= \alpha\theta^{-(\alpha+1)} I(\alpha > 1) \prod_{i=1}^{j} I(0 < x_i < \theta)(1/\theta) \\ &= \alpha\theta^{-(j+\alpha+1)} I(\theta > \max(1, x_1, \ldots, x_j)). \end{aligned}$$

Hence, $g(\theta|x_1,\ldots,x_j)$ is proportional to $\theta^{-(j+\alpha+1)}I(\theta > m_j)$, and so is $\mathcal{P}a(j+\alpha, m_j)$.
(b) $y_j(x_1,\ldots,x_j) = \mathrm{P}(M_n = m_j|M_j = m_j)I(X_j = M_j)$, and

$$\begin{aligned}\mathrm{P}(M_n = M_j|X_1,\ldots,X_j) &= \mathrm{E}\{\mathrm{P}(M_n = M_j|\theta, X_1,\ldots,X_j)|X_1,\ldots,X_j\} \\ &= \int_{M_j}^{\infty} (M_j/\theta)^{n-j} f(\theta|\alpha+j, M_j)\,d\theta \\ &= (\alpha+j)/(\alpha+n),\end{aligned}$$

independent of $X_1,\ldots,X_j$.
(c) If you have a new candidate at stage j, that is if $X_j = M_j$, it is optimal to stop if and only if

$$(\alpha+j)/(\alpha+n) \ge \mathrm{P}(\text{win with best strategy from stage } j+1 \text{ on}).$$

The right side of this inequality is a nonincreasing function of j, since any strategy available at stage $j+2$ is also available at stage $j+1$. Since the left side of the inequality is an increasing function of j, an optimal rule may be found among rules N_r for some $r \ge 1$: reject the first $r-1$ applicants and accept the next applicant for which $X_j = m_j$, if any.
(d) The probability of a win, $P_r = \mathrm{P}(\text{win}|N_r)$, may be computed as follows.

$$P_r = \sum_{j=r}^{n} \mathrm{P}(\text{select } j|N_r)\mathrm{P}(j \text{ is best}|\text{select } j) = \frac{\alpha+r-1}{\alpha+n}\sum_{r}^{n}\frac{1}{\alpha+j-1},$$

since $\mathrm{P}(j \text{ is best}|\text{select } j) = (\alpha + j/(\alpha+n)$, and

$$\begin{aligned}\mathrm{P}(\text{select } j|N_r) &= \mathrm{P}(M_{j-1} = M_{r-1} \text{ and } M_j > M_{j-1}) \\ &= ((\alpha+r-1)/(\alpha+j-1))(1-(\alpha+j-1)/(\alpha+j)) \\ &= (\alpha+r-1)/((\alpha+j-1)(\alpha+j)).\end{aligned}$$

To find the value of r that maximizes P_r, look at the differences,

$$\begin{aligned}P_{r+1} - P_r &= \frac{\alpha+r}{\alpha+n}\sum_{r+1}^{n}\frac{1}{\alpha+j-1} - \frac{\alpha+r-1}{\alpha+n}\sum_{r}^{n}\frac{1}{\alpha+j-1} \\ &= \frac{1}{\alpha+n}\left[\sum_{r+1}^{n}\frac{1}{\alpha+j-1} - 1\right].\end{aligned}$$

Since this is decreasing in r, the optimal r is the first r such that the term in square brackets is less than or equal to 0.

5. (a) If we reach stage j and the jth applicant is a candidate and if we accept him, we win if none of the remaining $K-j$ applicants is a candidate. Since the distribution of K given $K \ge j$ is uniform on the set $\{j,\ldots,n\}$, this happens with probability

$$\begin{aligned}p_j &= \frac{1}{n-j+1}\sum_{k=j}^{n}\mathrm{P}(X_{j+1} > 1,\ldots,X_k > 1) \\ &= \frac{1}{n-j+1}\sum_{k=j}^{n}\frac{j}{j+1}\frac{j+1}{j+2}\cdots\frac{k-1}{k} = \frac{j}{n-j+1}\sum_{k=j}^{n}\frac{1}{k}.\end{aligned}$$

Thus, $y_j = p_j \mathrm{I}\{K \geq j, X_j = 1\}$.

(b) We show that for every j, if it is optimal to stop at j with a candidate, then it is optimal to stop at $j+1$ with a candidate as well. Let W_j be the optimal expected return if we continue from j, given that we reach j. This sequence of constants is nonincreasing since continuing from $j+1$ is always an option if we continue from j. Moreover, it is optimal to stop at j with a candidate if $p_j \geq W_j$. We show below that p_j is increasing. Then the result follows since $p_{j+1} > p_j \geq W_j \geq W_{j+1}$.

$$\begin{aligned} p_{j+1} - p_j &= \frac{j+1}{n-j} \sum_{k=j+1}^{n} \frac{1}{k} - \frac{j}{n-j+1} \sum_{k=j}^{n} \frac{1}{k} \\ &= \left(\frac{j+1}{n-j} - \frac{j}{n-j+1} \right) \sum_{k=j+1}^{n} \frac{1}{k} - \frac{1}{n-j+1} \\ &= \frac{n+1}{(n-j+1)(n-j)} \sum_{k=j+1}^{n} \frac{1}{k} - \frac{1}{n-j+1}. \end{aligned}$$

This is positive if and only if

$$\frac{1}{n-j} \sum_{k=j+1}^{n} \frac{1}{k} > \frac{1}{n+1}.$$

This is always true since the left side is an average of terms each of which is greater than $1/(n+1)$.

(c) If N_r is used, the conditional probability of win given $K = k$ is zero if $k < r$ and, from (3) of the text, $((r-1)/k)\sum_{i=r}^{k} 1/(i-1)$ if $k \geq r$. Since K is uniform on $\{1, \ldots, n\}$,

$$\mathrm{E}(Y_{N_r}) = \frac{1}{n} \sum_{k=r}^{n} \frac{r-1}{k} \sum_{i=r}^{k} \frac{1}{i-1}.$$

(d) If n is large and r/n is approximately x, then the expected return of part 1(c) and 2(c) is a Riemann approximation to an integral,

$$\begin{aligned} \frac{r-1}{n} \sum_{k=r}^{n} \frac{1}{k} \sum_{j=r}^{k} \frac{1}{j-1} &\sim x \int_x^1 \frac{1}{y} \int_x^y \frac{1}{z}\, dz\, dy \\ &= x \int_x^1 \frac{1}{y} \log \frac{y}{x}\, dy = x(\log x)^2/2. \end{aligned}$$

This has derivative $(\log x)^2 + 2(\log x)$ which has a unique root at a maximum value at $\log x = 2$ or $x = e^{-2} = .135\ldots$. The optimal rule is to pass up about 13.5% of the possible applicants and select the next candidate. The optimal expected return is $2e^{-2} = .271\ldots$.

6. (a) Let T_j denote the time of the first relatively best object after j, and $T_j = n+1$ if there are none. Then if applicant j is relatively best and selected, then the expected proportion of time this applicant remains relatively best is

$$p_j = \mathrm{E}\,\frac{T_j - j}{n} = \sum_{k=j}^{n} \frac{1}{n}\mathrm{P}(T_j > k) = \frac{1}{n}\sum_{k=j}^{n}\frac{j}{k}.$$

(b) If W_j represents the optimal expected return if we continue from j, then the W_j are nonincreasing constants as in problem 5(b). However, the p_j are not monotone:

$$p_{j+1} - p_j = \frac{j+1}{n}\sum_{k=j+1}^{n}\frac{1}{k} - \frac{j}{n}\sum_{k=j}^{n}\frac{1}{k} = \frac{1}{n}\sum_{k=j+1}^{n}\frac{1}{k} - \frac{1}{n}.$$

This is positive if and only if $\sum_{j+1}^{n}(1/k) > 1$, valid for small j but not for large. Hence, we can see that p_j is monotone increasing for j less than some j_n and monotone decreasing for j greater than j_n. So it is certainly optimal to select a candidate at any stage $j \geq j_n$ since continuing will certainly decrease our expected payoff. On the other hand, for $j < j_n$ if it is optimal to stop at j with a candidate (because $y_j \geq W_j$), then it is optimal to stop at $j+1$ with a candidate, since $y_{j+1} > y_j \geq W_j \geq W_{j+1}$. This shows that there is a threshold rule that is optimal.

(c) The expected return of the threshold rule N_r is

$$\mathrm{E}Y_{N_r} = \sum_{j=r}^{n}\mathrm{P}(T_{r-1} = j)p_j = \sum_{j=r}^{n}\frac{r-1}{(j-1)j}\frac{1}{n}\sum_{k=j}^{n}\frac{j}{k} = \frac{r-1}{n}\sum_{k=r}^{n}\frac{1}{k}\sum_{j=r}^{k}\frac{1}{j-1}$$

exactly the same answer as 5(c).

7. For $n = 1$, there is no choice. The first observation is selected and the expected return is 5/2.

Suppose $n = 2$. If the first observation is 1, you clearly continue and the expected return is 3. If the first observation is 2, the expected return if you continue is 8/3, so you should continue. If the first observation is 3 or 4, you should stop. The expected return is $(3 + (8/3) + 3 + 4)/4 = 19/6$.

Suppose $n = 3$. If the first observation is 1 or 2, you should continue until you get a 3 or a 4, and your expected return is 7/2. If the first observation is 3, continuing and stopping at the next stage only with the 4 gives an expected payoff of 19/6; since this is greater then 3, it is best to continue. If the first observation is a 4, you should stop. The expected return is $[(7/2) + (7/2) + (19/6) + 4]/4 = 85/24$.

If $n = 4$, you can wait for the best value. The expected payoff is 4.

8.(a) We have $A_0 = 0$, $A_1 = 1$ and for $j \geq 1$,

$$\begin{aligned} A_{j+1} &= \int_0^{A_j} A_j e^{-x}\,dx + \int_{A_j}^{\infty} x e^{-x}\,dx \\ &= A_j + e^{-A_j}. \end{aligned}$$

(b) Let $B_n = e^{A_n} - (n+1)$, so that $A_n = \log(n+1+B_n)$. From the recursion formula of part (a), we have $\log(n+1+B_n) = \log(n+B_{n-1}) + 1/(n+B_{n-1})$ for $n \geq 1$, which implies,

$$\begin{aligned} B_n - B_{n-1} &= (n+B_{n-1})\left[\exp\{\frac{1}{n+B_{n-1}}\} - 1\right] - 1 \\ &= \frac{1}{2(n+B_{n-1})} + \frac{1}{3!(n+B_{n-1})^2} + \cdots \end{aligned}$$

We start with $B_0 = 0$ and find inductively that $B_n \geq 0$ for all n, so that $A_n \geq \log(n+1)$ for all n. This equation also implies that $B_n - B_{n-1} \leq 1/n$ for all n sufficiently large so that $B_n < \log(n) + c$ for some constant c. The conclusion now follows from $0 \leq A_n - \log(n+1) = \log(n+1+B_n) - \log(n+1) \leq \log(1 + (\log(n)+c)/(n+1)) \to 0$.

(c) If $a_n = (1+a_{n-1}^2)/2$ is the answer to the optimal cutoff for one choice, then the optimal cutoff for two choices left is s_n, where

$$s_n - a_n = (1 + (s_{n-1} - a_{n-1})^2)/2 - (a_n - a_{n-1}).$$

9.(a) We accept the last observation, X_n if we get that far, so $A_0 = 0$. We accept X_{n-1} if it is at least $A_1 = \mathrm{E}X_n = 1/2$. Inductively we accept X_{n-j} if it is at least A_j, where

$$A_j = \int_0^{A_{j-1}} A_{j-1}\frac{1}{j}\,dx + \int_{A_{j-1}}^{j} \frac{x}{j}\,dx = \frac{j}{2} + \frac{A_{j-1}^2}{2j}.$$

(b) Since A_n is the average of n and A_{n-1}^2/n and $A_0 = 0$ we have inductively that $A_n \leq n$ for all n. Let $B_n = n+1-A_n$. Then $B_0 = 1, B_1 = 3/2, B_2 = 31/16$, and inductively for $n \geq 1$,

$$B_n = B_{n-1} + 1 - \frac{B_{n-1}^2}{2n}.$$

We are to show that $B_n/\sqrt{2n} \to 1$. We first show by induction that $B_n \leq \sqrt{2(n+1)}$. This is true for $n = 0$, and if true for n, then, since $B + 1 - B^2/(2n)$ is an increasing function of B for $B < n$, $B_{n+1} \leq \sqrt{2n} + 1 - 1 < \sqrt{2(n+1)}$.

So now letting $C_n = B_n/\sqrt{2(n+1)}$, we find $C_0 = 1/\sqrt{2} = .707\cdots$, $C_1 = .75$, $C_2 = .791\cdots$, and

$$C_n = \frac{\sqrt{2n}C_{n-1} + 1 - C_{n-1}^2}{\sqrt{2n+2}}.$$

If $f_n(C_{n-1})$ denotes the right side of this expression, then $f_n(c)$ is concave and nondecreasing in (0,1) for $n \geq 2$ with $f_n(0) > 0$ and $f_n(1) < 1$. So there exists a unique fixed point r_n of f_n and C_n is between r_n and C_{n-1}. Moreover, r_n is the root in (0,1) of the equation $c^2 + (\sqrt{2n+2} - \sqrt{2n})c - 1 = 0$, and so the r_n are increasing since the coefficients $\sqrt{2n+2} - \sqrt{2n}$ are decreasing. Since $C_2 < r_2 = .800\cdots$, we have by induction that the C_n are increasing, and hence converging to a limit, say $C_n \to b \leq 1$. The proof will be complete when we show that $b = 1$. Suppose $b < 1$, that is suppose $B_n \le b\sqrt{2(n+1)}$ for all n. Then from the recursion for the B_n, we have $B_n = B_0 + \sum_1^n (B_j - B_{j-1}) = 1 + \sum_1^n (1 - B_{j-1}^2/(2j)) \ge 1 + n(1-b^2)$. But this contradicts the fact that $B_n < \sqrt{2(n+1)}$.

