

### A Normal Approximation Theorem of Stein.

In Stein (1972), a new approach to the central limit theorem was proposed. The result was remarkable because it gave a uniform upper bound to the approximation as in the Berry-Esseen bound, but the distributional assumptions were completely general, allowing dependent and nonidentical distributions. This study was continued in Stein (1986), from which the following theorem is taken, pg. 110.

**Stein's Theorem.** Let  $\mathcal{T}$  be a finite index set, and for  $t \in \mathcal{T}$ , let  $Z_t$  be a random variable with  $EZ_t = 0$  and  $EZ_t^4 < \infty$ . Let  $W = \sum_{t \in \mathcal{T}} Z_t$ . Suppose that for every  $t \in \mathcal{T}$ , there is a set  $S_t \subset \mathcal{T}$  such that

$$E\left[\sum_{t \in \mathcal{T}} Z_t \sum_{s \in S_t} Z_s\right] = 1. \quad (1)$$

Then

$$\begin{aligned} \sup_w |\mathbb{P}(W \leq w) - \Phi(w)| &\leq 2 \sqrt{E\left[\left(\sum_{t \in \mathcal{T}} \sum_{s \in S_t} (Z_t Z_s - \sigma_{ts})\right)^2\right]} \\ &\quad + \sqrt{\frac{\pi}{2}} E \sum_{t \in \mathcal{T}} \left|E(Z_t | \{Z_s\}_{s \notin S_t})\right| \\ &\quad + 2^{3/4} \pi^{-1/4} \sqrt{E \sum_{t \in \mathcal{T}} |Z_t| \left(\sum_{s \in S_t} Z_s\right)^2} \end{aligned} \quad (2)$$

where  $\sigma_{ts} = EZ_t Z_s$  and  $\Phi(w)$  represents the standard normal distribution function.

Though it looks formidable, it's not as bad as it seems. The interpretation is as follows. We work with the centered variables without loss of generality, so that  $EZ_t = 0$  for all  $t$ . The set  $S_t$  should contain  $s$  for those  $Z_s$  with which  $Z_t$  is most strongly dependent, so we certainly put  $t$  in  $S_t$ . In particular, if for all  $t$ ,  $Z_t$  and  $\{Z_s\}_{s \notin S_t}$  are independent then the second term on the left of (2) is zero. Equation (1) is merely a choice of scale. (If the sum in (1) is  $c > 0$ , just divide all  $Z_t$  by  $\sqrt{c}$ .)

**Exercise.** Let  $X_1, \dots, X_n$  be i.i.d. with mean zero, variance  $\sigma^2$ , third absolute moment  $\rho = E|X|^3$ , and fourth moment  $\mu_4 = EX^4$ . Using  $Z_t = X_t/(\sqrt{n}\sigma)$  and  $S_t = \{t\}$ , find the uniform upper bound of (2) and compare to the Berry-Esseen bound.

### References.

Stein, Charles (1972), "A bound for the error in the normal approximation to the distribution of a sum of dependent random variables", *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **2**, 583-602.

Stein, Charles (1986), *Approximate Computations of Expectations*. Lecture Notes - Monograph Series, IMS, vol. 7.

**Solution.** If  $S_t = \{t\}$ , then  $\sum_{s \in S_t} Z_s = Z_t$  and (1) becomes

$$\sum_{t \in \mathcal{T}} \mathbb{E}(Z_t^2) = 1. \quad (1')$$

If in addition the  $Z_t$  are independent, then the second term on the left of (2) is zero, and the summation in the first term becomes

$$\mathbb{E}\left[\left(\sum_{t \in \mathcal{T}} (Z_t^2 - \sigma_{tt}^2)\right)^2\right] = \mathbb{E}\left[\sum_t \sum_s (Z_t^2 - \sigma_{tt}^2)(Z_s^2 - \sigma_{ss}^2)\right] = \sum_{t \in \mathcal{T}} \mathbb{E}[(Z_t^2 - \sigma_{tt}^2)^2] = \sum_{t \in \mathcal{T}} \text{Var}(Z_t^2),$$

and the summation in the third term becomes  $\sum_{t \in \mathcal{T}} |Z_t|^3$ . Therefore, (2) becomes

$$\sup_w |\mathbb{P}(W \leq w) - \Phi(w)| \leq 2 \sqrt{\sum_{t \in \mathcal{T}} \text{Var}(Z_t^2)} + 2^{3/4} \pi^{-1/4} \sqrt{\mathbb{E} \sum_{t \in \mathcal{T}} |Z_t|^3} \quad (2')$$

This gives a uniform upper bound on the normal approximation valid for independent non-identically distributed variables.

Suppose that  $Z_t = X_t/(\sqrt{n}\sigma)$  where the  $X_1, \dots, X_n$  are independent with mean zero, variance  $\sigma^2$ , third absolute moment  $\rho = \mathbb{E}|X_t|^3$ , and fourth moment  $\mu_4 = \mathbb{E}X_t^4$ , for  $t = 1, \dots, n$ . Then (1') is satisfied,  $\text{Var}(Z_t^2) = \text{Var}(X_t^2/(n\sigma^2)) = (1/(n^2\sigma^4))\text{Var}(X_t^2) = (\mu_4 - \sigma^4)/(n^2\sigma^4)$ , and  $\mathbb{E}|Z_t|^3 = (1/(n^{3/2}\sigma^3))\mathbb{E}(|X_t|^3) = \rho/(n^{3/2}\sigma^3)$ . The uniform upper bound (2') becomes

$$\sup_w |\mathbb{P}(W \leq w) - \Phi(w)| \leq \frac{2}{\sqrt{n}} \sqrt{\frac{\mu_4 - \sigma^4}{\sigma^4}} + \left(\frac{8}{n\pi}\right)^{1/4} \sqrt{\frac{\rho}{\sigma^3}} \quad (2'')$$

This is not nearly as good as the Berry-Esseen bound for the i.i.d. case. It does not even have the correct order of convergence, as Stein himself pointed out.