Solutions to Exercise Set 8.

17.4. (a)

$$p_n = P(H_1|X_1,\ldots,X_n) = \frac{P(H_1,X_1,\ldots,X_n)}{P(X_1,\ldots,X_n)} = \frac{p_0\prod_1^n f_1(X_i)}{p_0\prod_1^n f_1(X_i) + (1-p_0)\prod_1^n f_0(X_i)}$$
$$= \left(1+\frac{1-p_0}{p_0}\prod_1^n\frac{f_0(X_i)}{f_1(X_i)}\right)^{-1}.$$

(b) Assuming H_0 is true, we are to show

$$-\frac{1}{n}\log(p_n) = \frac{1}{n}\log\left(1+\frac{1-p_0}{p_0}\prod_{i=1}^n\frac{f_0(X_i)}{f_1(X_i)}\right) \xrightarrow{a.s.} K(f_0,f_1).$$

Since $\frac{1}{n}\sum_1^n \log(f_0(X_i)/f_1(X_i)) \xrightarrow{a.s.} K(f_0,f_1) > 0$, we have $\prod_1^n(f_0(X_i)/f_1(X_i)) = \exp\{\sum_1^n \log(f_0(X_i)/f_1(X_i))\} = \exp\{n(\frac{1}{n}\sum_1^n \log(f_0(X_i)/f_1(X_i))\} \xrightarrow{a.s.} +\infty$. Therefore,

$$\frac{1}{n}\log\left(1+\frac{1-p_0}{p_0}\prod_{i=1}^n\frac{f_0(X_i)}{f_1(X_i)}\right) \sim \frac{1}{n}\log\left(\frac{1-p_0}{p_0}\prod_{i=1}^n\frac{f_0(X_i)}{f_1(X_i)}\right)$$
$$= \frac{1}{n}\log\left(\frac{1-p_0}{p_0}\right) + \frac{1}{n}\sum_{i=1}^n\log\left(\frac{f_0(X_i)}{f_1(X_i)}\right) \xrightarrow{a.s.} K(f_0,f_1).$$

18.6. (a) By the Central Limit Theorem, $\sqrt{n}((\overline{X}_n,\overline{Y}_n)-(\mu_x,\mu_y)) \xrightarrow{\mathcal{L}} \mathcal{N}((0,0),\Sigma)$. Then we apply Cramér's Theorem with $g(x,y)=x/y$ and $\dot{g}(x,y)=(1/y,-x/y^2)$, so that $\dot{g}(\mu_x,\mu_y)=(1/\mu_y)(1,-\theta)$. We find

$$\sqrt{n}(\theta_n^*-\theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\frac{1}{\mu_y^2}(1,-\theta)\begin{pmatrix}\sigma_x^2 & \sigma_{xy}\\ \sigma_{xy} & \sigma_y^2\end{pmatrix}\begin{pmatrix}1\\-\theta\end{pmatrix})$$
$$= \mathcal{N}(0,\frac{1}{\mu_y^2}(\sigma_x^2-2\theta\sigma_{xy}+\theta^2\sigma_y^2)).$$

(b) For the exponential distribution of Y, $\mu_y=1$ and $\sigma_y^2=1$. Then $\mathrm{E}(X)=\mathrm{E}(\mathrm{E}(X|Y))=\mathrm{E}(\theta Y)=\theta$. Similarly, $\mathrm{E}(X^2)=\mathrm{E}(\mathrm{E}(X^2|Y))=\mathrm{E}(1+\theta^2Y^2)=1+2\theta^2$, $\sigma_x^2=1+\theta^2$, $\mathrm{E}(XY)=\mathrm{E}(\mathrm{E}(XY|Y))=\mathrm{E}(Y\mathrm{E}(X|Y))=\mathrm{E}(\theta Y^2)=2\theta$, and $\sigma_{xy}=\theta$. So in this case, $\sqrt{n}(\theta_n^*-\theta)\xrightarrow{\mathcal{L}}\mathcal{N}(0,1)$.