10.Pretend you pay nc for gas anyway. Then you pay $(n-j)(1-c)$ if you stop at $j < n$, and $(j-n)(1+c)$ if you stop at $j > n$. Change scale by measuring in units of $(1-c)$: you pay $n-j$ for stopping at $j \le n$ and $(j-n)\alpha$ for stopping at $j > n$ where $\alpha = (1+c)/(1-c)$. The problem becomes as before except that if you go beyond the destination you expect to pay $\alpha/(1-p)$. Thus, the recursion is as before but the initial condition is now $P_0 = \alpha p/(1-p)$. Induction this time leads to $P_r = (r+1) + ((\alpha+1)p^{r+1} - 1)/(1-p)$. The optimal value of r is smallest $r \ge 0$ such that $p^{r+1} \ge 1/(\alpha+1) = (1-c)/2$.

11.(a) We have $W_T = 0$, $W_{T-1} = p_T s_T$ and for $j \le T$,

$$W_{j-1} = p_j \max\{s_j, W_j\} + (1-p_j)W_j = W_j + p_j(s_j - W_j)^+.$$

(b) An optimal rule stops at j if $s_j \ge W_j$. From (a), the W_j are nonincreasing, and from assumption the s_j are nondecreasing. So the rule N_r is optimal for $r = \min\{j : s_j \ge W_j\}$.

12.(a) We have $W_T(s) \equiv 0$ and for $j \le T$,

$$\begin{aligned} W_{j-1}(s) &= p_j \mathrm{E}\max(s+Z_j, W_j(s+Z_j)) + (1-p_j)\mathrm{E}W(s+Z_j) \\ &= \mathrm{E}W(s+Z_j) + p_j \mathrm{E}(s+Z_j - W_j(s+Z_j))^+. \end{aligned}$$

(b) It is optimal to stop at stage j with $S_j = s$ if $s \ge W_j(s)$. $W_j(s)$ is nondecreasing and continuous in s, since any stopping rule used with s gives at least a great a return when used with $s+\delta$ but not more that δ greater, where δ is an arbitrary positive number. Hence there is a number r_j such that it is optimal to stop at stage j with $S_j = s$ if $s \ge r_j$. That the r_j are nonincreasing follows from $W_{j-1}(s) \ge \mathrm{E}W_j(s+Z_j) \ge W_j(s)$, since the Z_j are nonnegative.

13. $V_n^{(n)} = Y_n = n - c\sum_1^n Z_j$. Using the independence of the Z_j and the fact that $\mathrm{E}Z_{n-j} = 1/(j+1)$, we find

$$\begin{aligned} V_{n-1}^{(n)} &= \max\{Y_{n-1}, \mathrm{E}(Y_n|Z_1,\ldots,Z_{n-1})\} \\ &= Y_{n-1} + \max\{0, \mathrm{E}(1 - cZ_n|Z_1,\ldots,Z_{n-1})\} \\ &= Y_{n-1} + \max\{0, 1-c\} = Y_{n-1} + c_{n-1}, \end{aligned}$$

say. We would only continue at the last stage if $c \le 1$. Similarly, by backward induction,

$$V_{n-j}^{(n)} = Y_{n-j} + c_{n-j},$$

where

$$c_{n-j} = \max\{0, 1 - c/j + c_{n-j+1}\}.$$

It is seen that $c_1, c_2, \ldots$ forms a decreasing sequence until c_j hits zero, and then it stays at zero. The optimal rule continues until c_j hits zero. So, find the first $k \le n$ such that $1 - c/k \le 0$ and stop at $n-k$. For example, if $c = 5.5$, catch all but 5 fish and stop.

14. (a) $q_j = \mathrm{P}\{S_j \ge 0, S_j - S_1 \ge 0, \ldots, S_j - S_{j-1} \ge 0\} = \mathrm{P}\{X_j \ge 0, X_j + X_{j-1} \ge 0, \ldots, X_j + \cdots + X_1 \ge 0\} = \mathrm{P}\{S_1 \ge 0, \ldots, S_j \ge 0\} = \mathrm{P}\{S_1 \le 0, \ldots, S_j \le 0\} = p_j$, using the symmetry of the distribution of the X's.

(b) Let $q(n,z)$ denote the probability of ending on a maximum if there are n stages to go and the deficit is z, $q(n,z) = \mathrm{P}(S_n = \max\{M_n, z\})$, defined for integers $n \ge 0$ and real numbers $z \ge 0$. Note that $q_n = q(n,0)$. Then $q(n,z)$ satisfies the recurrence,

$$q(n,z) = \mathrm{E}q(n-1,(z-X)^+) \tag{1}$$

with initial condition

$$q(0,z) = \begin{cases} 1 & \text{if } z = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Let $V_j^{(n)}(z)$ denote the expected return from an optimal policy for the problem with n stages when we are already at stage j with $S_j = s$ and $M_j = s + z$. Since we should never stop with $S_j < M_j$, the dynamic programming equations are

$$V_j^{(n)}(z) = \begin{cases} \mathrm{E}V_{j+1}^{(n)}((z-X)+) & \text{if } z > 0 \\ \max\left\{p_{n-j}, \mathrm{E}V_{j+1}^{(n)}((-X)^+)\right\} & \text{if } z = 0 \end{cases} \tag{3}$$

with boundary condition $V_n^{(n)}(z) = 1$ if $z = 0$ and $V_n^{(n)}(z) = 0$ otherwise.

We now proceed by backward induction to show that $V_j^{(n)}(z) = q(n-j,z)$. This is true for $j = n$ by the boundary conditions. Suppose true for $k+1 \le j \le n$. Then (3) for $j = k$ becomes

$$\begin{aligned} V_k^{(n)}(z) &= \begin{cases} \mathrm{E}q(n-k-1,(z-X)^+) & \text{if } z > 0 \\ \max\{p_{n-k}, \mathrm{E}q(n-k-1,(-X)^+)\} & \text{if } z = 0 \end{cases} \\ &= \begin{cases} q(n-k,z) & \text{if } z > 0 \\ \max\{p_{n-k}, q(n-k,0)\} & \text{if } z = 0 \end{cases} \end{aligned} \tag{4}$$

using (1) and (2). From part (a) we have $p_{n-k} = q_{n-k} = q(n-k,0)$. This shows $V_k^{(n)}(z) = q(n-k,z)$ and completes the induction. It also shows that when $S_k = M_k$, it is optimal to stop or to continue. If $S_k < M_k$, it is always optimal to continue.

(c) Use the Reflection Principle. First note that S_{2n-1} cannot be 0, so that $p_{2n-1} = p_{2n}$ for all n. Let us compute p_{2n}. We break the sample space into three subsets, $E_1 = \{S_1 \le 0, \ldots, S_{2n} \le 0\}$, $E_2 = \{S_{2n} > 0\}$, and $E_3 = \{S_{2n} \le 0, \text{ but } S_j > 0 \text{ for some } j\}$. We are to show $\mathrm{P}(E_1) = \binom{2n}{n} 2^{-2n}$.

The Reflection Principle says $\mathrm{P}(E_2) = \mathrm{P}(E_3)$. This is seen as follows. For any path $(S_1, S_2, \ldots, S_{2n})$ in E_2, (with $S_{2n} > 0$) there is a first k such that $S_k = 1$. This path reflected about the line $S_j = 1$ from $k+1$ on, namely $(S_1, \ldots, S_k, 2-S_{k+1}, \ldots, 2-S_{2n})$, has the same probability and is in E_3. Moreover this mapping is one-to one. Hence, $\mathrm{P}(E_2) = \mathrm{P}(E_3)$.

From this we may compute $\mathrm{P}(E_1) = 1 - \mathrm{P}(E_2) - \mathrm{P}(E_3) = 1 - \mathrm{P}(E_2) - \mathrm{P}(E_2) = 1 - \mathrm{P}(S_{2n} > 0) - \mathrm{P}(S_{2n} < 0) = \mathrm{P}(S_{2n} = 0) = \binom{2n}{n} 2^{-2n}$.

(d) For the double exponential distribution, $M(t) = 1/(1-t^2)$ for $|t| < 1$. The lack of memory property of the exponential distribution implies that S_N has an exponential distribution, independent of N, and $\mathrm{E}e^{tS_N} = 1/(1-t)$ for $|t| < 1$. Therefore, the Wald identity gives $\mathrm{E}e^{tS_N}\mathrm{E}M(t)^{-N} = (1-t)^{-1}\mathrm{E}(1-t^2)^N = 1$, or $\mathrm{E}(1-t^2)^N = 1-t$. Substituting u for $1-t^2$ gives $\mathrm{E}u^N = 1 - \sqrt{1-u}$. The power series expansion of $\sqrt{1-u}$ is $\sqrt{1-u} = 1 - \sum_1^\infty f_n u^n$ where

$$f_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} = \frac{1}{2^{2n-1}} \binom{2n-1}{n} \frac{1}{(2n-1)}.$$

Hence, $\mathrm{E}u^N = \sum_1^\infty f_n u^n$ and $f_n = \mathrm{P}(N = n)$. Then, since $p_n = \mathrm{P}(N > n)$,

$$p_n = 1 - \sum_1^n f_j = \frac{1}{2^{2n-1}} \binom{2n-1}{n}$$

as may be seen by induction.

15.(a) and (b) Let $g(\alpha) = \mathrm{E}\max(X, \alpha)$ when X has distribution $F(x|\theta)$.

$$\begin{aligned} g(\alpha) &= \alpha F(\alpha|\theta) + \int_\alpha^1 x\, dF(x|\theta) = \alpha F(\alpha|\theta) + xF(x|\theta)\Big|_\alpha^1 - \int_\alpha^1 F(x|\theta)\, dx \\ &= \alpha F(\alpha|\theta) + 1 - \alpha F(\alpha|\theta) - \int_\alpha^1 \frac{(1-\theta)^2}{(1-x\theta)^2}\, dx = 1 - \frac{(1-\theta)^2}{\theta(1-x\theta)}\Big|_\alpha^1 \\ &= 1 - \frac{1-\theta}{\theta} + \frac{(1-\theta)^2}{\theta(1-\theta\alpha)} = 1 - \frac{(1-\theta)(1-\alpha)}{(1-\theta\alpha)}. \end{aligned}$$

When $\alpha = 0$ this gives $\mathrm{E}(X) = \theta$.
(c)

$$\begin{aligned} g_1(g_2(\alpha)) &= 1 - \frac{(1-\theta_1)(1-g_2(\alpha))}{(1-\theta_1 g_2(\alpha))} \\ &= 1 - \frac{(1-\theta_1)(1-\theta_2)(1-\alpha)}{(1-\theta_2\alpha) - \theta_1((1-\theta_2\alpha) - (1-\theta_2)(1-\alpha))} \\ &= 1 - \frac{(1-\theta_1)(1-\theta_2)(1-\alpha)}{1 - \theta_1\theta_2(1-2\alpha) - (\theta_1+\theta_2)\alpha)}. \end{aligned}$$

Since this is symmetric in θ_1 and θ_2, we have $g_1(g_2(\alpha)) = g_2(g_1(\alpha))$.
(d) So g_i and g_j commute for all i and j. This implies that $A_n = g_n(g_{n-1}(\cdots(g_1(\alpha))\cdots))$ does not depend on the order of the subscripts. Since A_n represents the optimal payoff, it doesn't matter in what order the X_i are observed.

Solutions to the Exercises of Chapter 3.

1. (a) Let $N = N_1^{(J)}$. To show N is regular, we must show

$$\text{for all} \quad n, \quad \mathrm{E}\{Y_N|\mathcal{F}_n\} > Y_n \quad \text{a.s. on} \quad \{N > n\}. \tag{1}$$

We proceed by backward induction. For $n = J$, this is automatic since $\{N > J\}$ is empty. For $n = J-1$: on $\{N > J-1\}$ we have $\{N_1 > J-1\}$, i.e.

$$Y_{J-1} < \mathrm{E}(Y_J|\mathcal{F}_{J-1}) = \mathrm{E}(Y_N|\mathcal{F}_{J-1}),$$

since $N = J$ on $\{N > J-1\}$.

Now assume the induction hypothesis (1) for $n = k+1$. We show (1) for $n = k$. On $\{N > k\}$, we have $N_1 > k$, so that

$$\begin{aligned} Y_k &< \mathrm{E}(Y_{k+1}|\mathcal{F}_k) \\ &= \mathrm{E}(Y_{k+1}I(N = k+1)|\mathcal{F}_k) + \mathrm{E}(Y_{k+1}I(N > k+1)|\mathcal{F}_k). \end{aligned}$$

The first term on the right is equal to $\mathrm{E}(Y_N I(N = k+1)|\mathcal{F}_k)$. The second term is bounded, using the induction hypothesis, by

$$\begin{aligned} \mathrm{E}(Y_{k+1}I(N > k+1)|\mathcal{F}_k) &\le \mathrm{E}(\mathrm{E}\{Y_N|\mathcal{F}_{k+1}\}I(N > k+1)|\mathcal{F}_k) \\ &= \mathrm{E}(\mathrm{E}\{Y_N I(N > k+1)|\mathcal{F}_{k+1}\}|\mathcal{F}_k) \\ &= \mathrm{E}(Y_N I(N > k+1)|\mathcal{F}_k). \end{aligned}$$

Combining these, we get $Y_k < \mathrm{E}(Y_N|\mathcal{F}_k)$ as was to be shown.

(b) We are to show that on $\{N_1 > n\}$, $Y_n < \mathrm{E}(Y_{N_1}|\mathcal{F}_n)$. First note that part (a) implies that for all $J > n$,

$$Y_n < \mathrm{E}(Y_{n+1}|\mathcal{F}_n) \le \sum_{j=n+1}^{J} \mathrm{E}(Y_{N_1}I(N_1 = j)|\mathcal{F}_n) + \mathrm{E}(Y_J I(N_1 > J)|\mathcal{F}_n) \quad \text{on} \quad \{N_1 > n\}.$$

Now, let $J \to \infty$. The first term on the right converges to $\sum_{j=n+1}^{\infty} \mathrm{E}(Y_{N_1}I(N_1 = j)|\mathcal{F}_n)$ by A1. By Fatou-Lebesgue and A2, the limsup of the second term is bounded above by $\mathrm{E}(Y_\infty I(N_1 = \infty)|\mathcal{F}_n)$. Combining these, we have on $\{N_1 > n\}$,

$$\begin{aligned} Y_n < \mathrm{E}(Y_{n+1}|\mathcal{F}_n) &\le \sum_{j=n+1}^{\infty} \mathrm{E}(Y_{N_1}I(N_1 = j)|\mathcal{F}_n) + \mathrm{E}(Y_\infty I(N_1 = \infty)|\mathcal{F}_n) \\ &= \mathrm{E}(Y_{N_1}|\mathcal{F}_n) \end{aligned}$$

as was to be shown.

