Solutions to Exercises 5.6.1 through 5.6.8, and 5.6.12.

5.6.1. This problem is invariant under a change of location, $g_c(x_1, x_2) = (x_1+c, x_2+c)$, and a maximal invariant is $Y = X_1 - X_2$. Under H_0, Y has a $\mathcal{N}(0,2)$ distribution with density

$$f_0(y) = \frac{1}{\sqrt{4\pi}} e^{-y^2/4}.$$

Under H_1, Y has a $\mathcal{C}(0,2)$ distribution with density

$$f_1(y) = \frac{2}{\pi(y^2+4)}.$$

The uniformly most powerful invariant rule for testing H_0 against H_1 is the Neyman-Pearson rule that rejects H_0 if the ratio

$$f_0(y)/f_1(y) = \frac{\sqrt{\pi}}{4}(y^2+4)e^{-y^2/4}$$

is too small. This ratio is a function of y^2 and so is symmetric about zero. Moreover, it is a decreasing function of y^2 on the positive axis, as may be checked by taking derivatives. Thus, the best invariant rule rejects H_0 if y^2 is too large, or equivalently if $|X_1 - X_2| > K$ for some constant K. This shows that

$$\phi(x_1, x_2) = \begin{cases} 0 & \text{if } |x_1 - x_2| < K \\ 1 & \text{if } |x_1 - x_2| > K \end{cases}$$

is a UMP invariant test of H_0 against H_1.

5.6.2. We must find the joint density of $\mathbf{Y} = (Y_1, \ldots, Y_{n-1})$ under H_0 and H_1, where $Y_i = X_i - X_n$ for $i = 1, \ldots, n-1$. Under H_0, this is the multivariate normal density,

$$f_0(\mathbf{y}) = (2\pi)^{-n/2}(\det \Sigma)^{-1/2} \exp\{-(1/2)\mathbf{y}^T\Sigma^{-1}\mathbf{y}\}$$

where Σ is the covariance matrix (3.34) of the text. To find the density of $\mathbf{Y}$ under H_1, we may put $\theta = 0$ and first find the joint density of $Y_1, \ldots, Y_{n-1}, X_n$ as

$$f_1(y_1, \ldots, y_{n-1}, x_n) = \exp\{-\sum_1^{n-1}[e^{(y_i+x_n)} + (y_i + x_n)] + e^{x_n} + x_n\}$$

Then we integrate out the variable x_n using the change of variable $u = \exp\{x_n\}$.

$$\begin{aligned} f_1(\mathbf{y}) &= \exp\{\sum_1^{n-1} y_i\} \int \exp\{-e^{-x_n}[1 + \sum_1^{n-1} e^{y_i}] + nx_n\}\, dx_n \\ &= \exp\{\sum_1^{n-1} y_i\} \int_0^\infty \exp\{-u[1 + \sum_1^{n-1} e^{y_i}]\}u^{n-1}\, du \\ &= \exp\{\sum_1^{n-1} y_i\}\Gamma(n)/[1 + \sum_1^{n-1} e^{y_i}]^n \end{aligned}$$

Therefore, the best invariant test of H_0 against H_1 rejects H_0 if

$$W = \exp\{\sum_1^{n-1} y_i\} \exp\{-(1/2)\mathbf{y}^T\Sigma^{-1}\mathbf{y}\}/[1 + \sum_1^{n-1} e^{y_i}]^n$$

is too large, where too large refers to the distribution of W under H_0.

5.6.3. Sufficiency reduces the problem to the consideration of the sufficient statistic, $(\overline{X}_n, S^2)$, where $S^2 = \sum_1^n (X_i - \overline{X})^2$. The problem is invariant under location changes, which in terms of the sufficient statistic becomes transformations of the form $g_c(\overline{X}, S^2) = (\overline{X} + c, S^2)$ for arbitrary real c. The maximal invariant is just S^2, whose distribution depends only on σ^2. The problem of finding a UMP invariant test of H_0 against H_1 reduces to the problem of finding a UMP test of H_0 against H_1 based on S^2. But since the distribution of S^2 has monotone likelihood ratio in σ^2 (in fact, $S^2 \in \sigma^2\chi^2_{n-1}$, an exponential family of distributions), the usual test that rejects H_0 for small values of S^2 is UMP invariant for this problem.