(c) In case (b), the joint density of (X,Y) is $f(x,y|\theta)=(2\pi)^{-1/2}e^{-y}e^{-(x-\theta y)^2/2}$ for $y>0$. We find

$$(\partial/\partial\theta)\log f(x,y|\theta)=(x-\theta y)y.$$

From this we can see that the maximum likelihood estimate of θ is $\hat{\theta}_n = (\sum X_i Y_i / \sum Y_i^2)$. From $(\partial/\partial\theta)^2 f(x,y|\theta) = -y^2$, we see that Fisher information is $\mathcal{I}(\theta) = \mathrm{E}(Y^2) = 2$. Therefore, $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1/2)$. The asymptotic efficiency of θ_n^* relative to $\hat{\theta}_n$ is only 50%.

19.3. (a) The density is $f(x|\mu,\sigma) = \sigma^{-1}\exp\{-e^{-(x-\mu)/\sigma} - (x-\mu)/\sigma\}$. Let $Y = (X-\mu)/\sigma$. Since $\boldsymbol{\theta} = (\mu,\sigma)$ is a location-scale parameter, the distribution of Y does not depend on $\boldsymbol{\theta}$. (It is $G_3(y)$.) We have $\partial Y/\partial\mu = -1/\sigma$ and $\partial Y/\partial\sigma = -Y/\sigma$. Using $\ell = -\log\sigma - e^{-y} - y$, we find

$$\frac{\partial\ell}{\partial\mu} = -\frac{e^{-y}}{\sigma} + \frac{1}{\sigma} \qquad \text{and} \qquad \frac{\partial\ell}{\partial\sigma} = -\frac{1}{\sigma} - \frac{ye^{-y}}{\sigma} + \frac{y}{\sigma}$$

so that $\mathrm{E}e^{-Y} = 1$ and $\mathrm{E}Y = \mathrm{E}Ye^{-Y} + 1$. We also have

$$\frac{\partial^2\ell}{\partial\mu^2} = -\frac{e^{-y}}{\sigma^2}, \quad \frac{\partial^2\ell}{\partial\mu\partial\sigma} = -\frac{ye^{-y}}{\sigma^2} + \frac{e^{-y}}{\sigma^2} - \frac{1}{\sigma^2} \quad \text{and} \quad \frac{\partial^2\ell}{\partial\sigma^2} = \frac{1}{\sigma^2} - \frac{y^2e^{-y}}{\sigma^2} + \frac{2ye^{-y}}{\sigma^2} - \frac{2y}{\sigma^2}.$$

Taking expectations and simplifying, we find

$$\mathcal{I}(\mu,\sigma) = \frac{1}{\sigma^2}\begin{pmatrix} 1 & \mathrm{E}Ye^{-Y} \\ \mathrm{E}Ye^{-Y} & \mathrm{E}Y^2e^{-Y}+1 \end{pmatrix} = \frac{1}{\sigma^2}\begin{pmatrix} 1 & -.42278 \\ -.42278 & 1.82367 \end{pmatrix}.$$

If $Z = -\log(Y)$, then Z has an exponential distribution with mean 1 and $\mathrm{E}Ye^{-Y} = \mathrm{E}Y - 1 = \gamma - 1$ and Maple tells us that $\mathrm{E}Y^2e^{-Y} = \mathrm{E}(\log(Z)^2 Z) = (\pi^2/6) - 2\gamma + \gamma^2$, from which the above numbers follow.

(b) If $\hat{\theta}_n$ is an unbiased estimate of $g(\mu,\sigma) = \mu/\sigma$, then with $\dot{g}(\mu,\sigma) = (1/\sigma, -\mu/\sigma^2)$, we have

$$\mathrm{Var}_\theta(\hat{\theta}_n) \geq \frac{1}{n}\dot{g}\mathcal{I}^{-1}\dot{g}^T = \frac{1}{n}[1.1087 - .5140\frac{\mu}{\sigma} + .6079\frac{\mu^2}{\sigma^2}].$$

(The minimum value occurs at $\mu/\sigma = .4228$.)