(c) In Example 3.1, A2 is satisfied and A1 is not. Moreover, the 1-sla will continue as long as successes occur and stop at the first failure, so $N_1 = \min\{n \geq 0 : Y_n = 0\}$ and $Y_{N_1} = 0$. Thus, $Y_n \geq \mathrm{E}(Y_{N_1}|\mathcal{F}_n) = 0$ on $\{N_1 > n\}$, so N_1 is not regular. In Example 3.2, A1 is satisfied and A2 is not. Here, the 1-sla is simply $N_1 = \infty$, and again $Y_n \geq \mathrm{E}(Y_{N_1}|\mathcal{F}_n) = 0$. So N_1 is not regular.

2. We are to show that $Y_n < \mathrm{E}(Y_{N_\infty}|\mathcal{F}_n)$ on $\{N_\infty > n\}$. First note that on $\{N_\infty > n\}$,

$$\begin{aligned} Y_n &< \mathrm{E}(Y_\infty|\mathcal{F}_n) \\ &= \mathrm{E}(Y_\infty I(N_\infty = \infty)|\mathcal{F}_n) + \sum_{j=n+1}^{\infty} \mathrm{E}(Y_\infty I(N_\infty = j)|\mathcal{F}_n). \end{aligned}$$

The first term on the right is equal to $\mathrm{E}(Y_{N_\infty} I(N_\infty = \infty)|\mathcal{F}_n)$. Since for every j, $Y_j I(N_\infty = j) \geq \mathrm{E}(Y_\infty|\mathcal{F}_j) I(N_\infty = j)$, if we take expectation conditional on $\mathcal{F}_n$, we have for $j > n$,

$$\mathrm{E}(Y_j I(N_\infty = j)|\mathcal{F}_n) \geq \mathrm{E}(\mathrm{E}\{Y_\infty I(N_\infty = j)|\mathcal{F}_j\}|\mathcal{F}_n)$$

or

$$\mathrm{E}(Y_{N_\infty} I(N_\infty = j)|\mathcal{F}_n) \geq \mathrm{E}(Y_\infty I(N_\infty = j)|\mathcal{F}_n).$$

Combining these, we find on $\{N_\infty > n\}$,

$$\begin{aligned} Y_n &< \mathrm{E}(Y_{N_\infty} I(N_\infty = \infty)|\mathcal{F}_n) + \sum_{j=n+1}^{\infty} \mathrm{E}(Y_{N_\infty} I(N_\infty = j)|\mathcal{F}_n) \\ &= \mathrm{E}(Y_{N_\infty} I(N_\infty > n)|\mathcal{F}_n) \qquad \text{since A1 implies } \mathrm{E}Y_{N_\infty} < \infty \\ &= \mathrm{E}(Y_{N_\infty}|\mathcal{F}_n). \end{aligned}$$

3. (a) We follow the general ideas of Exercise 1(a,b). First we show that a related truncated rule, $T(J)$, that does not allow stopping at any finite $n > J$ is regular. $T(J)$ is defined by

$$T(J) = \min\{n \geq 0 : Y_n \geq \sup\nolimits_{n<j\leq J} \mathrm{E}(Y_j|\mathcal{F}_n) \text{ and } Y_n \geq \mathrm{E}(Y_\infty|\mathcal{F}_n)\}.$$

That is, we show that for all $n \leq J$,

$$Y_n < \mathrm{E}(Y_{T(J)}|\mathcal{F}_n) \quad \text{on} \quad \{T(J) > n\}. \tag{2}$$

For $n = J$, this is easy since on $\{T(J) > J\}$, we have $Y_J < \mathrm{E}(Y_\infty|\mathcal{F}_J) = \mathrm{E}(Y_{T(J)}|\mathcal{F}_J)$. Assuming, as the induction hypothesis, that (2) is true for $n = k+1$, we must show that (2) holds for $n = k$. On $\{T(J) > k\}$, either there exists a t with $k < t \leq J$ such that $Y_k < \mathrm{E}(Y_t|\mathcal{F}_k)$ or $Y_k < \mathrm{E}(Y_\infty|\mathcal{F}_k)$, so it suffices to show that both (i) $\mathrm{E}(Y_\infty|\mathcal{F}_k) \leq \mathrm{E}(Y_{T(J)}|\mathcal{F}_k)$ on $\{T(J) > k\}$, and (ii) for all t with $k < t \leq J$, we have $\mathrm{E}(Y_t|\mathcal{F}_k) \leq \mathrm{E}(Y_{T(J)}|\mathcal{F}_k)$, on $\{T(J) > k\}$. Details are similar to Exercise 1(a).

To show that T_1 is regular, note that on $\{T_1 > n\}$, either $Y_n < \mathrm{E}(Y_\infty|\mathcal{F}_n)$ or $Y_n < \mathrm{E}(Y_t|\mathcal{F}_n)$ for some $n < t < \infty$. Therefore it suffices to show that both $\mathrm{E}(Y_\infty|\mathcal{F}_n) \le \mathrm{E}(Y_{T_1}|\mathcal{F}_n)$ and for $k < t < \infty$, $\mathrm{E}(Y_t|\mathcal{F}_n) \le \mathrm{E}(Y_{N_{1\infty}}|\mathcal{F}_n)$ on $\{T_1 > n\}$. But from the above paragraph, we know for $J > n$, that on $\{T_1 > n\}$, both $\mathrm{E}(Y_\infty|\mathcal{F}_n) \le \mathrm{E}(Y_{T(J)}|\mathcal{F}_n)$ and $\mathrm{E}(Y_{n+1}|\mathcal{F}_n) \le \mathrm{E}(Y_{T(J)}|\mathcal{F}_n)$. Therefore it is sufficient to show that on $\{T_1 > n\}$,

$$\limsup_{J\to\infty} \mathrm{E}(Y_{T(J)}|\mathcal{F}_n) \le \mathrm{E}(Y_{T_1}|\mathcal{F}_n).$$

Details of this are similar to Exercise 1(b).

(b) Since both N_1 and T_1 are regular, we have from Lemma 2 that $\mathrm{E}Y_{\max\{N_1,T_1\}} \ge \max\{\mathrm{E}Y_{N_1}, \mathrm{E}Y_{T_1}\}$. But if N_1 says continue, then so does T_1. Hence, $\max\{N_1, T_1\} = T_1$ and $\mathrm{E}Y_{T_1} \ge \mathrm{E}Y_{N_1}$. Similarly, $\mathrm{E}Y_{T_1} \ge \mathrm{E}Y_{N_\infty}$.

4. (a) Suppose for independent $X_1, X_2, \ldots$ that $\mathrm{P}(X_j = 2^j) = \mathrm{P}(X_j = -2^j) = 1/2$ for all j, and consider the stopping rule, $N_r = \min\{j \ge r : X_j > 0\}$. Then N_r has a geometric distribution on $\{r, r+1, \ldots\}$ and so has finite expectation. Moreover, $Y_{N_1} \equiv 1$ so $\mathrm{E}(Y_{N_1} = 1$. Similarly, $\mathrm{E}S_{N_r} = 2^{r-1}$. This shows that $\sup_N \mathrm{E}S_N = \infty$.

(b) The equations (16) still hold provided the interchange of summations is valid. However, if X_j is replaced by $|X_j|$ in (16), we see that we need $\sum_{j=1}^\infty \mathrm{P}\{N \ge n\}\mathrm{E}(|X_j|) < \infty$. For this to be true for all stopping rules N with $\mathrm{E}N < \infty$, we need that the $\mathrm{E}(|X_j|)$ are bounded, i.e. $\sup_j \mathrm{E}(|X_j|) < \infty$. Under this condition, (33) holds.

5. We need a mass at m, the median of $\max\{X_1, X_2\}$, so let's try $X_1 = 1$ and make $m = 1$. This only requires $\mathrm{P}(X_2 \le 1) > 1/2$. Then $m = 1$, $p = \mathrm{P}(M_2 > 1) = \mathrm{P}(X_2 > 1) < 1/2$, $q = \mathrm{P}(M_2 < 1) = 0$, and $\beta = \mathrm{E}(X_2 - 1)^+$. Then $\mathrm{E}(M_2) = 1 + \beta$, $\mathrm{E}(X_{s(1)}) = \mathrm{E}(X_2)$, and $\mathrm{E}(X_{t(1)}) = 1$. We can make β as large as we like. So $\mathrm{E}(M_2) = (1+\beta)\mathrm{E}(X_{t(1)}) > 2\mathrm{E}(X_{t(1)})$ when $\beta > 1$. Similarly, we can make $\mathrm{E}(X_2)$ as small as we like. So $\mathrm{E}(M_2) > 1 > 2\mathrm{E}(X_2)$ when $\mathrm{E}(X_2) < 1/2$.

6. By the Borel-Cantelli Lemma, $\sum_{i=1}^\infty \mathrm{P}(X_i \ge 0) = \infty$ implies that $\mathrm{P}(X_i \ge 0 \text{ i.o.}) = 1$. So $M^* = \mathrm{E}(\sup_n X_n)$ exists and is equal to $\mathrm{E}(\sup_n |X_n|)$. Let $N < \infty$ be an arbitrary stopping rule for the sequence X_n^+, let $N' = N$ if $X_N^+ > 0$, and let $N' = \min\{j > N : X_j \ge 0\}$ otherwise. Then $N' < \infty$ is a stopping rule for the sequence X_i and $\mathrm{E}(X_{N'}) \ge \mathrm{E}(X_N^+)$. Thus $\sup_{N'<\infty} \mathrm{E}(X_{N'}) \ge \sup_{N<\infty} \mathrm{E}(X_N^+) \ge (1/2)M^*$.

7. Take $X_1 = v$, $\mathrm{P}(X_2 = 1) = v$, $\mathrm{P}(X_2 = 0) = 1 - v$ and $X_3 = X_4 = \cdots = 0$. Then the best the decision maker can do is stop at the first observation and get $V^* = v$. But since $\mathrm{P}(\max\{X_1, X_2\} = v) = 1 - v$ and $\mathrm{P}(\max\{X_1, X_2\} = 1) = v$, we have $M^* = v(1-v) + v = 2v - v^2$.

Solutions to the Exercises of Chapter 4.

1. From Theorem 1, since the second moment of the distribution of X is finite, the optimal rule exists and is the same for sampling with and without recall. It has the form $N^* = \min\{n \ge 1 : X_n \ge V^*\}$, where V^* satisfies the equation

$$\begin{aligned} c &= \int_{V^*}^{\infty} (x - V^*)\, dF(x) \\ &= \begin{cases} \exp\{-\sigma V^*\}/\sigma & \text{for } V^* > 0 \\ 1/\sigma - V^* & \text{for } V^* \le 0. \end{cases} \end{aligned}$$

So, $V^* = -\log(c\sigma)/\sigma$ for $c < 1/\sigma$, and $V^* = 1/\sigma - c$ for $c \ge 1/\sigma$.

2.(a) For A1, note that $\sup_n Y_n \le \sum_1^\infty \beta^n |X_n|$, so $\mathrm{E}(\sup_n Y_n) \le \sum_1^\infty \beta^n \mathrm{E}(|X_n|) = \mathrm{E}(|X|)\beta/(1-\beta) < \infty$. For A2, note $\limsup_n Y_n \le \lim_n \beta^n \sum_1^n |X_j| = \lim_n n\beta^n [\sum_1^n |X_j|/n] = 0$, since $\sum_1^n |X_j|/n \to \mathrm{E}(|X|)$ from the law of large numbers, and $n\beta^n \to 0$.

(b) Since the X_j are independent, $v_n = \mathrm{E}(V_{n+1}^* | \mathcal{F}_n)$ are constants for $n = 0, 1, \ldots$. By time invariance, $v_{n+1} = \beta v_n$. Moreover, using the optimality equation,

$$v_0 = \mathrm{E}(V_1^*) = \mathrm{E}(\max(\beta X_1, v_1)) = \beta \mathrm{E}(\max(X_1, v_0).$$

Therefore the optimal rule is given by the principle of optimality, $N^* = \min\{n \ge 0 : X_n \ge v_0\}$, where v_0 satisfies the above equation.

(c) If $X \in \mathcal{U}(0,1)$, then $0 < v_0 < 1$, and so v_0 satisfies the equation

$$v = \beta \int_0^v v\, dx + \beta \int_v^1 x\, dx = \beta(1 + v^2)/2.$$

This gives $v_0 = (1 \pm \sqrt{1 - \beta^2})/\beta$, and since $v_0 < 1$, the minus sign must be used. The optimal rule is therefore $N = \min\{n \ge 1 : X_n \ge (1 - \sqrt{1 - \beta^2})/\beta\}$.

If $X \in \mathcal{U}(-1,1)$, again $0 < v_0 < 1$, and v_0 satisfies

$$v = \beta \int_{-1}^v v/2\, dx + \beta \int_v^1 x/2\, dx = \beta(1 + 2v + v^2)/4.$$

The root between 0 and 1 is $v_0 = (2 - \beta - 2\sqrt{1 - \beta})/\beta$.

3. $Y_n \le \beta^n \max(X_1, \ldots, X_n) \le \max(\beta X_1, \ldots, \beta^n X_n) \le \sum_1^\infty \beta^j |X_j|$. Thus, A1 follows as in problem 2, and A2 follows similarly. As in the case in the text of selling an option with a cost, the principle of optimality will never lead to using the recall option. The best rule will be identical to the no recall rule of problem 2, with the same results for $\mathcal{U}(0,1)$ and $\mathcal{U}(-1,1)$.

4.(a) To show A1, note that $\sup_n Y_n = \max\{Z_1,\ldots,Z_{K-1},\mu\} = \mu + \max\{Z_1-\mu,\ldots,Z_{K-1}-\mu,0\} \le \mu + \sum_1^{K-1}(Z_j-\mu)^+$; so using Wald's equation, $\mathrm{E}\sup_n Y_n \le \mu + \mathrm{E}\{\sum_1^{K-1}(Z_j-\mu)^+\} = \mu + \mathrm{E}(Z-\mu)^+\mathrm{E}(K-1) < \infty$. A2 is clear since $Y_n \to \mu$ a.s.

