Solution to Exercises 6.3.3 through 6.3.5.

6.3.3. (a) The joint density of $Y_1,\ldots,Y_{k-1}$ under H_i for $i \neq 0$ is given by (6.22). Let $Z_1 = |Y_1|, Z_2 = Y_2\text{sgn}Y_1,\ldots,Z_{k-1} = Y_{k-1}\text{sgn}Y_1$. This is a two-to-one map and the density of $Z_1,\ldots,Z_{k-1}$ is given for $z_1 > 0$ by

$$\begin{aligned} f_{Z_1,\ldots,Z_{k-1}}(z_1,\ldots,z_{k-1}) &= f_{Y_1,\ldots,Y_{k-1}}(z_1,\ldots,z_{k-1}) + f_{Y_1,\ldots,Y_{k-1}}(-z_1,\ldots,-z_{k-1}) \\ &= \frac{1}{\sqrt{k}(2\pi)^{(k-1)/2}}\exp\{-\frac{k}{2}s_z^2 - \frac{\Delta^2(k-1)}{2k}\}[\exp\{\Delta(z_i-\bar{z})\} + \exp\{-\Delta(z_i-\bar{z})\}] \end{aligned}$$

where $s_z^2 = (1/k)\sum_1^k(z_j-\bar{z})$ and $z_k = 0$.

(b) The term in square brackets is just $2\cosh(\Delta(z_i-\bar{z}))$ and since $\cosh(x)$ is symmetric in x, the density of the Z's depends only on $|\Delta| = \Delta_0 > 0$. Since it is increasing on $[0,\infty)$, we have

$$\phi(0|x) = 1 \qquad \text{if} \qquad \max_j |z_j - \bar{z}| \geq c$$

and $\phi(0|x) = 0$ otherwise. But since $|z_i - \bar{z}| = |x_i - \bar{x}|$ for all i, we have

$$\phi(0|x) = \begin{cases} 1 & \text{if } \max_j |x_j-\bar{x}| \leq c \\ 0 & \text{otherwise} \end{cases}$$

and

$$\phi(i|x) = \begin{cases} 1 & \text{if } |x_i-\bar{x}| = \max_j |x_j-\bar{x}| > c \\ 0 & \text{otherwise} \end{cases}$$

where c is chosen so that P(accept $H_0|H_0) = 1-\alpha$. This test is best out of the invariant rules satisfying (a) and (b) of Theorem 1 and does not depend on Δ_0.

6.3.4. (a) The joint distribution of $X_1,\ldots,X_k$ under H_i has density

$$f_{X_1,\ldots,X_k}(x_1,\ldots,x_k|i) = \frac{1}{\Gamma(\alpha)^k\beta^{k\alpha}}\exp\{-\frac{1}{\beta}\sum x_j\}(\prod x_j)^{\alpha-1}\cdot\frac{1}{\lambda}\exp\{\frac{x_i(\lambda-1)}{\beta\lambda}\}$$

where all $x_j > 0$. We make the change of variable $Z_j = X_j/X_k$ for $j = 1,\ldots,k-1$ and $Y = X_k$, and we let Z_k be a dummy variable equal to 1. The inverse transformation is $X_j = YZ_j$ for $j = 1,\ldots,k$ with Jacobian y^{k-1}. The joint density of $Z_1,\ldots,Z_{k-1},Y$ is

$$f_{Z_1,\ldots,Z_{k-1},Y}(z_1,\ldots,z_{k-1},y|i) = \frac{1}{\Gamma(\alpha)^k\beta^{k\alpha}}\exp\{-\frac{y}{\beta}\sum z_j\}(\prod z_j)^{\alpha-1}\cdot\frac{1}{\lambda}\exp\{\frac{y}{\beta}\frac{z_i(\lambda-1)}{\lambda}\}y^{k\alpha-1}$$

Integrating out y over $(0,\infty)$, we find the joint density of the maximal invariant to be

$$f_{Z_1,\ldots,Z_{k-1}}(z_1,\ldots,z_{k-1}|i) = \frac{1}{\lambda\Gamma(\alpha)^k}(\prod z_j)^{\alpha-1}(\sum z_j - \frac{z_i(\lambda-1)}{\lambda})^{-k\alpha}$$

for all $z_j > 0$.

