Stat 200C, Spring 2010 Ferguson

Solutions to Exercise Set 10.

24.1. (a) The chi-square is $\chi^2(q_1, q_2, q_3) = \sum_{i=1}^{r}\sum_{j=1}^{c}(n_{ij} - p_{ij})^2/p_{ij}$, where

$$p_{ij} = \begin{cases} q_3 & \text{if } 2 < i < r \text{ and } 2 < j < c. \\ q_1 & \text{if } i = 1 \text{ and } j = 1, \text{ or } i = 1 \text{ and } j = c, \text{ or } i = r \text{ and } j = 1, \text{ or } i = r \text{ and } j = c. \\ q_2 & \text{otherwise.} \end{cases}$$

There are 4 corner cells, $2r + 2c - 8$ edge cells and $(r - 2)(c - 2)$ central cells, so we have $4q_1 + 2(r+c-4)q_2 + (r-2)(c-2)q_3 = 1$. The likelihood function is $L \propto q_1^{N_1} q_2^{N_2} q_3^{N_3}$, where N_1 is the number of observations falling in the corners, N_2 is the number on the edges, and N_3 is the number in the central cells. Therefore the Maximum Likelihood Estimates are

$$\hat{q}_1 = \frac{1}{4}\frac{N_1}{n}$$
$$\hat{q}_2 = \frac{1}{2(r + c - 4)}\frac{N_2}{n}$$
$$\hat{q}_3 = \frac{1}{(r - 2)(c - 2)}\frac{N_3}{n}$$

The test rejects H_1 if $\chi^2(\hat{q}_1, \hat{q}_2, \hat{q}_3)$ is too large, in reference to the χ^2 distribution with $rc - 3$ degrees of freedom.

(b) Under H_0, we have $q_1 = q_2 = q_3 = 1/(rc)$. So $\chi^2(1/(rc), 1/(rc), 1/(rc))$ is the χ^2 used to test H_0 against all alternatives. It has $rc - 1$ degrees of freedom. For testing H_0 against H_1, we reject H_0 if the difference, $\chi^2(1/(rc), 1/(rc), 1/(rc)) - \chi^2(\hat{q}_1, \hat{q}_2, \hat{q}_3)$ is too large. This has 2 degrees of freedom, the number of restrictions going from H_1 to H_0.

24.4. (a) The maximum likelihood estimates of the p_{ij} under H_1 are $\hat{p}_{1j} = \hat{p}_{2j} = (n_{1j} + n_{2j})/(2n) = n_{\cdot j}/(2n)$. The chi-square statistic is

$$\chi^2_{H_1} = \sum_{i=1}^{2}\sum_{j=1}^{c}\frac{(n_{ij} - (n_{\cdot j}/2))^2}{n_{\cdot j}/2}.$$

There are $2c$ cells and $2c - 1$ d.f. originally, and $c - 1$ paramters estimated, so there are c d.f. finally. The test rejects H_1 if $\chi^2_{H_1} > \chi^2_{c;\alpha}$ where α is the level of significance.

(b) This is a straight Pearson's χ^2:

$$\chi^2_{H_0} = \sum_{i=1}^{2}\sum_{j=1}^{c}\frac{(n_{ij} - n/(2c))^2}{n/(2c)}.$$

The test rejects H_0 if $\chi^2_{H_0} > \chi^2_{2c-1;\alpha}$.

(c) The test rejects H_0 in favor of H_1 if $\chi^2_{H_0} - \chi^2_{H_1} > \chi^2_{c-1;\alpha}$.

24.6. (a) The chi-square for the problem is

$$\chi^2 = \sum_{k=1}^{K}\sum_{i=1}^{I}\sum_{j=1}^{J} \frac{(n_{ijk} - n_{..k}\pi_{ijk})^2}{n_{..k}\pi_{ijk}}$$

For testing a fixed set of values of the π_{ijk}, it is the sum of K independent chi-squares with $(IJ-1)$ degrees of freedom, and so has $K(IJ-1)$ degrees of freedom.

(b) Under the hypothesis $H_0 : \pi_{ijk} = p_i q_{jk}$, the maximum likelihood estimates are $\hat{p}_i = n_{i..}/n_{...}$ and $\hat{q}_{jk} = n_{.jk}/n_{..k}$. With these substituted in the chi-square, we have

$$\chi^2 = \sum_{k=1}^{K}\sum_{i=1}^{I}\sum_{j=1}^{J} \frac{(n_{ijk} - n_{i..}n_{.jk}/n_{...})^2}{n_{i..}n_{.jk}/n_{...}}$$

There have been $I-1$ parameters p_i estimated and $K(J-1)$ parameters q_{jk} estimated. Therefore the number of degrees of freedom of the chi-square is the difference,

$$K(IJ-1) - (I-1) - K(J-1) = (I-1)(JK-1).$$