(b) Since the Z's are independent, we have on $\{K>n\}$, $\mathrm{E}(V^*_{n+1}|\mathcal{F}_n) = \mathrm{E}(V^*_{n+1}|K>n)$ is a constant independent of $Z_1,\ldots,Z_n$. The time invariance implies that this constant is independent of n, and equal to V^*, say. From the optimality equation,

$$\begin{aligned} V^* &= \mathrm{E}(V^*_{n+1}|K>n) = (1-\beta)\mu + \beta\mathrm{E}(\max(Z_{n+1},V^*)) \\ &= (1-\beta)\mu + \beta\mathrm{E}((Z-V^*)^+) + \beta V^*. \end{aligned}$$

This equation may be rewritten in the form

$$V^* = \mu + (\beta/(1-\beta))\mathrm{E}((Z-V^*)^+),$$

which has a unique solution since the left side is increasing in V^* and the right side is nonincreasing in V^*.

(c) If $Z \in \mathcal{U}(0,1)$, then $\mathrm{E}((Z-V^*)^+) = \int_{V^*}^1 (z-V^*)\,dz = (1-V^*)^2/2$, so the above equation becomes, with $\mu=0$,

$$V^* = \frac{\beta}{1-\beta}\frac{(1-V^*)^2}{2} \qquad \text{or} \qquad V^{*2} - \frac{2}{\beta}V^* + 1 = 0.$$

This has a root in the interval (0,1) at the point

$$V^* = \frac{1}{\beta}(1-\sqrt{1-\beta^2}).$$

If $Z \in \mathcal{U}(-1,1)$, similar calculations give $V* = (2-\beta-2\sqrt{1-\beta})/\beta$.

5. (a) Since the distribution of X is elementary, the optimal rule is $N^* = \min\{n \ge 0 : S_n \ge s_0\}$, where $s_0 = \phi(\beta)/(1-\phi(\beta))$, and $\phi(\beta)$ satisfies the equation $G(\phi(\beta)) = 1/\beta$, G being the generating function of X, $G(\theta) = \mathrm{E}(\theta^{-X}) = p\theta^{-1} + (1-p)\theta$. This equation becomes

$$(1-p)\phi(\beta)^2 - \phi(\beta)/\beta + p = 0.$$

The root in $(0,1)$ is $\phi(\beta) = (1-\sqrt{1-4\beta^2(1-p)p})/(2\beta(1-p))$. The optimal rule stops as soon as S_n is at least $\phi(\beta)/(1-\phi(\beta))$.

(b) If $p \ge 1/2$, the probability that S_n eventually hits 1 is 1. This implies that $\mathrm{P}(N^* < \infty) = 1$. Suppose $p < 1/2$. Let $T = \min\{n \ge 0 : S_n = 1\}$. Then $\mathrm{P}(T<\infty) = p\mathrm{P}(T<\infty|X_1=1) + (1-p)\mathrm{P}(T<\infty|X_1=-1) = p + (1-p)\mathrm{P}(T<\infty)^2$. Solving for $\mathrm{P}(T<\infty)$ in $(0,1)$, we find that $\mathrm{P}(T<\infty) = p/(1-p)$. Let $k = \lceil s_0 \rceil$, so that $N^* = \min\{n \ge 0 : S_n = k\}$. Note that N^* is finite if and only if the event $\{T<\infty\}$ occurs k times. Hence, $\mathrm{P}(N<\infty) = \mathrm{P}(T<\infty)^k = (p/(1-p))^{\lceil s_0 \rceil}$.

(c) Note that $s_0 \le 1$ if and only if $\phi \le 1/2$. So the question becomes when is $\phi \le 1/2$ for all $\beta < 1$. Since ϕ is increasing in β, we take the limit, $\lim_{\beta\to 1}\phi(\beta) = (1-\sqrt{1-4p(1-p)})/2(1-p) = p/(1-p)$ for $p < 1/2$. For $p \le 1/3$, this is less than or equal to 1, so the optimal rule stops at the first time that $S_n = 1$ no matter what β is. When $1/3 < p < 1/2$, we have $\phi > 1/2$ for β sufficiently close to 1, so the optimal rule may continue past $S_n = 1$.

6.(a) Let E_k be the event that S_n eventually hits k, for $k = 1, 2, \ldots$. Then, $E_1 \supset E_2 \supset \cdots$, and $\mathrm{P}(M \ge k) = \mathrm{P}(E_k)$. Moreover, $\mathrm{P}(E_1) = \mathrm{P}(T < \infty)$, $\mathrm{P}(E_2) = \mathrm{P}(E_1)\mathrm{P}(E_2|E_1) = \mathrm{P}(T < \infty)^2$, etc. Hence, $\mathrm{P}(M \ge k) = \mathrm{P}(E_k) = \mathrm{P}(T < \infty)^k$, the geometric distribution.

(b) Let $\phi(\beta) = \mathrm{E}(\beta^T) = \sum_{n=1}^{\infty}\beta^n\mathrm{P}(T = n)$, and note that as $\beta \to 1^-$, $\phi(\beta) \to \mathrm{P}(T < \infty)$. If the distribution of X is elementary, we have $G(\phi(\beta)) = 1/\beta$, as in §4.2. Letting $\beta \to 1^-$, we find $\mathrm{P}(T < \infty)$ is the root in $(0, 1)$ of the equation, $G(\mathrm{P}(T < \infty)) = 1$. After solving this for $\mathrm{P}(T < \infty)$, we find $\mathrm{E}M$ as the mean of the geometric, $\mathrm{E}M = \mathrm{P}(T < \infty)/(1 - \mathrm{P}(T < \infty))$.

(c) If $G(\theta) = p/\theta + (1-p)\theta$, we solve $G(\theta) = 1$ for the root in $(0, 1)$, and find $\mathrm{P}(T < \infty) = p/(1-p)$.

7.(a) To show A1, write

$$\begin{aligned} M_n - nc &= \max(S_1, \ldots, S_n) - nc \\ &= \max(S_1 - nc, \ldots, S_n - nc) \\ &\le \max(S_1 - c, \ldots, S_n - nc) \\ &= \max(S_1', \ldots, S_n') \end{aligned}$$

where $\mathrm{E}S_1' < 0$ and $\mathrm{E}(S_1'^2) < \infty$. Hence by Theorem 2, $\mathrm{E}\sup(M_n - nc) \le \mathrm{E}\sup S_n' < \infty$, and A1 is satisfied. Since $Y_n \to -\infty$ a.s. by the strong law of large numbers, A2 is satisfied.

(b) The time invariance of the problem implies that V_n has the form, $V_n = M_n - nc + f(M_n - S_n)$ for some decreasing function f. The optimal rule is therefore $N^* = \min\{n \ge 0 : Y_n \ge V_n\} = \min\{n \ge 0 : f(M_n - S_n) \le 0\} = \min\{n \ge 0 : M_n - S_n \ge \gamma\}$ for some γ.

(c) We may take γ to be an integer. Let $N_\gamma = \min\{n \ge 0 : M_n - S_n = \gamma\}$, and let $V(k)$ denote the return using N_γ starting at $k - \gamma$, $V(k) = \mathrm{E}\{Y_{N_\gamma}|M_0 = 0, S_0 = k - \gamma\}$. Then,

$$\begin{aligned} &V(0) = 0 \\ &V(k) = (1/2)V(k-1) + (1/2)V(k+1) - c \qquad \text{for} \quad 0 < k < \gamma \\ &V(\gamma) = (1/2)V(\gamma - 1) + (1/2)(V(\gamma) + 1) - c. \end{aligned}$$

Let $x = V(1)$ and solve the central equations successively for $V(2), \ldots, V(\gamma)$ to find $V(k) = kx + k(k-1)c$. Now use the last equation to solve for x to find $x = 1 - 2c\gamma$, and thus $V(\gamma) = \gamma - \gamma(\gamma + 1)c$. Now choose γ to maximize $V(\gamma)$; the last γ such that $V(\gamma) - V(\gamma - 1) = 1 - 2\gamma c \ge 0$ is the optimal one. This gives the optimal γ as the largest integer less than or equal to $1/2c$.

8.(a) Y_n is bounded above, so A1 holds. $Y_n \to -\infty$, so A2 holds.

(b) If $S_n = s$ and it is optimal to continue, it is also optimal to continue at $S_n = s'$ where $s < s' < a$. It is clearly optimal to stop when $S_n \geq a$. Hence, the rule $N_\gamma = \min\{n \geq 0 : S_n < \gamma \quad \text{or} \quad S_n \geq a\}$ is optimal for some γ.

(c) We may take both γ and a to be integers and we may take the optimal rule to be $N_\gamma = \min\{n \geq 0 : S_n \leq \gamma \quad \text{or} \quad S_n \geq a\}$. For $\gamma \leq k \leq a$, let $V(k)$ be the expected return using N_γ starting at k. Then,

$$\begin{aligned}
&V(\gamma) = 0 \\
&V(k) = (1/2)V(k-1) + (1/2)V(k+1) - c \quad \text{for} \quad \gamma < k < a \\
&V(a) = 1.
\end{aligned}$$

This is the classical gambler's ruin problem, with P(hit $a|$ start at k) $= (k-\gamma)/(a-\gamma)$ and expected duration $= \mathrm{E}(N_\gamma) = (k-\gamma)(a-k)$. So, $V(k) = (k-\gamma)/(a-\gamma) - (k-\gamma)(a-k)c = 1 - ck(a-k) + (a-k)[c\gamma - 1/(a-\gamma)]$. The optimal γ is the one that maximizes $c\gamma - 1/(a-\gamma)$, namely $a - \{\sqrt{1/4 + 1/c}\}$, where $\{x\}$ denotes the integer closest to x.

9.(a) $Y_n = \max\{S_n - nc, -S_n - nc\}$, so $\sup Y_n = \max\{\sup(S_n - nc), \sup(-S_n - nc)\} \leq \sup(S_n - nc) + \sup(-S_n - nc)$. Thus, by the theorem of Kiefer and Wolfowitz, $\mathrm{E}(\sup Y_n) < \infty$ and A1 is satisfied. Moreover, $\limsup Y_n = \limsup n(|S_n/n| - c) = -\infty$, since $S_n/n \to 0$.

(b) Let $V_0^*(x)$ denote the optimal return starting with $S_0 = x$. Then

$$\begin{aligned}
V_n^* &= \operatorname{ess\,sup}_{N \geq n} \mathrm{E}(|S_N| - Nc|\mathcal{F}_n) \\
&= \operatorname{ess\,sup}_{N \geq n} \mathrm{E}(|S_n + S'_{N-n}| - Nc|\mathcal{F}_n) = V_0^*(S_n) - nc,
\end{aligned}$$

where S'_{N-n} denotes the sum of the X_j from $n+1$ to N. The optimal rule is $N^* = \min\{n \geq 0 : Y_n \geq V_n^*\} = \min\{n \geq 0 : |S_n| \geq V_0^*(S_n)\} = \min\{n \geq 0 : V_0^*(S_n) - |S_n| \leq 0\}$. Note that $V_0^*(x) - |x|$ is symmetric about 0 and is nonincreasing on $(0, \infty)$ since for $x \geq 0$, $V_0^*(x) - |x| = \sup_{N \geq 0} \mathrm{E}(|x + S_N| - x - Nc)$ and $|x + S_N| - x$ is nonincreasing in x. Hence the optimal rule is of the form $N_\gamma = \min\{n \geq 0 : |S_n| \geq \gamma\}$ for some $\gamma \geq 0$.

(c) $\mathrm{E}(Y_{N_\gamma}) = \mathrm{E}(|S_{N_\gamma}|) - c\mathrm{E}(N_\gamma)$. We may take γ to be an integer. When stopping occurs, S_{N_γ} is $\pm\gamma$, so $\mathrm{E}(|S_{N_\gamma}|) = \gamma$. Moreover, this is the gambler's ruin problem symmetric about 0, so $\mathrm{E}(N_\gamma) = \gamma^2$. Thus, $\mathrm{E}(Y_{N_\gamma}) = \gamma - c\gamma^2$. This is maximized by choosing γ to be the integer closest to $1/(2c)$.

10.(a) Let $M = \sup_{n \geq T}(S_n - S_T)$. Then $\mathrm{E}M < \infty$ from Theorem 2. Then

$$\begin{aligned}
\mathrm{E}\{\sup_n S_n | T = t\} &= \mathrm{E}\{\max(S_0, S_1, \ldots, S_{t-1}, S_{t-1} + M) | T = t\} \\
&\leq \mathrm{E}\{\max(S_0, S_1, \ldots, S_{t-1}) | T = t\} + \mathrm{E}M \\
&\leq t\mathrm{E}X^+ + \mathrm{E}M.
\end{aligned}$$

Then A1 follows from $\mathrm{E}(\sup_n Y_n) \le \mathrm{E}(\sup_n S_n) = \mathrm{E}(\mathrm{E}(\sup_n S_n|T)) \le \mathrm{E}T\, E_0 X^+ + \mathrm{E}M < \infty$, where E_0 represents the expectation under $f_0(x)$. To show A2, it is sufficient to show that $S_n \to -\infty$ a.s. This follows since $\mathrm{P}(S_n \to -\infty|T = t) = 1$ for all t so that $\mathrm{P}(S_n \to -\infty) = \mathrm{E}\{\mathrm{P}(S_n \to -\infty|T)\} = 1$.

