Stat 200C, Spring 2010 Ferguson

Solutions to Exercise Set 7.

14.1 (a) $F(a_n+b_nx)^n = (1+e^{-(a_n+b_nx)})^n \to \exp\{-\lim_{n\to\infty} ne^{-(a_n+b_nx)}\}$. If we take $b_n=1$, we get $F(a_n+b_nx)^n \to \exp\{-\lim_{n\to\infty} ne^{-a_n}e^{-x}\}$, which is equal to $\exp\{e^{-x}\} = G_3(x)$ if we take $ne^{a_n}=1$. This amounts to taking $a_n=\log(n)$, so that we have $M_n - \log(n) \xrightarrow{\mathcal{L}} G_3$.

(b) Since the tail goes down at a power rate, we should be in case (a) where $a_n=0$. We have , with $a_n=0$,

$$F(a_n+b_nx)^n = \left(1-\frac{\log(b_nx+1)}{b_nx}\right)^n \to \exp\left\{-\lim_{n\to\infty}\frac{n\log(b_nx+1)}{b_nx}\right\}.$$

We see that b_n must tend to infinity, implying that $\log(b_nx+1)\sim\log(b_nx)\sim\log(b_n)$. To cancel the n in the nunmerator, let $b_n=nc_n$ for some c_n. Then

$$F(a_n+b_nx)^n \to \exp\left\{-\frac{1}{x}\lim_{n\to\infty}\frac{\log(nc_n)}{c_n}\right\}$$

If $c_n=\log(n)$, then $\lim_{n\to\infty}\log(nc_n)/c_n = \lim_{n\to\infty}(\log(n)+\log\log(n))/\log(n) = 1$. Therefore, we have $b_n = n\log(n)$, and $M_n/(n\log(n)) \xrightarrow{\mathcal{L}} G_{1,1}$.

(c) We have and $F(x) = (1/2)(\sin(x)+1)$ for $|x|<\pi/2$. Expanding $\sin(x)$ about $\pi/2$ gives $\sin(x) \sim 1-(x-(\pi/2))^2/2$. Therefore, $F(a_n+b_nx)^n \sim [(1/2)(1-(a_n+b_nx-(\pi/2))^2/2+1]^n = (1-(a_n+b_nx-(\pi/2))^2/4)^n$. Take $a_n=\pi/2$ to find $F(a_n+b_nx)^n \to \exp\{-\lim_{n\to\infty} b_n^2x^2/4\}$. Take $(b_n)^2/4 = 1/n$, or $b_n = 2/\sqrt{n}$ and find $\sqrt{n}(M_n - (\pi/2))/2 \xrightarrow{\mathcal{L}} e^{-x^2}$ for $x<0$, which is $G_{2,2}$.