Solutions to Exercises 5.10.1 through 5.10.6.

5.10.1. Let $\mathbf{x} \in E_r$, $\theta \in E_r$ $c_0 \geq 0$, and $\Sigma > 0$.

(a) *If.* Putting $\mathbf{a} = \Sigma^{-1}(\mathbf{x}-\theta)$ into the right side gives $((\mathbf{x}-\theta)^T\Sigma^{-1}(\mathbf{x}-\theta))^2 \leq c_0(\mathbf{x}-\theta)^T\Sigma^{-1}(\mathbf{x}-\theta)$. If $(\mathbf{x}-\theta)^T\Sigma^{-1}(\mathbf{x}-\theta) \neq 0$, then this may be cancelled from both sides, resulting in the desired inequality. If $(\mathbf{x}-\theta)^T\Sigma^{-1}(\mathbf{x}-\theta) = 0$, the desired inequality is obviously true.
Only if. By Schwartz inequality, $(\mathbf{a}^T(\mathbf{x}-\theta))^2 = ((\mathbf{a}^T\Sigma^{1/2})(\Sigma^{-1/2}(\mathbf{x}-\theta))^2 \leq (\mathbf{a}^T\Sigma\mathbf{a})(((\mathbf{x}-\theta)^T\Sigma^{-1}(\mathbf{x}-\theta))$. Hence, $(\mathbf{a}^T(\mathbf{x}-\theta))^2 \leq c_0(\mathbf{a}^T\Sigma\mathbf{a})$.

(b) *If.* Since $1^T(\mathbf{x}-\theta) = 0$, we may put $\mathbf{a} = (\mathbf{x}-\theta)$ into the right side and cancel.
Only if. By Schwartz, $(\mathbf{a}^T(\mathbf{x}-\theta))^2 \leq (\mathbf{a}^T\mathbf{a})((\mathbf{x}-\theta)^T(\mathbf{x}-\theta))$ etc.

(c) *If.* Since $1^T\Sigma^{-1}(\mathbf{x}-\theta) = 0$, we may put $\mathbf{a} = \Sigma^{-1}(\mathbf{x}=\theta)$ into the right side and cancel.
Only if. By Schwartz, $(\mathbf{a}^T(\mathbf{x}-\theta))^2 \leq (\mathbf{a}^T\Sigma\mathbf{a})((\mathbf{x}-\theta)^T\Sigma^{-1}(\mathbf{x}-\theta))$ etc.

5.10.2. The confidence ellipsoid (5.133) is

$$\{\mu : \sum_{i=1}^{I}(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot} - \mu_i)^2 \leq (I-1)\hat{\sigma}^2 F_{(I-1),IJ-I-J+1;\alpha}/J\}$$

Since $\sum_i(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot} - \mu_i) = 0$, Exercise 1(b) states this is equivalent to the multiple comparison

$$(\sum_{i=1}^{I} a_i(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot} - \mu_i))^2 \leq (\sum_{i=1}^{I} a_i^2)(I-1)\hat{\sigma}^2 F_{(I-1),IJ-I-J+1;\alpha}/J$$

for all vectors $\mathbf{a}$ for which $\sum_i a_i = 0$. But since $\sum_i a_i\overline{X}_{\cdot\cdot} = 0$, this reduces to (5.143).

5.10.3. (a) The UMP invariant confidence set of confidence coefficient $1-\alpha$, given by (5.132), reduces to

$$\{\theta : \sum_i\sum_j(\overline{X}_{i\cdot} - \theta_i)^2 \leq I\hat{\sigma}^2 F_{I,N-I;\alpha}\}$$

where $N = \sum_i n_i$, and $\hat{\sigma}^2 = (1/(N-I))\sum\sum(X_{ij} - \overline{X}_{i\cdot})^2$. The summation over j in the confidence ellipsoid may be replaced by n_i, and the expression has the form of Exercise 1(a) with Σ^{-1} being the diagonal matrix with $n_1, \ldots, n_I$ down the diagonal. This exercise states that the confidence ellipsoid is equivalent to the multiple comparison

$$(\sum_i a_i(\overline{X}_{i\cdot} - \theta_i))^2 \leq (\sum_i a_i^2/n_i)I\hat{\sigma}^2 F_{I,N-I;\alpha}$$

for all vectors $\mathbf{a}$.

(b) The confidence ellipsoids reduce to

$$\{\mu : \sum_i n_i(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot} - \mu_i)^2 \leq (I-1)\hat{\sigma}^2 F_{I-1,N-I;\alpha}\}.$$

Here $\sum_i(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot} - \mu_i) = 0$, so that we may apply Exercise 1(c) to obtain the multiple comparison statement

$$(\sum_i a_i\overline{X}_{i\cdot} - \sum_i a_i\mu_i)^2 \leq (\sum_i a_i^2/n_i)(I-1)\hat{\sigma}^2 F_{I-1,N-I;\alpha}.$$

for all $\mathbf{a}$ such that $\sum_i a_i = 0$.