(b)By direct calculation,

$$\mathrm{P}((T \le n|X_1, \dots, X_n) = \frac{\sum_{t=1}^n \pi^{t-1} \prod_t^n \lambda(X_i)}{\sum_{t=1}^n \pi^{t-1} \prod_t^n \lambda(X_i) + \sum_{t=n+1}^\infty \pi^{t-1}},$$

from which we may compute

$$Q_{n+1} = \mathrm{P}(T \le n+1|X_1, \dots, X_{n+1}) = \frac{[Q_n + (1-Q_n)(1-\pi)]\lambda(X_{n+1})}{[Q_n + (1-Q_n)(1-\pi)]\lambda(X_{n+1}) + (1-Q_n)\pi}.$$

(c) Let $V^*(q)$ denote the optimal return for the problem modified by allowing an initial $Q_0 = \mathrm{P}(T = 0) = q$, so that $\mathrm{P}(T = t) = (1-q)(1-\pi)\pi^{t-1}$ for $T \ge 1$. The rule given by the principle of optimality is $N^* = \min\{n \ge 0 : Y_n \ge V_n^*\}$, where $V_n^* = \operatorname{ess\,sup}_{N \ge n} \mathrm{E}(Y_N|\mathcal{F}_n) = Y_n + V^*(Q_n)$. Hence, $N^* = \min\{n \ge 0 : V^*(Q_n) \le 0\}$. To show that $N^* = \min\{n \ge 0 : Q_n \ge \gamma\}$ for some γ, it is sufficient to show that $V^*(q)$ is nonincreasing in q. Since $V^*(q) = \operatorname{ess\,sup}_N \mathrm{E}(S_N - aQ_N - c)$, it is sufficient to show that $\mathrm{E}S_N$ is nonincreasing in q and $\mathrm{E}Q_N$ is nondecreasing in q for any stopping rule N. The latter is clear since each Q_n is increasing in q by induction using the answer to (b). To see that $\mathrm{E}S_N$ is nonincreasing, we make the computation

$$\begin{aligned}\mathrm{E}S_N &= \sum_{t=0}^\infty \mathrm{P}(T = t)\mathrm{E}(S_N|T = t) \\ &= q\mathrm{E}(S_N|T = 0) + (1-q)\sum_{t=1}^\infty (1-\pi)\pi^{t-1}\mathrm{E}(S_N|T = t).\end{aligned}$$

The coefficient of q in this expectation is

$$\sum_{t=1}^\infty (1-\pi)\pi^{t-1}[\mathrm{E}(S_N|T = 0) - \mathrm{E}(S_N|T = t)].$$

From the generalization of Wald's equation found in Exercise 4 of Chapter 3, we find that $\mathrm{E}(S_N|T = 0) = \mu_1\mathrm{E}(N)$ and, for $t \ge 1$, $\mathrm{E}(S_N|T = t) = \mathrm{E}\sum_{i=1}^N \xi_i$ where ξ_i is μ_0 for $i < t$ and μ_1 for $i \ge t$. Therefore, $\mathrm{E}(S_N|T = 0) - \mathrm{E}(S_N|T = t) = \sum_{i=1}^N (\mu_1 - \xi_i) \le 0$, completing the proof.

11. (a) The Optimality Equation is $V_{xy} = \mathrm{E}(\max\{W_n, V_{xy}\} - c$. This becomes

$$V_{xy} = \mathrm{E}(\max\{X_n + Y_n, X_n + V_y, Y_n + V_x, V_{xy}\}) - c.$$

The optimal strategy given by the Principle of Optimality is to continue sampling until $W_n \geq V_{xy}$, and then to stop and sell both objects if $X_n + Y_n = W_n$, the x-object alone if $X_n + V_y = W_n$, and the y-object alone if $Y_n + V_x = W_n$.

(b) The Optimality Equation becomes

$$V_{xy} = \mathrm{E}(\max\{X_n + Y_n, X_n + .5, Y_n + .5, V_{xy}\}) - .125$$

Here $1 < V_{xy} < 1.5$. In the unit square, the max is V_{xy} on the set $\{y < V_{xy} - .5, x < V_{xy} - .5, x + y < V_{xy}\}$, the max is $x + .5$ on the set $\{x > V_{xy}, y < .5\}$, the max is $y + .5$ on the set $\{y > V_{xy} - .5, x < .5\}$, and the max is $x + y$ on the set $\{x > .5, y > .5, x + y > V_{xy}\}$.

The expectation on the first set is $V_{xy}[(V_{xy} - .5)^2 - .5(V_{xy} - 1)^2] = V_{xy}(V_{xy}^2 - .5)/2$.

The expectation on the second set is $.5\int_{V_{xy}-.5}^{1}\int_0^{.5}(x + .5)\,dy\,dx = .25(2.25 - V_{xy}^2)$, and the same on the third set.

The expectation on the fourth set is the integral of $x + y$ over the set $\{x > .5, y > .5\}$, minus the integral of $x + y$ over the triangle $\{x > .5, y > .5, x + y < V_{xy} - .5\}$, i.e.

$$\int_{.5}^{1}\int_{.5}^{1}(x + y)\,dy\,dx = 3/8$$

minus

$$\int_{.5}^{V_{xy}-.5}\int_{.5}^{V_{xy}-x}(x + y)\,dy\,dx = (V_{xy} - 1)^2(V_{xy} + .5)/3$$

The Optimality Equation becomes

$$\begin{aligned} V_{xy} + .125 &= \frac{1}{2}V_{xy}(V_{xy}^2 - .5) + \frac{1}{2}(2.25 - V_{xy}^2) + \frac{3}{8} - \frac{1}{3}(V_{xy} - 1)^2(V_{xy} + .5) \\ &= \frac{1}{6}V_{xy}^3 - \frac{1}{4}V_{xy} + \frac{4}{3} \end{aligned}$$

This reduces to $4V_{xy}^3 - 30V_{xy} + 29 = 0$ whose solution between 1 and 1.5 is $V_{xy} = 1.19313$.

(c) The Optimality Equation becomes $V_{xy} = \mathrm{E}(\max\{X_n + Y_n, V_{xy}\}) - .5$. Here $0 < V_{xy} < 1$. The integral of V_{xy} over the region $\{0 < x, 0 < y, x + y < V_{xy})$ is $V_{xy}^3/2$. The integral of $x + y$ over the unit square minus the region $\{0 < x, 0 < y, x + y < V_{xy})$ is $1 - (1/3)V_{xy}^3$. The Optimality Equation is therefore $V_{xy} = V_{xy}^3/2 + 1 - V_{xy}^3/3 - .5$. This gives the cubic equation, $V_{xy}^3 - 6V_{xy} + 3 = 0$, whose solution is $V_{xy} = .52398$. If we sell the objects separately, we accept the first offers that come in. However if we sell them together, we accept the first vector of offers such that $X_n + Y_n > .52398$.

Solutions to the Exercises of Chapter 5.

1. (a) The joint density of X_1 and θ is

$$\begin{aligned} f(x,\theta) &= g(\theta)f(x|\theta) \\ &= \frac{1}{\theta}e^{-x/\theta}\frac{\lambda^\alpha}{\Gamma(\alpha)}e^{-\lambda/\theta}\frac{1}{\theta^{\alpha+1}} \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)}e^{-(\lambda+x)/\theta}\frac{1}{\theta^{\alpha+2}} \end{aligned}$$

The marginal density of X_1 is

$$f(x) = \int_0^\infty f(x,\theta)\,d\theta = \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}} \quad \text{for} \quad x>0,$$

the inverse power, or Pareto, distribution. The posterior density of θ given X_1 is

$$g(\theta|x) = f(x,\theta)/f(x) = \frac{(\lambda+x)^{\alpha+1}}{\Gamma(\alpha+1)}e^{-(\lambda+x)/\theta}\frac{1}{\theta^{\alpha+2}},$$

the inverse gamma distribution, $\mathcal{G}^{-1}(\alpha+1,\lambda+x)$. Thus, as each observation arrives, the parameters are updated by adding one to the alpha parameter and adding the x-value to the lambda parameter, giving $\mathcal{G}^{-1}(\alpha+n,\lambda+S_n)$ as the posterior distribution of θ given $X_1,\ldots,X_n$.

(b) If $\theta \in \mathcal{G}^{-1}(\alpha,\lambda)$, then

$$\mathrm{E}\theta = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int e^{-\lambda/\theta}\theta^{-\alpha}\,d\theta = \frac{\lambda^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\lambda^{\alpha-1}} = \frac{\lambda}{\alpha-1}.$$

Similarly, $\mathrm{E}\theta^2 = \lambda^2/((\alpha-1)(\alpha-2))$, and so $\mathrm{Var}\,\theta = \lambda^2/((\alpha-1)^2(\alpha-2))$.

(c)If you stop at stage 0, you lose $Y_0 = \mathrm{Var}\,(\theta)$. If you take one observation and stop, you lose

$$c + \mathrm{E}\frac{(\lambda+X_1)^2}{\alpha^2(\alpha-1)} = c + \frac{\lambda^2 + 2\lambda\mathrm{E}X + \mathrm{E}X^2}{\alpha^2(\alpha-1)}.$$

Now, $\mathrm{E}X = \mathrm{E}\{\mathrm{E}(X|\theta)\} = \mathrm{E}\theta = \lambda/(\alpha-1)$, and $\mathrm{E}X^2 = \mathrm{E}\{\mathrm{E}(X^2|\theta)\} = \mathrm{E}\{2\theta^2\} = 2\lambda^2/((\alpha-1)(\alpha-2))$, so the above quantity reduces to

$$c + \frac{\lambda^2}{\alpha(\alpha-1)(\alpha-2)}.$$

Therefore, the 1-sla calls for stopping at stage 0 if Y_0 is no greater than this, or equivalently, if

$$\frac{\lambda^2}{\alpha(\alpha-1)^2(\alpha-2)} \le c.$$

(d) At stage 1, the conditional expected loss given X_1 of the optimal rule truncated at stage 2 is

$$c+\min\{\frac{(\lambda+X_1)^2}{\alpha^2(\alpha-1)}, c+\frac{(\lambda+X_1)^2}{(\alpha+1)\alpha(\alpha-1)}\}.$$

The 2-sla calls for stopping at stage 0 if Y_0 is no greater than the expectation of this. As in the problem in the text, we can simplify this by subtracting the expectation of the first term of the minimum from both sides of the inequality to find that the 2-sla calls for stopping at stage 0 if

$$(*)\qquad \frac{\lambda^2}{\alpha(\alpha-1)^2(\alpha-2)} \le c-\mathrm{E}(\frac{(\lambda+X_1)^2}{(\alpha+1)\alpha^2(\alpha-1)}-c)^+.$$

We now evaluate this expectation using the marginal distribution of X_1 as found in (a). Case 1, $\lambda^2 \ge c(\alpha+1)\alpha^2(\alpha-1)$: Then the quantity inside the expectation sign is positive whatever be $X_1>0$, and since from (c), $\mathrm{E}(\lambda+X_1)^2=\lambda^2\alpha/(\alpha-2)$, (*) reduces to

$$\frac{\lambda^2}{\alpha(\alpha-1)^2(\alpha-2)} \le 2c-\frac{\lambda^2}{(\alpha+1)\alpha(\alpha-1)(\alpha-2)}$$

In case 1, this is never satisfied, so we always take two observations whatever be X_1. Case 2, $\lambda^2 < c(\alpha+1)\alpha^2(\alpha-1)$: Let $A=c(\alpha+1)\alpha^2(\alpha-1)$. Then

$$\begin{aligned}
\mathrm{E}((\lambda+X_1)^2-A)^+ &= \int_{\sqrt{A}-\lambda}^{\infty}((\lambda+x)^2-A)(\frac{\lambda}{\lambda+x})^{\alpha+1}\frac{\alpha}{\lambda}\,dx \\
&= \int_{\sqrt{A}-\lambda}^{\infty}(\frac{\lambda}{\lambda+x})^{\alpha-1}\alpha\lambda\,dx - A\int_{\sqrt{A}-\lambda}^{\infty}(\frac{\lambda}{\lambda+x})^{\alpha+1}\frac{\alpha}{\lambda}\,dx \\
&= \frac{\alpha\lambda^2}{\alpha-2}(\frac{\lambda}{\sqrt{A}})^{\alpha-2} - A(\frac{\lambda}{\sqrt{A}})^{\alpha} \\
&= \frac{2A}{\alpha-2}(\frac{\lambda}{\sqrt{A}})^{\alpha}.
\end{aligned}$$

Equation (*) now reduces to

$$\begin{aligned}
\frac{\lambda^2}{\alpha(\alpha-1)^2(\alpha-2)} &\le c-\frac{\mathrm{E}((\lambda+X_1)^2-A)^+}{(\alpha+1)\alpha^2(\alpha-1)} \\
&= c-(2c/(\alpha-2))(\lambda^2/(c(\alpha+1)\alpha^2(\alpha-1)))^{\alpha/2}.
\end{aligned}$$

2. (a) The posterior distribution of θ given $X_1,\ldots,X_n$ is easily seen to be the gamma distribution, $\mathcal{G}(n+1,1/(S_n+\lambda))$. Hence, $\mathrm{E}(X_{n+1}|X_1,\ldots,X_n)=\mathrm{E}(1/\theta|X_1,\ldots,X_n)=(S_n+\lambda)/n$. If we stop at stage n, we recieve $\beta^n S_n$; if we continue one stage more and stop, we expect to receive $\mathrm{E}(\beta^{n+1}S_{n+1}|S_n)=\beta^{n+1}(S_n+\mathrm{E}(X_{n+1}|S_n))=\beta^{n+1}(S_n+(S_n+\lambda)/n)$. The 1-sla stops the first time the former is greater than the latter, namely

$$\begin{aligned}
N_1 &= \min\{n\ge 1:\beta^n S_n > \beta^{n+1}(S_n+(S_n+\lambda)/n)\} \\
&= \min\{n\ge 1:(1-\beta)S_n/(S_n+\lambda) > \beta/n\}.
\end{aligned}$$

Since $S_n/(S_n+\lambda)$ is increasing a.s. and β/n is decreasing, the problem is monotone.
(b) Note that $\mathrm{E}(X_j) = \mathrm{E}(1/\theta) = \infty$ but that given $X_1 = x_1$, the posterior distribution of θ is $\mathcal{G}(2, 1/(\lambda + x_1))$, so that for $j > 1$, $\mathrm{E}(X_j|X_1 = x_1) = \mathrm{E}(1/\theta|X_1 = x_1) = \lambda + x_1$. Now, $\beta^n S_n \le \sum_1^n \beta^j X_j \le \sum_1^k \beta^j X_j$. Hence, $\mathrm{E}(\sup_n |\beta^n X_n||x_1) \le \mathrm{E}(\sum_1^k \beta^j X_j|X_1) = \sum_1^k \beta^j \mathrm{E}(X_j|X_1) = (\lambda + X_1)\beta/(1-\beta)$, so that the conditions of Corollary 2 are satisfied in this conditional problem.

3. (a)

$$\begin{aligned} N_1 &= \min\{n \ge 0 : Y_n \ge \mathrm{E}(Y_{n+1}|\mathcal{F}_n\} \\ &= \min\{n \ge 0 : (T \le n) \quad \text{or} \quad (T > n \text{ and } u(S_n) \ge \int_0^\infty u(S_n + x) r(S_n + x)\, dF(x))\} \end{aligned}$$

so the 1-sla stops as soon as

$$u(S_n) \int_0^\infty (1 - r(S_n + x)\, dF(x) \ge \int_0^\infty (u(S_n + x) - u(S_n)) r(S_n + x)\, dF(x)$$

The problem is monotone since the left side of this inequality is a.s. nondecreasing (S_n is a.s. nondecreasing since $X_j \ge 0$ a.s., $u(S_n)$ is nondecreasing a.s. since u is nondecreasing, and the integral of $1 - r(S_n + x)$ over x is nondecreasing since r is nonincreasing) and the right side is a.s. nonincreasing ($u(S_n + x) - u(S_n)$ is nonincreasing a.s. since u is concave).