(b) Out of the class of invariant rules satisfying conditions (a) and (b) of Theorem 1, the rule of (6.18) and (6.19) with $X_1,\ldots,X_k$ replaced by $Z_1,\ldots,Z_k$, maximizes the common value of P(accept $H_i|H_i$). Here,

$$V = \max_i \frac{f_{Z_1,\ldots,Z_{k-1}}(z_1,\ldots,z_{k-1}|i)}{f_{Z_1,\ldots,Z_{k-1}}(z_1,\ldots,z_{k-1}|0)} = \max_i\left(1 - \frac{z_i(\lambda-1)}{\sum z_j\lambda}\right)^{-k\alpha} > c$$

if and only if $\max_i(z_i/\sum z_j) > c'$ for some c'. Replacing z_j by x_j and letting $M = \max_i x_i/\sum x_j$, we find the optimal rule to be

$$\phi(0|x) = \begin{cases} 1 & \text{if } M \leq c' \\ 0 & \text{otherwise} \end{cases}$$

and

$$\phi(i|x) = \begin{cases} 1 & \text{if } x_i/\sum x_j = M > c' \\ 0 & \text{otherwise.} \end{cases}$$

where c' is chosen so that condition (a) is satisfied. This rule is independent of λ provided $\lambda > 1$.

6.3.5. Assume that Δ is known and $\Delta > 0$. The hypotheses are all simple and the invariant priors are of the form $\tau_p(H_0) = 1 - pk(k-1)/2$, and $\tau_p(H_{h,i}) = p$ for some $0 < p \leq 2/(k(k-1))$. Under H_0, the density of the observations is $\prod_1^k \varphi(x_j)$ where $\varphi(x)$ is the density of $\mathcal{N}(0,1)$. Under $H_{h,i}$, this density is $\left(\prod_1^k \varphi(x_j)\right)(\varphi(x_h - \Delta)/\varphi(x_h))(\varphi(x_i - \Delta)/\varphi(x_i))$. The ratio $\varphi(x-\Delta)/\varphi(x)$ is equal to $\exp\{\Delta x - \Delta^2/2\}$. Assuming zero/one loss, the Bayes rule reduces to the following.

$$\phi(0|x) = \begin{cases} 1 & \text{if } \exp\{M - \Delta^2\} \leq (1/p) - (k(k-1)/2) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\phi(h,i|x) = \begin{cases} 1 & \text{if } \exp\{\Delta(x_h + x_i) - \Delta^2\} = \exp\{M - \Delta^2\} > (1/p) - (k(k-1)/2) \\ 0 & \text{otherwise} \end{cases}$$

where $M = \max_{h \neq i} \Delta(x_h + x_i)$. As in the proof of Theorem 1, we may replace the property of Bayesian optimality by Neyman-Pearson type optimality. Assume that $\Delta > 0$ and let $M' = \max_{h \neq i}(x_h + x_i)$. Out of all rules satisfying (a) P(accept $H_0|H_0) \geq 1 - \alpha$, and (b) P(accept $H_{h,i}|H_{h,i}$) is independent of h and i, the rule,

$$\phi(0|x) = \begin{cases} 1 & \text{if } M' \leq c \\ 0 & \text{otherwise} \end{cases}$$

and

$$\phi(h,i|x) = \begin{cases} 1 & \text{if } (x_h + x_i) = M' > c \\ 0 & \text{otherwise} \end{cases}$$

where c is chosen so that P(accept $H_0|H_0) = 1 - \alpha$, maximizes the common value of P(accept $H_{h,i}|H_{h,i}$) subject to (a) and (b). This rule is independent of Δ provided $\Delta > 0$.