5.6.4. Using the form of the density of the noncentral t_ν-distribution given by (3.17), we have

$$\frac{f_T(t|\delta)}{f_T(t|0)} \propto \exp\{-\frac{\nu\delta^2}{2(t^2+\nu)}\} \int_0^\infty \exp\{-\frac{1}{2}(x - \frac{\delta t}{\sqrt{t^2+\nu}})^2\} x^\nu\, dx$$
$$= \exp\{-\frac{\delta^2}{2}\} \int_0^\infty \exp\{-\frac{x^2}{2} + \frac{\delta t x}{\sqrt{t^2+\nu}}\} x^\nu\, dx$$

Since $t/\sqrt{t^2+\nu}$ is increasing in t, this ratio is increasing in t for fixed $\delta > 0$.

5.6.5. Using $f_{|T|}(t|\delta) = f_T(t|\delta) + f_T(-t|\delta)$ for $t > 0$, we have

$$\frac{f_{|T|}(t|\delta)}{f_{|T|}(t|0)} \propto \exp\{-\frac{\nu\delta^2}{2(t^2+\nu)}\} \left[\int_0^\infty \exp\{-\frac{1}{2}(x - \frac{\delta t}{\sqrt{t^2+\nu}})^2\} x^\nu\, dx + \int_0^\infty \exp\{-\frac{1}{2}(x - \frac{\delta t}{\sqrt{t^2+\nu}})^2\} x^\nu\, dx\right]$$
$$= \exp\{-\frac{\delta^2}{2}\} \int_0^\infty \left[\exp\{\frac{\delta t x}{\sqrt{t^2+\nu}}\} + \exp\{-\frac{\delta t x}{\sqrt{t^2+\nu}}\}\right] \exp\{-\frac{x^2}{2}\} x^\nu\, dx$$
$$= \exp\{-\frac{\delta^2}{2}\} \int_0^\infty 2\cosh(\frac{\delta t x}{\sqrt{t^2+\nu}}) \exp\{-\frac{x^2}{2}\} x^\nu\, dx$$

Since $t/\sqrt{t^2+\nu}$ is increasing in t, and since $\cosh(x)$ is increasing in x for $x > 0$, this ratio is increasing in t for fixed $\delta > 0$.

5.6.6. The sufficient statistics are $\overline{X}$, $\overline{Y}$, and $S^2 = \sum_1^m (X_i - \overline{X})^2 + \sum_1^n (Y_i - \overline{Y})^2$. Under the general hypothesis, these are independent and have distributions $\mathcal{N}(\mu, \sigma^2/m)$, $\mathcal{N}(\eta, \sigma^2/n)$ and $\sigma^2\chi^2_{m+n-2}$. The problem of testing H_0 against H_1 is invariant under change of location and scale, and a maximal invariant is $T = (\overline{X} - \overline{Y})/\sqrt{S^2/(m+n-2)}$. The distribution of T depends on the parameters, μ, η and σ, only through $\delta = (\mu - \eta)/\sigma$. In fact, under the general hypothesis, T has the noncentral t-distribution, $t_{m+n-2}(\delta)$. The hypotheses become $H_0 : \delta \le 0$, and $H_1 : \delta > 0$. Now using Exercise 5.6.4, we see that the usual t-test that rejects H_0 when T is too large is UMP invariant.

For testing H_0' against H_1', the problem is invariant not only under change of location and scale, but also under multiplication of all observations by minus one. Now, $|T|$ is a maximal invariant whose distribution depends only on $|\delta|$, and the hypotheses become $H_0' : |\delta| = 0$ and $H_1' : |\delta| > 0$. Using Exercise 5.6.5, we see that the usual two-sided t-test that rejects H_0' when $|T|$ is too large is UMP invariant.

5.6.7. Let X and Y be independent with $X \in \mathcal{G}(1, \lambda^{-1})$ and $Y \in \mathcal{G}(1, \mu^{-1})$, and let $\theta = \lambda/\mu$.