Since r is nonincreasing and not identically one, there is some z_0 and $\delta > 0$ such that $r(z) \le 1 - \delta$ for $z > z_0$. Since $X_j \ge 0$ and $S > 0$, we have $S_n \to \infty$ a.s. Hence, $\mathrm{P}(T = n+1|T > n) = \mathrm{E}(r(S_n))$ will be eventually be less than $1 - \delta/2$, say, so that T is finite with probability one, and in fact has all moments finite. Hence, A3 is satisfied. To check A1, first note that the concavity of u imply it is bounded above by a linear function, $u(z) \le a + bz$ for some a and $b > 0$. Then, $\sup_n Y_n \le u(S_T) \le a + bS_T$, so that

$$\mathrm{E}(\sup_n Y_n) \le a + b\mathrm{E}(S_T) = a + b\mu\mathrm{E}(T) < \infty.$$

Now, the conditions of Corollary 2, with $W \equiv 0$, yield the optimality of the 1-sla.
(b)

$$N_1 = \min\{n \ge 0 : (T \le n) \quad \text{or} \quad (T > n \quad \text{and} \quad u(n) \ge u(n+1) \int_0^\infty r(S_n + x)\, dF(x))\}.$$

The problem monotone because $u(n)/u(n+1)$ is nondecreasing, r is nonincreasing, and S_n is nondecreasing a.s. A3 is satisfied by the argument given in part (a), and A1 is satisfied since $\mathrm{E}(\sup_n Y_n) \le \mathrm{E}(u(T))$, which is finite by assumption. Hence by Corollary 2, the 1-sla is optimal.

4. (a)
$$\begin{aligned} N_1 &= \min\{n \ge 0 : Y_n \ge \mathrm{E}(M_{n+1} - (n+1)c|\mathcal{F}_n)\} \\ &= \min\{n \ge 0 : \mathrm{E}(M_{n+1} - M_n|\mathcal{F}_n) \le c\}. \end{aligned}$$

and

$$\begin{aligned}\mathrm{E}(M_{n+1} - M_n|\mathcal{F}_n) &= \int_{M_n}^{\infty} (x - M_n)\, d\mathrm{E}(F(x)|\mathcal{F}_n) \\ &= \int_{M_n}^{\infty} (x - M_n)\, d(p_n F_0(x) + (1 - p_n)\, dF_n^*(x)) \\ &= p_n \int_{M_n}^{\infty} (x - M_n)\, dF_0(x).\end{aligned}$$

This is decreasing a.s. in n since p_n is decreasing and M_n is nonincreasing a.s. Since $X_1, X_2, \ldots$ are identically distributed with finite second moment, Theorem 4.1 implies that the conditions of Corollary 2 are satisfied and that the 1-sla is optimal.

(b)
$$\begin{aligned}N_1 &= \min\{n \geq 0 : Y_n \geq \mathrm{E}(\beta^{n+1} M_{n+1}|\mathcal{F}_n)\} \\ &= \min\{n \geq 0 : (1 - \beta)M_n \leq \beta\mathrm{E}(M_{n+1} - M_n|\mathcal{F}_n)\}.\end{aligned}$$

As long as $M_n < 0$, N_1 never stops. So we may rewrite N_1 as

$$N_1 = \min\{n \geq 0 : M_n > 0 \quad \text{and} \quad \mathrm{E}((M_{n+1}/M_n - 1)|\mathcal{F}_n) \leq (1 - \beta)/\beta\}.$$

Since

$$\begin{aligned}\mathrm{E}((M_{n+1}/M_n - 1)|x_n) &= \int^i nfty_{M_n} (x/M_n - 1)\, d\mathrm{E}(F(x)|\mathcal{F}_n) \\ &= p_n \int_{M_n}^{\infty} (x/M_n - 1)\, dF_0(x),\end{aligned}$$

and since this is a decreasing function of M_n, the problem is monotone. To see that the 1-sla is optimal, we use Corollary 2 with $W \equiv 0$, and note that $\mathrm{E}(\sup|Y_n|) \leq \mathrm{E}(\sup \beta^n \sum_1^n |X_j|) \leq \mathrm{E}(\sum_1^\infty \beta^j |X_j| = (\beta/(1 - \beta))\mathrm{E}(|X|) < \infty$, and also that $\lim Y_n = \lim(n\beta^n)(M_n/n) \leq \lim(n\beta^n)(\sum_1^n |X_j|/n) = 0$ a.s.

5. If you continue at stage $n \geq r$, you will improve the sum of the expected worths if and only if $X_{n+1} \geq X_r^{(n)}$; we then drop $X_r^{(n)}$ and accept X_{n+1} so the gain would be $(X_{n+1} - X_r^{(n)})^+$. Hence,

$$\begin{aligned}\mathrm{E}(Y_{n+1} - Y_n|\mathcal{F}_n) &= \mathrm{E}((X_{n+1} - X_r^{(n)})^+|\mathcal{F}_n) - c \\ &= g(X_r^{(n)}) - c,\end{aligned}$$

where $g(x) = \int_x^\infty (z - x)\, dF(x)$. Since

$$N_1 = \min\{n \geq r : Y_n \geq \mathrm{E}(Y_{n+1}|x_n)\} = \min\{n \geq r : g(X_r^{(n)}) \geq c\},$$

the problem is monotone since $g(x)$ is nonincreasing. An argument similar to that in Example 2 shows that the T_n are a.s. nonincreasing and that $T_n \to 0$, so that $\mathrm{E}T_n \to 0$ and the 1-sla is optimal.

6. (a)
$$\begin{aligned}N_1 &= \min\{n \geq 0 : g(X_n) \leq \mathrm{E}(g(X_{n+1})|x_n)\} \\ &= \min\{n \geq 0 : g(X_n) \leq \int g(X_n + z)\, dF(z)\}.\end{aligned}$$

Hence, the problem is monotone if $g(x) - \int g(x+z)\,dF(z)$ is monotone nonincreasing in x, i.e. we must show that if $x \le x'$ then $g(x') - g(x) \le \int [g(x'+z) - g(x+z)]\,dF(z)$. This would follow if we show that if $x \le x'$ then $g(x') - g(x) \le g(x'+z) - g(x+z)$ for all $z > 0$, since Z is assumed to be a positive random variable.

Lemma. *If $g(x)$ is convex, then for all $y > 0$ and $z > 0$, $g(x+y) - g(x) \le g(x+y+z) - g(x+z)$.*

Proof. Conxevity implies both $g(x+y) \le (z/(y+z))g(x) + (y/(y+z))g(x+y+z)$ and $g(x+z) \le (y/(y+z))g(x) + (z/(y+z))g(x+y+z)$. Summing gives the inequality of the lemma.

(b) A1 is satisfied since this is a minimum cost problem with nonnegative cost. A3 is satisfied as noted in the statement of the problem. $T_n = \sup_{j \ge n}(-Y_j + Y_n) = \sup_{j \ge n}(g(X_n) - g(X_j)) \le g(0)$, so the T_n are u.i.

7. (a) We must find

$$
\begin{aligned}
\mathrm{E}(Y_{n+1}|S_n = x) &= \mathrm{P}(\text{win if stop at next available place}|\text{start at } x < t). \\
&= \int_0^\infty \mathrm{P}(\text{win if stop at } x+z)\lambda \exp\{-\lambda z\}\,dz \\
&= \int_0^{t-x} \exp\{-2\lambda(t-x-z)\}\lambda\exp\{-\lambda z\}\,dz + \int_{t-x}^{2(t-x)} \lambda\exp\{-\lambda z\}\,dz \\
&= 2\exp\{-\lambda(t-x)\}(1-\exp\{-\lambda(t-x)\}.
\end{aligned}
$$

The 1-sla stops if $S_n \ge t$, or (when $S_n < t$) if $\exp\{-2\lambda(t-S_n)\} \ge \mathrm{E}(Y_{n+1}|S_n)$, so that

$$
\begin{aligned}
N_1 &= \min\{n \ge 0 : S_n \ge t \text{ or } [S_n < t \text{ and } \exp\{-\lambda(t-S_n)\} \ge 2/3]\} \\
&= \min\{n \ge 0 : S_n \ge t - (1/\lambda)\log(3/2)\}.
\end{aligned}
$$

It may be noted that the 1-sla stops when the probability of win is at least 4/9.
(b) Since the S_n are nondecreasing a.s., the problem is monotone. Since the payoff is bounded, all the conditions of Theorem 2 are satisfied and the 1-sla is optimal.

8. (a) We are given $M \in \mathcal{P}(\lambda)$, and for $n = 1, 2, \ldots$, $(X_{n+1}|M, X_1, \ldots, X_n) \in \mathcal{B}(M_n, p_{n+1})$. An analysis similar to that in the text shows that $M_n|x_n \in \mathcal{P}(\lambda(1-p_1)\cdots(1-p_n))$. We simplify the problem by replacing Y_n by its expectation given $\mathcal{F}_n$:

$$
\begin{aligned}
Y_n &= nc_1 + c_2\mathrm{E}(M_n|\mathcal{F}_n) \\
&= nc_1 + c_2\lambda(1-p_1)\cdots(1-p_n).
\end{aligned}
$$

Since the payoffs are constants, the optimal rule is a fixed sample size rule, namely, stop at that n for which Y_n is smallest. The 1-sla is

$$
\begin{aligned}
N_1 &= \min\{n \ge 0 : Y_n \le Y_{n+1}\} \\
&= \min\{n \ge 0 : \lambda p_{n+1}\prod_1^n (1-p_j) \le c_1/c_2\}.
\end{aligned}
$$

If $p_{n+1}\prod_1^n(1-p_j)$ is nonincreasing in j, then the problem is monotone and the 1-sla is optimal. This condition is equivalent to the condition that $p_{n+1}/(1+p_{n+1}) \le p_n$ for all n. In particular, if the p_n are nonincreasing, the 1-sla is optimal.
(b) We are given $M \in \mathcal{B}(W,\pi)$, and for $n=1,2,\ldots$, $X_{n+1}|M,X_1,\ldots,X_n \in \mathcal{B}(M_n,p_{n+1})$. A somewhat lengthy prior to posterior analysis shows that $M_n|\mathcal{F}_n \in \mathcal{B}(W_n,\pi_n)$, where $W_n = W - X_1 - \cdots - X_n$, and $\pi_n = \pi\prod_1^n(1-p_j)/[(1-\pi)+\pi\prod_1^n(1-p_j)]$. This time the 1-sla becomes

$$\begin{aligned} N_1 &= \min\{n \ge 0 : \mathrm{E}(X_{n+1}|\mathcal{F}_n) \le c_1/c_2\} \\ &= \min\{n \ge 0 : p_{n+1}\mathrm{E}(M_n|\mathcal{F}_n) \le c_1/c_2\} \\ &= \min\{n \ge 0 : p_{n+1}W_n\pi_n \le c_1/c_2\} \end{aligned}$$

Since the W_n are a.s. nonincreasing, the problem will be monotone if the $p_{n+1}\pi_n$ are nonincreasing. The π_n are automatically nonincreasing, so the 1-sla will be optimal if the p_n are nonincreasing.

9. Let Z be 1 if the object is in the box and 0 otherwise, so that $\mathrm{P}(Z=1)=\pi$. Let $X_1,X_2,\ldots$ be the indicators of success on the searches, so that given $Z,X_1,X_2,\ldots$ are i.i.d. Bernoulli, with $\mathrm{P}(X_j=1|Z=0)=0$, and $\mathrm{P}(X_j=1|Z=1)=p$. Then, $Y_n = M_n - nc$, where $M_n=\max\{X_1,\ldots,X_n\}$.
(a) The 1-sla is

$$\begin{aligned} N_1 &= \min\{n \ge 0 : M_n - nc \ge \mathrm{E}(M_{n+1}|\mathcal{F}_n) - (n+1)c\} \\ &= \min\{n \ge 0 : M_n = 1 \quad \text{or} \quad \mathrm{P}(X_{n+1}=1|M_n=0) \le c\}. \end{aligned}$$

The problem is monotone if $\mathrm{P}(X_{n+1}=1|M_n=0)$ is nonincreasing a.s. in n. This probability is

$$\mathrm{P}(X_{n+1}=1|M_n=0) = \frac{\mathrm{P}(M_n=0,X_{n+1}=1)}{P(M_n=0)} = \frac{\pi(1-p)^n p}{1-\pi+\pi(1-p)^n}$$

This is decreasing in n. A1 is satisfied, since the Y_n are bounded above, and A3 is satisfied if $Y_\infty = -\infty$. Since the loss is of the form of Corollary 2, the 1-sla is optimal.
(b) Same N_1 as above, but this time

$$P(X_{n+1}=1|M_n=0) = \frac{P(M_n=0,X_{n+1}=1)}{P(M_n=0)} = \frac{\mathrm{E}(\pi(1-p)^n p)}{\mathrm{E}(1-\pi+\pi(1-p)^n)}$$

This is nonincreasing in n if for all $n \ge 0$,

$$\mathrm{E}(\pi(1-p)^n p)\mathrm{E}(1-\pi+\pi(1-p)^{n+1}) \ge \mathrm{E}(\pi(1-p)^{n+1}p)\mathrm{E}(1-\pi+\pi(1-p)^n).$$

Some algebra reduces this to

$$\mathrm{E}(\pi(1-p)^n p^2)(\mathrm{E}(1-\pi)+\mathrm{E}(\pi(1-p)^n) \ge (\mathrm{E}(\pi(1-p)^n p))^2.$$

The left side is $\ge \mathrm{E}(\pi(1-p)^n p^2)\mathrm{E}(\pi(1-p)^n)$, which is $\ge$ the right side by Schwartz' inequality.

