Solutions to Exercises 5.8.2 through 5.8.4 and 5.8.7.

5.8.2. Let $\varphi(x)$ denote the density of the standard normal distribution and let $\theta_1 < \theta_2$. The likelihood ratio is

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \begin{cases} \varphi(x-\theta_2+1)/\varphi(x-\theta_1+1) & \text{if } \theta_1 < \theta_2 < 0 \\ \varphi(x-\theta_2)/\varphi(x-\theta_1+1) & \text{if } \theta_1 < \theta_2 = 0 \\ \varphi(x-\theta_2-1)/\varphi(x-\theta_1+1) & \text{if } \theta_1 < 0 < \theta_2 \\ \varphi(x-\theta_2-1)/\varphi(x-\theta_1) & \text{if } \theta_1 = 0 < \theta_2 \\ \varphi(x-\theta_2-1)/\varphi(x-\theta_1-1) & \text{if } 0 < \theta_1 < \theta_2 \end{cases}$$

In all cases, this is increasing in x, so X has monotone likelihood ratio. The acceptance region of the UMP size α test of $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$ is the interval,

$$A(\theta_0) = \begin{cases} (-\infty, \theta_0 - 1 + z_\alpha] & \text{if } \theta_0 < 0 \\ (-\infty, \theta_0 + z_\alpha] & \text{if } \theta_0 = 0 \\ (-\infty, \theta_0 + 1 + z_\alpha] & \text{if } \theta_0 > 0 \end{cases}$$

Therefore, the level $1-\alpha$ family of confidence sets,

$$S(x) = \begin{cases} [x + 1 - z_\alpha, \infty) & \text{if } x \le z_\alpha - 1 \\ [0, \infty) & \text{if } z_\alpha - 1 < x \le z_a \\ (0, \infty) & \text{if } z_\alpha < x \le z_\alpha + 1 \\ [x - 1 - z_\alpha, \infty) & \text{if } z_\alpha + 1 < x \end{cases}$$

has the property that $\mathrm{P}_{\theta'}\{S(X) \text{ contains } \theta\}$ is uniformly minimum for $\theta < \theta'$.

5.8.3. By Exercise 5.2.7(d), $A(\theta_0) = (\theta_0 \sqrt[n]{\alpha}, \theta_0]$, is the acceptance region of a UMP test of size α for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. By Theorem 1, the family of confidence regions, $S(x) = [x, x/\sqrt[n]{\alpha})$ is a UMA family of confidence intervals for θ, at level $1-\alpha$.

5.8.4. By making the change of variable $X_i' = X_i - \theta_0$ in Exercise 5.4.7, we see that

$$A(\theta_0) = \{(\mathbf{X}, \mathbf{Y}) : |\overline{X} - \overline{Y} - \theta_0| < s\sqrt{\frac{1}{m} + \frac{1}{n}} t_{m+n-2;\alpha/2}\}$$

is the acceptance region of a UMP unbiased test of size α for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. By Theorem 2, the family of confidence intervals,

$$S(\mathbf{X}, \mathbf{Y}) = (\overline{X} - \overline{Y} - s\sqrt{\frac{1}{m} + \frac{1}{n}} t_{m+n-2;\alpha/2}, \overline{X} - \overline{Y} + s\sqrt{\frac{1}{m} + \frac{1}{n}} t_{m+n-2;\alpha/2})$$

is a UMA unbiased family of confidence intervals for θ, at level $1-\alpha$.

5.8.7. The density of X is $f_X(x|\theta) = \exp\{\pi(\theta)\}\exp\{-x\}\mathrm{I}(\pi(\theta) < x)$ where

$$\pi(\theta) = \begin{cases} \theta - 1 & \text{if } \theta < 0 \\ 0 & \text{if } \theta = 0 \\ \theta + 1 & \text{if } \theta > 0 \end{cases}$$

The negation of X has a distribution of the form given in the solution of Exercise 5.2.7(e). Using that exercise, we may generalize the problem to a sample $X_1, \ldots, X_n$ from this distribution, with sufficient statistic, $T = \max(X_1, \ldots, X_n)$. The region $A(\theta_0) = \{\mathbf{x} : \pi(\theta) < t < b\}$ is the acceptance region of a UMP size α test of $\theta = \theta_0$ against $\theta \neq \theta_0$, provided b satisfies $\exp(\pi(\theta_0)) \int_b^\infty \exp(-x)\,dx = \sqrt[n]{\alpha}$. Solving for b, we find $b = \pi(\theta_0) - \log(\alpha)/n$. This leads to acceptance regions which are intervals of t,

$$A(\theta_0) = \begin{cases} (\theta_0 - 1, \theta_0 - 1 + c) & \text{if } \theta_0 < 0 \\ (0, c) & \text{if } \theta_0 = 0 \\ (\theta_0 + 1, \theta_0 + 1 + c) & \text{if } \theta_0 > 0 \end{cases}$$

where $c = -\log(\alpha)/n$. Suppose $c < 1$. Then the associated level $1-\alpha$ UMA confidence sets for θ are

$$S(\mathbf{x}) = \begin{cases} (t+1-c, t+1) & \text{if } t \le -1 \\ (t+1-c, 0) & \text{if } -1 < t < -1+c \\ \text{empty} & \text{if } -1+c \le t \le 0 \\ \{0\} & \text{if } 0 < t < c \\ \text{empty} & \text{if } c \le t \le 1 \\ (0, t-1) & \text{if } 1 < t < 1+c \\ (t-1-c, t-1) & \text{if } t \ge 1+c \end{cases}$$

The cases $1 \le c < 2$ and $c \ge 2$ lead to

$$S(\mathbf{x}) = \begin{cases} (t+1-c, t+1) & \text{if } t \le -1 \\ (t+1-c, 0) & \text{if } -1 < t \le 0 \\ (t+1-c, 0] & \text{if } 0 < t \le -1+c \\ \{0\} & \text{if } -1+c < t < 1 \\ [0, t-1) & \text{if } 1 \le t < c \\ (0, t-1) & \text{if } c \le t < 1+c \\ (t-1-c, t-1) & \text{if } t \ge 1+c \end{cases} \quad \text{and} \quad S(\mathbf{x}) = \begin{cases} (t+1-c, t+1) & \text{if } t \le -1 \\ (t+1-c, 0) & \text{if } -1 < t \le 0 \\ (t+1-c, 0] & \text{if } 0 < t \le 1 \\ (t+1-c, t-1+c) & \text{if } 1 < t < -1+c \\ [0, t-1+c) & \text{if } -1+c \le t < c \\ (0, t-1+c) & \text{if } c \le t < 1+c \\ (t-1-c, t-1+c) & \text{if } t \ge 1+c \end{cases}$$

respectively. The case $n = 1$ and $\alpha = e^{-1/2}$ leads to $c = 1/2$. The fact that the confidence interval can be empty (with probability about .238 when $\theta = 0$) underscores the absurdity of trying to interpret a confidence interval as an estimate.