5.10.4. (a) The multiple comparison given by (5.140) is

$$(\sum_i a_i\overline{X}_{i\cdot} - \sum_i a_i\theta_i)^2 \leq (\sum_i a_i^2)(I-1)\hat{\sigma}^2 F_{I-1,IJ-I;\alpha}/J = (\sum_i a_i^2)5F_{3,20;\alpha}$$

for all $\mathbf{a}$ such that $\sum_i a_i = 0$. When $\alpha = .10$ and $\sum_i a_i^2 = 2$, the square root of the right side reduces to $\sqrt{23.8} = 4.88\cdots$. Only $\overline{X}_{1\cdot} - \overline{X}_{3\cdot}$ and $\overline{X}_{1\cdot} - \overline{X}_{4\cdot}$ are further apart than this. So the ordering is

$$\theta_1 \quad < \quad \theta_2 \quad < \quad \theta_3 \quad < \quad \theta_4$$

(b) When $\alpha = .01$, the square root of the right side is $\sqrt{49.4} = 7.03$, so this time all confidence intervals for the differences, $\theta_i - \theta_j$, contain the origin.

(c) To compute the F-test, we first compute $\overline{X}_{\cdot\cdot} = (0+4+5+7)/4 = 4$ and then $\sum\sum(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot})^2 = 6[4^2+0^2+1^2+3^2] = 156$. The F-statistic is $F = (156/3)/10 = 5.2$. This is greater than $F_{3,20;.01} = 4.94$ so H_0 is rejected.

(d) It may be seen that the worst case occurs when the a_i are proportional to the $\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot}$, namely $a_1 = -4$, $a_2 = 0$, $a_3 = 1$ and $a_4 = 3$. For this linear combination, we get the confidence region $(3\theta_4 + \theta_3 - 4\theta_1 - 26)^2 \le (\sum a_i^2) 5F_{I-1,IJ-I;\alpha} = (26)24.2 = 629.2$. This leads to the confidence interval $.9 \le 3\theta_4 + \theta_3 - 4\theta_1 \le 51.1$.

5.10.5. (a) The sum of squares to be minimized is $S(\beta) = \sum(X_i - \beta_0 - \beta_1 z_i - \beta_2 z_i)^2$. The formulas are somewhat simpler if we take $\sum_i z_i^2 = n$ instead of $\sum_i z_i^2 = 1$. Let $m_4 = (1/n)\sum_i z_i^4$ and assume $m_4 > 1$. Setting the three partial derivatives equal to zero, we have $\hat\beta_0 + \hat\beta_2 = \overline{X}$, $\hat\beta_1 = s_{xz}$ and $\hat\beta_0 + m_4\hat\beta_2 = s_{xz^2}$. Solving simultaneously, we find

$$\hat\beta_1 = s_{xz}, \quad \hat\beta_2 = (s_{xz^2} - \overline{X})/(m_4 - 1), \quad \text{and} \quad \hat\beta_0 = \overline{X} - \hat\beta_2.$$

(b) With β_1 and β_2 fixed, the least squares estimate of β_0 is $\hat{\hat\beta}_0 = \overline{X} - \beta_2$. Hence the numerator sum of squares for the F is

$$\begin{aligned}
&\sum(\hat\beta_0 + \hat\beta_1 z_i + \hat\beta_2 z_i^2 - \hat{\hat\beta}_0 - \beta_1 z_i - \beta_2 z_i^2)^2 \\
&= \sum((\hat\beta_1 - \beta_1)z_i + (\hat\beta_2 - \beta_2)(z_i^2 - 1))^2 \\
&= n(\hat\beta_1 - \beta_1)^2 + n(m_4 - 1)(\hat\beta_2 - \beta_2)2
\end{aligned}$$

The confidence ellipsoid for (β_1, β_2) then becomes

$$(\hat\beta_1 - \beta_1)^2 + (m_4 - 1)(\hat\beta_2 - \beta_2)2 \le (2/n)\hat\sigma^2 F_{2,n-3;\alpha}$$

where $\hat\sigma^2 = S(\hat\beta)/(n-3)$.

(c) The multiple comparison statements become

$$(a_1\beta_1 + a_2\beta_2 - (a_1\hat\beta_1 + a_2\hat\beta_2))^2 \le (a_1^2 + \frac{a_2^2}{m_4 - 1})(2/n)\hat\sigma^2 F_{2,n-3;\alpha}$$

for all (a_1, a_2).

5.10.6. The F-test statement is

$$\mathrm{P}\{\sum\sum(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot} - \mu_i)^2 + \sum\sum \overline{X}_{\cdot j} - \overline{X}_{\cdot\cdot} - \eta_j)^2 \le c\hat\sigma^2\} = 1 - \alpha,$$

where $c = (I+J-2)F_{I+J-2,(I-1)(J-1);\alpha}$ and $\hat\sigma^2 = \frac{1}{(I-1)(J-1)}\sum\sum(X_{ij} - \overline{X}_{i\cdot} - \overline{X}_{\cdot j} + \overline{X}_{\cdot\cdot})^2$. By the argument given for the final example in the text, this probability statement is equivalent to the statement that the probability is $1 - \alpha$ that

$$(\sum a_i\overline{X}_{i\cdot} + \sum b_j\overline{X}_{\cdot j} - \sum a_i\mu_i - \sum b_j\eta_j)^2 \le c\hat\sigma^2(\frac{1}{I}\sum a_i^2 + \frac{1}{J}\sum b_j^2)$$

for all $\mathbf{a} \in E_I$ and all $\mathbf{b} \in E_J$ such that $\sum a_i = 0$ and $\sum b_j = 0$.