Solution to Exercises 7.4.6 and 7.4.7.

7.4.6.(a) The problem is invariant under scale changes, $g_b(x_1,\ldots,x_j) = (bx_1,\ldots,bx_j)$ for $b > 0$, with $\bar{g}_b(\theta) = b\theta$ and $\tilde{g}_b(a) = ba$. Since $\bar{\mathcal{G}}$ is transitive, all invariant rules will have constant risk. Since $\tilde{\mathcal{G}}$ is transitive too, every invariant rule must take at least one observation. We see from the density, $f(x|\theta) = (\theta^{-\alpha}/\Gamma(\alpha))e^{-x/\theta}x^{\alpha-1}$ with α known, that $T_j = \sum_1^j X_i$ is a sufficient statistic for θ for a sample of size j. The terminal decision rule can be restricted to functions of the sufficient statistic.

(b) To show that the maximal invariant, $\mathbf{Y}_j = (Y_2,\ldots,Y_j) = (X_2/X_1,\ldots,X_j/X_1)$, and T_j are independent, it is easier to use Basu's Theorem (the next exercise) but here is a direct demonstration. The change of variable from $X_1,\ldots,X_j$ to $X_1,Y_2,\ldots,Y_j$ has Jacobean x_1^{j-1}, so the joint density of $X_1,Y_2,\ldots,Y_j$ is

$$f(x_1,y_2,\ldots,y_j) = \frac{1}{\Gamma(\alpha)^j\theta^{j\alpha}}\exp\{-\frac{1}{\theta}\,x_1(1+t_2\cdots+t_j)\}x_1^{j(\alpha-1)+(j-1)}\prod_2^j y_i^{\alpha-1}.$$

Now make the change of variable $T_j = X_1(1+Y_2+\cdots+Y_j)$ for X_1 (Jacobean $1/(1+y_2+\cdots+y_j)$) and find the joint density of $T_j,Y_2,\ldots,Y_j$ to be

$$f(t,y_2,\ldots,y_j) = \frac{1}{\Gamma(\alpha)^j\theta^{j\alpha}}\exp\{-\frac{t}{\theta}\}\,t^{j\alpha-1}(1+y_2+\cdots+y_j)^{-j\alpha}\prod_2^j y_i^{\alpha-1}.$$

Since this factors into a function of t and $(y_2,\ldots,y_j)$, it follows that T_j and $\mathbf{Y}_j$ are independent. Hence under the hypotheses of Theorem 4, the best invariant rule is a fixed sample size rule.

(c) The best invariant terminal rules are $d_j(T_j) = T_j/b$ where b is chosen to minimize $\mathrm{E}_1(1-(T_j/b)^2)^2$, where E_1 refers to expectation when $\theta = 1$. This leads to $b^2 = \mathrm{E}_1 T_j^4/\mathrm{E}_1 T_j^2$. Since $T_j \in \mathcal{G}(j\alpha,1)$ when $\theta = 1$, we have $b^2 = (j\alpha)(j\alpha+1)(j\alpha+2)(j\alpha+3)/[j\alpha(j\alpha+1)] = (j\alpha+2)(j\alpha+3)$. The minimum terminal loss is

$$\rho_j = \mathrm{E}_1(1-(T_j/b)^2)^2 = 1 - \frac{(j\alpha(j\alpha+1)}{(j\alpha+2)(j\alpha+3)} = \frac{4j\alpha+6}{(j\alpha+2)(j\alpha+3)}.$$

In summary, the best invariant rule takes a fixed sample of size J, where J is that j that minimizes $\rho_j + jc$, and then estimates θ to be $T_J/\sqrt{(j\alpha+2)(j\alpha+3)}$. Through some oversight, the value of α was not given so numerical values of the rule cannot be found. If $\alpha = 1$ and $c = 1/60$, then numerical computation gives $J = 12$.

7.4.7. Let $g(\mathbf{Y})$ be an arbitrary function of $\mathbf{Y}$. Since the distribution of $\mathbf{Y}$ does not depend on θ, $\mathrm{E}(g(\mathbf{Y}))$ does not depend on θ. By sufficiency $\mathrm{E}(g(\mathbf{Y})|T)$ does not depend on θ, so

$$\begin{aligned}\mathrm{E}_\theta[\mathrm{E}(g(\mathbf{Y}|T)) - \mathrm{E}(g(\mathbf{Y}))] &= \mathrm{E}_\theta[\mathrm{E}(g(\mathbf{Y}|T)] - \mathrm{E}(g(\mathbf{Y})) \\ &= \mathrm{E}(g(\mathbf{Y})) - \mathrm{E}(g(\mathbf{Y})) = 0\end{aligned}$$

for all θ. Since the sufficient statistic is complete, we have $\mathrm{E}(g(\mathbf{Y})|T) = \mathrm{E}(g(\mathbf{Y}))$ with probability one. This shows that T and $\mathbf{Y}$ are independent.