(c) Let $M_{j,n}$ be 1 if object j is found by search n, and 0 otherwise. Then, $Y_n = \sum_{j=1}^{k} x_j M_{j,n} - nc$. Hence,

$$N_1 = \min\{n \geq 0 : c \geq \sum_{j=1}^{k} (1 - M_{j,n}) x_j \pi_{j,n}\},$$

where $\pi_{j,n} = \pi_j (1-p_j)^n p_j / [1 - \pi_j + \pi_j (1-p_j)^n]$. As in (a), each of the $W_{j,n}$ is decreasing in n, and since each $1 - M_{j,n}$ is nonincreasing in n, the problem is monotone. The same argument as in (a) shows the 1-sla is optimal.

10. (a) Since $\mathrm{E}(h(K_{n+1})|\mathcal{F}_n) = Q_n h(K_n + 1) + (1 - Q_n) h(K_n)$, where Q_n is the sum of the p_j over the unobserved species after the first n observations, we have

$$\begin{aligned} N_1 &= \min\{n \geq 0 : h(K_n) - nc \geq \mathrm{E}(h(K_{n+1}) - (n+1)c|\mathcal{F}_n)\} \\ &= \min\{n \geq 0 : c \geq Q_n (h(K_n + 1) - h(K_n))\}. \end{aligned}$$

The problem is monotone since Q_n is nonincreasing a.s., $h(k+1) - h(k)$ is nonincreasing by assumption, and K_n is nondecreasing a.s. From Corollary 2, with $W_n = nc$ and $Z_n = h(K_n) \leq h(s)$, we see that the 1-sla is optimal.

(b) For random p_j, we find that $\mathrm{E}(h(K_{n+1})|\mathcal{F}_n) = \mathrm{E}(Q_n|\mathcal{F}_n)(h(K_n+1) - h(K_n)) + h(K_n)$, so the 1-sla is

$$N_1 = \min\{n \geq 0 : c \geq \mathrm{E}(Q_n|x_n)(h(K_n + 1) - h(K_n))\}.$$

In order that the problem be monotone, we need $\mathrm{E}(Q_n|\mathcal{F}_n)$ to be nonincreasing a.s. If the prior of $(p_1, \ldots, p_s)$ is $\mathcal{D}(\alpha_1, \ldots, \alpha_s)$, then the posterior given $\mathcal{F}_n$ is $\mathcal{D}(\alpha_1 + X_{1,n}, \ldots, \alpha_s + X_{s,n})$, where $X_{j,n}$ represents the number of observations of species j by time n. Hence, $\mathrm{E}(Q_n|\mathcal{F}_n) = M_n/(M + n)$, where M is the sum of the α_j over all j, and M_n is the sum of the α_j over the unobserved species after the first n observations. Since this is nonincreasing a.s., the 1-sla is monotone and the same argument as in (a) shows it is optimal.

(c) Let $s = 3$, take $h(k) = k$ for all k, and take the prior distribution of (p_1, p_2, p_3) to be

$$\begin{aligned} &\mathrm{P}\{(p_1, p_2, p_3) = (1, 0, 0)\} = 1 - \epsilon \\ &\mathrm{P}\{(p_1, p_2, p_3) = (1/3, 1/3, 1/3)\} = \epsilon, \end{aligned}$$

where ϵ is a small positive number. For $c < 1$, the 1-sla calls for taking the first observation. Suppose it is from species 1; then, $\mathrm{E}(Q_1|\mathcal{F}_1) = \mathrm{E}(p_2 + p_3|\mathcal{F}_1) = \epsilon/(3 - 2\epsilon)$, so the 1-sla calls for stopping if $c \geq \epsilon/(3 - 2\epsilon)$. Yet, if one continues and the second observation turn out to be species 2, then $\mathrm{E}(Q_2|\mathcal{F}_2) = 1/3$. So, for ϵ chosen so that $\epsilon/(3 - 2\epsilon) < 1/3$ (say, $\epsilon = 1/2$), and $c = \epsilon/(3 - 2\epsilon)$, the 1-sla calls for stopping (actually, the 1-sla is indifferent), but the 2-sla calls for continuing and does strictly better.

11. (a) For $n = 0, 1, \ldots, T$,

$$Y_n = \mathrm{E}(W_n|\mathcal{F}_n) = \left[\sum_{j=1}^{n}(n-j)X_j + \sum_{j=n+1}^{T}(T-j)\lambda_j\right]\delta.$$

$$\mathrm{E}(Y_{n+1}|\mathcal{F}_n) = \left[\sum_{j=1}^{n}(n+1-j)X_j + \sum_{j=n+2}^{T}(T-j)\lambda_j\right]\delta.$$

so that the 1-sla is

$$\begin{aligned} N_1 &= \min\{n \ge 0 : Y_n \le \mathrm{E}(Y_{n+1}|\mathcal{F}_n)\} \\ &= \min\{n \ge 0 : (T-(n+1))\lambda_{n+1} \le S_n\}, \end{aligned}$$

where $S_n = \sum_{j=1}^{n} X_j$. The right side is nondecreasing since the X_n are nonnegative. So if $(T-n)\lambda_n$ is nonincreasing, the problem is monotone, and since the horizon is finite, the 1-sla is optimal.

(b) If the prior distribution of λ is $\mathcal{G}(\alpha, 1/\beta)$, the posterior distribution of λ given $\mathcal{F}_n$ is $\mathcal{G}(\alpha + S_n, 1/(\beta+n))$. Since the mean of $\mathcal{G}(\alpha, 1/\beta)$ is α/β, we have

$$Y_n = \mathrm{E}(W_n|\mathcal{F}_n) = \sum_{j=1}^{n}(n-j)X_j + \frac{\alpha+S_n}{\beta+n}\sum_{j=n+1}^{T}(T-j)$$

$$\begin{aligned} \mathrm{E}(Y_{n+1}|\mathcal{F}_n) &= \sum_{j=1}^{n}(n+1-j)X_j + \frac{\alpha+\mathrm{E}(S_{n+1}|\mathcal{F}_n)}{\beta+n+1}\sum_{j=n+2}^{T}(T-j) \\ &= \sum_{j=1}^{n}(n+1-j)X_j + \frac{\alpha+S_n}{\beta+n}\sum_{j=n+2}^{T}(T-j) \end{aligned}$$

so that the 1-sla becomes

$$\begin{aligned} N_1 &= \min\{n \ge 0 : (T-(n+1))(\alpha+S_n) \le (\beta+n)S_n\} \\ &= \min\{n \ge 0 : \alpha(T-(n+1)) \le (\beta+2n-T+1)S_n\}. \end{aligned}$$

To see that the problem is monotone, suppose that the inequality in N_1 is satisfied at n. Then $\beta+2n-T+1$ must be positive, and will stay positive for larger n; thus, the right side is nondecreasing a.s. and the left side is decreasing, so the problem is monotone and the 1-sla is optimal.

12. (a)

$$\begin{aligned} N_1 &= \min\{n \ge 0 : Y_n \le \mathrm{E}(Y_{n+1}|\mathcal{F}_n)\} \\ &= \min\{n \ge 0 : cP(K > n|\mathcal{F}_n) + \mathrm{E}((n-K)^+|\mathcal{F}_n) \le \\ &\qquad\qquad cP(K > n+1|\mathcal{F}_n) + \mathrm{E}((n+1-K)^+|\mathcal{F}_n)\} \\ &= \min\{n \ge 0 : cP(K = n+1|\mathcal{F}_n) \le \mathrm{E}((n+1-K)^+ - (n-K)^+|\mathcal{F}_n)\} \\ &= \min\{n \ge 0 : U_n \ge c\}, \end{aligned}$$

where $U_n = P(K \le n|\mathcal{F}_n)/P(K = n+1|\mathcal{F}_n)$.
(b) We have $U_0 = 0$, $U_1 = \mathrm{P}(K=1|X_1)/\mathrm{P}(K=2|X_1) = \pi_1 f_1(X_1)/(\pi_2 f_0(X_1))$, and in general,

$$\begin{aligned} U_n &= \frac{\mathrm{P}(K \le n, X_1,\ldots,X_n)}{\mathrm{P}(K=n,X_1,\ldots,X_n)} \cdot \frac{\mathrm{P}(K=n,X_1,\ldots,X_n)}{\mathrm{P}(K=n+1,X_1,\ldots,X_n)} \\ &= [1 + \frac{\mathrm{P}(K \le n-1, X_1,\ldots,X_{n-1})f_1(X_n)}{\mathrm{P}(K=n,X_1,\ldots,X_{n-1})f_1(X_n)}] \cdot \frac{\pi_n f_1(X_n)}{\pi_{n+1} f_0(X_n)} \\ &= [1+U_{n-1}] \cdot (\pi_n/\pi_{n+1})\lambda(X_n), \end{aligned}$$

where $\lambda(x) = f_1(x)/f_0(x)$ is the likelihood ratio.
(c) We have $\pi_n = (1-\pi)\pi^{n-1}$ for $n = 1,2,\ldots$, $f_0(x) = \exp\{-x\}I(x>0)$, $f_1(x) = (1/\mu)\exp\{-x/\mu\}I(x>0)$, so $\lambda(x) = (1/\mu)\exp\{x(\mu-1)/\mu)\}$. Thus, $U_0 = 0$, and

$$U_n = [1+U_{n-1}](1/\pi)(1/\mu)\exp\{X_n(\mu-1)/\mu\}.$$

Since $\mu > 1$ and $X_n > 0$, we have $\exp\{X_n(\mu-1)/\mu\} > 1$. The problem is monotone if for all n, $U_{n-1} > c$ implies $U_n > c$. If $U_{n-1} > c$, then $U_n > (1+c)/(\pi\mu)$, which is $> c$ if and only if $1 + c > c\pi\mu$, as was to be shown.

13. (a) Let $A = \{(s,z) : 0 \le s \le t\}$. Then independent of the number, catch times and sizes of the fish caught in the interval $(0,t]$, the number of fish not caught by time t is $N(A^c)$, which has a Poisson distribution with parameter, $\lambda\mathrm{P}(A^c)$, where $\mathrm{P}(A^c) = \mathrm{P}(T > t) = S(t)$, and given $N(A^c)$, their catch times and sizes are i.i.d. $\mathrm{P}/\mathrm{P}(A^c)$, namely, $\mathrm{P}(T \le s, Z \le z|T > t) = (F(s,z) - F(t,z))/S(t)$.
(b) $\mathrm{E}Y_t = \mathrm{E}(\sum_j Z_j \mathrm{I}(T_j \le t) - tc)$ where the sum is over those j for which $T_j \le t$. Hence, from Wald's equation, $\mathrm{E}Y_t = \mathrm{E}N(A)\mathrm{E}(Z|T \le t) - tc = \lambda \int_0^t \int z f(s,z)\,dz\,ds - tc$. The optimal rule, if any, is given by the value of t, if any, that maximizes this. The derivative of $\mathrm{E}Y_t$ with respect to t is $\lambda \int z f(t,z)\,dz - c = \lambda\mathrm{E}(Z|T=t)f(t) - c$, where $f(t)$ is the density of T. If this is a decreasing function of t from a positive value to a negative value, as it is for parts (c) and (d), the optimal rule is given by the value of t that makes this derivative zero, and the rule is the infinitesimal look-ahead rule.
(c) Since $\mathrm{E}(Z|T=t) = \alpha\gamma(t+1)$, the derivative of $\mathrm{E}Y_t$ with respect to t is $\lambda\alpha\gamma(t+1)\theta/(t+1)^{\theta+1} - c$, from which we conclude that it is optimal to stop fishing at the fixed time,

$$t = ((\frac{\lambda\alpha\gamma\theta}{c})^{1/\theta} - 1)^+.$$

(d) The conditional distribution of Z given $T = t$ is easily found to be the gamma, $\mathcal{G}(\alpha+1, \beta/(1+\beta\gamma t))$ so that $\mathrm{E}(Z|T=t) = (\alpha+1)\beta/(1+\beta\gamma t)$. The marginal density of T is found to be inverse power, $f(t) = \alpha\beta\gamma/(1+\beta\gamma t)^{\alpha+1}$ for $t > 0$. The optimal rule is the fixed time

$$t = \frac{1}{\beta\gamma}((\frac{\lambda(\alpha+1)\beta^2\alpha\gamma}{c})^{1/(\alpha+2)} - 1)^+.$$

14. (a) If $S_n > L$, we might as well stop. The 1-sla as defined says we should stop. On $\{N > n\} = \{S_n \le L\}$, we have $\mathrm{E}(Y_{n+1}|\mathcal{F}_n) = Y_n + \mathrm{E}(r_{n+1}(S_{n+1})|\mathcal{F}_n) - b_{n+1}\mathrm{P}(S_{n+1} >$

$L|\mathcal{F}_n)$. The 1-sla calls for stopping iat n if this is less or equal to Y_n. Hence,

$$N_1 = \min\{n \geq 0 | S_n > L \text{ or } \mathrm{E}(r_{n+1}(S_{n+1})|\mathcal{F}_n) \leq b_{n+1}\mathrm{P}(S_{n+1} > L|\mathcal{F}_n)\}.$$

(b) The problem is monotone if the difference $\mathrm{E}(r_{n+1}(S_{n+1})|\mathcal{F}_n) - b_{n+1}\mathrm{P}(S_{n+1} > L|\mathcal{F}_n)$ is nonincreasing almost surely. This will be satisfied if the following two conditions are satisfied:

(1) $\mathrm{E}(r_{n+1}(S_{n+1})|\mathcal{F}_n)$ is nonincreasing a.s., and

(2) $b_{n+1}\mathrm{P}(X_{n+1} > L - S_n|\mathcal{F}_n)$ is nondecreasing a.s.

The expectation in condition (1) may be written $\mathrm{E}(r_{n+1}(z + X))$ where z is a constant representing S_n and X has the same distribution as X_{n+1}. Thus, (1) would be satisfied, for example, if $r_n(x)$ is nonincreasing in n for all x. The probability in (2) may be written as $1 - F(L - S_n)$. Since the X_n are nonnegative, the S_n are nondecreasing a.s., and so the probability in (2) is automatically nondecreasing a.s. Thus (2) would be satisfied, for example, if the b_n were nondecreasing.
(c) If L is random with distribution function G on $(0,\infty)$, the only change in N_1 is that the probability in $P(S_{n+1} > L|\mathcal{F}_n)$ must take this randomness into account. In particular, $\mathcal{F}_n$ contains the information whether $S_n > L$ or not. Thus the only change in the conditions for monotonicity is that (2) becomes (2′): $b_{n+1}\mathrm{P}(X_{n+1} > L - S_n|S_n, L > S_n)$ is nondecreasing a.s. The probability may be written as $\int_{S_n}^{\infty}(1 - F(L - S_n))dG(L)/(1 - G(S_n))$. For this to be nondecreasing in n a.s., it is sufficient that the conditional distribution of $L - s$ given $L > s$ be stochastically increasing in s.

15. If you stop at stage n, your expected payoff is given by Equation (17) with $Y_n = \theta$ if $T_n = \infty$, and $Y_\infty = \theta$. If you continue to the next success and stop, your expected payoff is $[\prod_{i=n+1}^{\infty}(1 - p_i)]\sum_{i=n+1}^{\infty}\frac{p_i}{1-p_i} + \theta\prod_{i=n+1}^{\infty}(1 - p_i)$. Therefore the 1-sla is

$$N_1 = \min\{n \geq 0 : X_n = 1 \text{ and } \sum_{i=n+1}^{\infty}\frac{p_i}{1 - p_i} \leq 1 - \theta\}.$$

This rule is clearly monotone. It is optimal by the same argument that shows the rule of Equation (19) is optimal.

16. Let X_i be the indicator of the event that the ith group has a relatively best applicant, $i = 1, \ldots, m$. The X_i are independent and we want to stop on the last success. The probability that X_i is 1 is the probability that among the first $k_1 + \cdots + k_i$ applicants, the best is found in the ith group, namely, $P(X_i = 1) = k_i/(k_1 + \cdots + k_i$. The odds of success are $k_i/(k_1 + \cdots + k_{i-1}$. Therefore, the sum-the-odds rule

$$N_1 = \min\{t \geq 1 : X_t = 1 \text{ and } \sum_{i=t+1}^{n}\frac{k_i}{k_1 + \cdots + k_{i-1}} \leq 1\}.$$

is optimal.

17. (a) First note that the distribution of the minimum of two exponentials, $\mathcal{E}(\theta_1)$ and $\mathcal{E}(\theta_2)$, is exponential, $\mathcal{E}(\theta_1+\theta_2)$. By induction, the minimum of n exponentials, $\mathcal{E}(\theta_1),\ldots,\mathcal{E}(\theta_n)$, is exponential, $\mathcal{E}(\sum_1^n \theta_i)$. Next note that $P(Z_2 < Z_1) = \theta_2/(\theta_1+\theta_2)$. Then, $P(X_n = 1) = P(Z_n < \min\{Z_1,\ldots,Z_{n-1}\} = \theta_n/(\theta_1+\cdots+\theta_n)$. Note that the condition $\sum_1^\infty \theta_i < \infty$ is equivalent to the condition that $\sum_1^\infty P(X_i = 1) < \infty$.

(b) To show independence, it is sufficient to show $P(X_1 = 1,\ldots,X_n = 1) = \prod_1^n P(X_i = 1)$ because all the variables are Bernoulli. But we have

$$\begin{aligned}
P(X_1 = 1,\ldots,X_n = 1) &= P(Z_n < Z_{n-1} < \cdots < Z_1) \\
&= \int_0^\infty \int_{z_n}^\infty \cdots \int_{z_2}^\infty (\prod_1^n \theta_i) \exp\{-\sum_1^n \theta_i z_i\}\, dz_1 \cdots dz_n \\
&= (\prod_1^\infty \theta_i)\frac{1}{\theta_1} \int_0^\infty \cdots \int_{z_3}^\infty \exp\{-\sum_3^n \theta_i z_i - (\theta_2+\theta_1)z_2\}\, dz_2 \cdots dz_n \\
&= \frac{\theta_1\theta_2\cdots\theta_n}{\theta_1(\theta_1+\theta_2)\cdots(\theta_1+\cdots+\theta_n)} \\
&= P(X_1 = 1)P(X_2 = 1)\cdots P(X_n = 1)
\end{aligned}$$

(c) The optimal stopping rule is the sum-the-odds rule,

$$N_1 = \min\{t \ge 1 : X_t = 1 \quad \text{and} \quad \sum_{i=t+1}^\infty \frac{\theta_i}{\theta_1+\cdots+\theta_{i-1}} \le 1\}.$$

18. (a) Since $\sum_{s+1}^n r_i \le 1$, all p_i are less than 1 for $i \ge s$. If $p_s = 1$, then the optimal rule stops at s and succeeds if and only if all remaining events are failures; hence, $V_n^* = \prod_{s+1}^n q_i$. If $p_s < 1$, then as in the derivation of Equation (18),

$$V_n^* = p_s \prod_{s+1}^n q_i + q_s p_{s+1} \prod_{s+2}^n q_i + \cdots = \left[\prod_s^n q_i\right] \sum_s^n r_i.$$

(b) We have $V_n^* = [\prod_{s+1}^n q_i](p_s + (1-p_s)\sum_{s+1}^n r_i)$. This is nondecreasing in p_s since $\sum_{s+1}^n r_i \le 1$. Thus we decrease V_n^* by making p_s smaller. But we must stop when we hit the constraint $\sum_s^n r_i \ge 1$.
(c) This is immediate using Lagrange multipliers, but here is a way to get the result using $G \le A$, the geometric mean of positive numbers is less than or equal to the arithmetic mean, with equality if and only if all the numbers are equal. Thus,

$$\left(\prod_s^n (1+r_i)\right)^{1/(n+s+1)} \le \frac{1}{n+s+1} \sum_s^n (1+r_i) = \frac{n+s+2}{n+s+1}$$

with equality if and only if all r_i are equal. Now note that $q_i = 1/(1+r_i)$, and write

$$\prod_s^n q_i = \frac{1}{\prod_s^n (1+r_i)} \geq \left(\frac{n-s+1}{n-s+2}\right)^{n-s+1} = \left(1 - \frac{1}{n-s+2}\right)^{n-s+1}$$

with equality if and only if all q_i are equal.
(d) It is well known (and easy to prove) that $(1+\frac{1}{n})^n$ is monotonically increasing to e as n increases to infinity. So $(1-\frac{1}{n+1})^n = (1+\frac{1}{n})^{-n}$ decreases to e^{-1}. This implies that $\prod_s^n q_i$ of the previous display is bounded above by the value of the right side when $s=1$. This gives the bound $V_n^* \geq (1-\frac{1}{n+1})^n$ which is achieved if and only if all p_i are equal to $1/(n+1)$.
(e) Let $\epsilon > 0$. Find n such that P(no successes after n) $\geq 1-\epsilon$. Such an n exists since $\sum p_i < \infty$. From (d), there is a stopping rule $N_n \leq n$ such that the probability that N_n stops at the last success in the first n trials is at least $(1+(1/n))^{-n}$. Then, the probability that N_n stops at the last success in all trials is at least $(1+(1/n))^{-n}(1-\epsilon) \to e^{-1}(1-\epsilon)$. This shows $V_\infty^* \geq e^{-1}(1-\epsilon)$. Since this is true for all $\epsilon > 0$, it is also true for $\epsilon = 0$, completing the proof.

Solutions to the Exercises of Chapter 6.

1. Maximizing the ratio $\mathrm{E}(X_N - c)/\mathrm{E}(N + d)$ reduces to solving the problem of maximizing the return $\mathrm{E}(X_N - c - \lambda(N + d))$ and choosing λ so that the optimal return is zero. Since the X_n have finite second moment, this problem is solved in §4.1 and the optimal rule is found to be stop as soon as $X_n - c - \lambda d$ is at least V^*, where V^* satisfies the equation $\mathrm{E}(X - c - \lambda d - V^*)^+ = \lambda$. Setting $V^* = 0$, we find that the optimal rule, N^*, for maximizing the rate of return is to accept the first offer that exceeds $c+\lambda d$, where λ satisfies

$$\mathrm{E}(X - c - \lambda d)^+ = \lambda.$$

The left side of this inequality is decreasing in λ with value $\mathrm{E}(X - c)^+$ at $\lambda = 0$, so there is a unique solution for λ. If $\mathrm{P}(X < c) = 1$, then $\lambda = 0$ and $N^* \equiv \infty$. Otherwise, the optimal rate of return is $\lambda > 0$.

If F is $\mathcal{U}(0, 1)$ and $c < 1$, then $\mathrm{E}(X - c - \lambda d)^+ = (1 - c - \lambda d)^2/2 = \lambda$, so that λ is the root of $d^2\lambda^2 - 2((1 - c)d + 1)\lambda + (1 - c)^2 = 0$ between 0 and $(1 - c)/d$, namely,

$$\lambda = [(1 - c)d + 1 - \sqrt{1 + (1 - c)d}]/d^2.$$

2.(a) The 1-sla is

$$\begin{aligned} N_1 &= \min\{n \ge 1 : Y_n - n\lambda \le \mathrm{E}(Y_{n+1} - (n + 1)\lambda|\mathcal{F}_n)\} \\ &= \min\{n \ge 1 : T \le n \quad \text{or} \quad (T > n \quad \text{and} \quad \lambda \le \mathrm{E}(Y_{n+1} - Y_n|S_n))\} \\ &= \min\{n \ge 1 : T \le n \quad \text{or} \quad (T > n \quad \text{and} \quad \lambda \le K\mathrm{P}(T = n + 1|S_n))\} \\ &= \min\{n \ge 1 : T \le n \quad \text{or} \quad (T > n \quad \text{and} \quad \lambda \ge Kq\exp(-(M - S_n)/\mu))\} \end{aligned}$$

Thus, $N_1 \equiv T$ if $\lambda \ge qK$, and, $N_1 = \min\{n \ge 1 : S_n \ge M - \mu\log(Kq/\lambda)\}$ if $\lambda < qK$. Since the S_n are increasing, the problem is monotone. Since $n\lambda - Y_n$ is bounded above by λT which is integrable, and since Y_n is a.s. nondecreasing, the 1-sla is optimal by Lemma 5.2.

(b) The expected loss is $\mathrm{E}(Y_N - N\lambda) = c + K\mathrm{P}(N = T) - \lambda\mathrm{E}(N)$. Since we must take at least one observation, we must distinguish two cases.

Case 1 : $M \le \mu\log(Kq/\lambda)$. Then $N \equiv 1$, and $\mathrm{P}(N = T) = \mathrm{P}(T = 1) = q\exp\{-M/\mu\}$, so the expected loss is $c + Kq\exp\{-M/\mu\} - \lambda$.

Case 2 : $M > \mu\log(Kq/\lambda)$. Then,

$$\mathrm{P}(N = T) = \mathrm{P}(S_N \ge M|S_N \ge M - \mu\log(Kq/\lambda)) = \exp(-(\mu\log(Kq/\lambda)/\mu) = \lambda/(Kq).$$

To find $\mathrm{E}(N)$, we use two formulas for $\mathrm{E}(S_N)$; $\mathrm{E}(S_N) = \mathrm{E}(N)\mathrm{E}(X) = \mathrm{E}(N)q\mu$, and $\mathrm{E}(S_N) = (M - \mu\log(Kq/\lambda)) + \mu$. (This uses the fact that when S_n does exceed $M - \mu\log(Kq/\lambda)$, it will do so on the average by μ). Thus, $\mathrm{E}(N) = (M/\mu - \log(Kq/\lambda) + 1)/q$, and the expected loss is $c + K(\lambda/(Kq)) - \lambda(M/\mu - \log(Kq/\lambda) + 1)/q = c + (\lambda/q)(M/\mu - \log(Kq/\lambda))$.

We must solve for the value of λ that makes the expected loss zero. In case 1, $\lambda \le Kq\exp\{-M/\mu\}$, so that the expected loss cannot be made equal to zero. Thus, the optimal rate of return may be found by solving the equation

$$\lambda(M - \mu\log(Kq/\lambda)) = c.$$

Since the left side is increasing in λ, there is a unique root of this equation in the interval $(Kq\exp\{-M/\mu\}, \infty)$.

3. To find N to maximize $\mathrm{E}(N)/\mathrm{E}(X_N + \tau)$, we first solve the problem of finding N to maximize $\mathrm{E}(N - \lambda(X_N + \tau))$ and then adjust λ so that this expectation is zero. As in Example 5 of §5.4, the optimal rule for this problem is the fixed sample size rule $N^* = \min\{j \ge 0 : j \ge n - \lambda\} = n - \lfloor\lambda\rfloor$. The expected return using this rule is $\mathrm{E}(N - \lambda(X_N + \tau)) = n - \lfloor\lambda\rfloor - \lambda(\mathrm{E}(X_{n-\lfloor\lambda\rfloor} + \tau))$. Since $\mathrm{E}(X_k) = 1/n + 1/(n-1) + \cdots + 1/(n-k+1)$, the optimal rate of return satisfies the equality

$$n - \lfloor\lambda\rfloor = \lambda(\tau + 1/n + \cdots + 1/(\lfloor\lambda\rfloor + 1)).$$

(This equation has a unique solution.) When $n = 10$ and $\tau = 1$, trial and error gives $\lfloor\lambda\rfloor = 3$, and $\lambda = 3.34028$, and the optimal rule is the fixed sample size rule $N^* = 7$.

4. First, we solve the problem of finding a stopping rule to maximize $c_1\mathrm{P}(S_N > a) - \lambda\mathrm{E}(N + c_2)$, and then adjust λ to make the optimal return equal to zero. From the solution to Exercise 4.8(c), we see that the optimal rule is $N^* = \min\{n \ge 1 : S_n = -\gamma \quad \text{or} \quad S_n = a\}$, where

$$\gamma = a + \{a(1/4 + c_1/\lambda)\},$$

where $\{x\}$ represents the integer closest to x (either integer if there are two such). The optimal return using this rule is (using formulas from the gambler's ruin problem),

$$c_1(a/(a+\gamma)) - \lambda(\gamma a + c_2).$$

We must set this to zero and solve for λ keeping in mind that $\gamma$is a function of λ. If $c_2 = 0$, then $\gamma = 1$ works: $\lambda = c_1/(a(a+1))$ and $\{a(1/4 + c_1/\lambda)\} = \{a(1/4 + a(a+1))\} = \{a + 1/2\} = a + 1 = a + \gamma$. The optimal rule is to stop as soon as S_n is negative.

For general c_2, a similar analysis shows that

$$\gamma = \{a(c_2 + 1/4)\}$$

works, and the optimal return is $\lambda = c_1\gamma/((a+\gamma)(a\gamma + c_2))$. It is interesting to note that the stopping value for negative values of S_n, (the point where you get discouraged), is independent of the goal, a, and of the reward for attaining it, c_1.