(a) The group of transformations, $g_c(x, y) = (cx, cy)$ for $c > 0$ leaves the distributions invariant since cX and cY are independent with $cX \in \mathcal{G}(1, c/\lambda)$ and $cY \in \mathcal{G}(1, c/\mu)$. The induced group of transformations, $\bar{g}_c(\lambda, \mu) = (\lambda/c, \mu/c)$, on the parameter space leaves the hypotheses $H_0 : \theta \le 1$ and $H_1 : \theta > 1$ invariant, so the problem in invariant. $T = Y/X$ is a maximal invariant with distribution depending only on θ. The joint density of X and T is $f_{X,T}(x, t|\lambda, \mu) = \lambda\mu e^{-(\lambda + t\mu)x} x$ for $x > 0$ and $t > 0$. The marginal density of T may be found from this to be $f_T(t|\theta) = \theta/(\theta + t)^2$ for $t > 0$. Here θ is a scale parameter for the distribution of T so that the UMP invariant test is to reject H_0 if $T > c$. To find c so that the test has size α, we solve

$$\alpha = \int_c^\infty \frac{\theta}{(\theta + t)^2}\, dt = \frac{\theta}{\theta + t}$$

with $\theta = 1$, and find $c = (1 - \alpha)/\alpha$.

(b) In testing $H_0' : \theta = 1$ against $H_1' : \theta \ne 1$, the distributions are invariant under scale changes as above, and also under the transformation $g(x, y) = (y, x)$, with $\bar{g}(\lambda, \mu) = (\mu, \lambda)$. H_0' and H_1' are left invarint under all these transformations so the problem is invariant. The maximal invariant is $U = \max\{T, 1/T\}$. The density of u is

$$f_U(u\theta) = f_T(u|\theta) + f_T(1/u|\theta)/u^2 = \frac{\theta}{(\theta + u)^2} + \frac{\theta}{(\theta u + 1)^2}$$

for $u > 1$. The distribution depends on θ only through $\max\{\theta, 1/\theta\}$ so we may assume $\theta \geq 1$. Below we show that $f_U(u|\theta)/f_U(u|1)$ is increasing in u. From this we may deduce that the UMP invariant size α test of H_0' against H_1' is to reject if $U > c$, where c satisfies

$$\alpha = \mathrm{P}(U > c|1) = \int_c^\infty \frac{2}{(1+u)^2}\, du = \frac{2}{1+c},$$

namely, $c = (2-\alpha)/\alpha$.

The likelihood ratio is

$$\frac{f_U(u|\theta)}{f_U(u|1)} = \frac{\theta}{2}\left[\left(\frac{u+1}{u+\theta}\right)^2 + \left(\frac{u+1}{\theta u+1}\right)^2\right].$$

We show the derivative of the term in square brackets is positive when $u > 1$ and $\theta > 1$.

$$\begin{aligned}\frac{d}{du}\left[\left(\frac{u+1}{u+\theta}\right)^2 + \left(\frac{u+1}{\theta u+1}\right)^2\right] &= 2\left(\frac{u+1}{u+\theta}\right)\frac{\theta-1}{(u+\theta)^2} + 2\left(\frac{u+1}{\theta u+1}\right)\frac{1-\theta}{(\theta u+1)^2} \\ &= 2(u+1)(\theta-1)\left[\frac{1}{(u+\theta)^3} - \frac{1}{(\theta u+1)^3}\right].\end{aligned}$$

This is positive if $(\theta u+1)^3 > (u+\theta)^3 > 0$, which is positive if $\theta u + 1 > u + \theta$, which is positive if $(\theta-1)u > \theta - 1$, which follows since $u > 1$ and $\theta > 1$.

5.6.8. (In the statement of the problem, θ_0 should be replaced by 0 four times.)

(a) The distributions are invariant under location changes, $g_c(x,y) = (x-c, y-c)$ and the induced group is $\bar{g}_c(\lambda,\mu) = (\lambda-c, \mu-c)$. With $\theta = \lambda - \mu$, the problem of testing $H_0 : \theta \leq 0$ against $H_1 : \theta > 0$ is invariant, and a maximal invariant is $Z = X - Y$ with a distribution depending only on θ. If $f_Z(z|\theta)/f_Z(z|0)$ is nondecreasing in z for $\theta > 0$, then the test that rejects H_0 when $Z > c$ is UMP invariant of size α if c is chosen so that $\mathrm{P}(Z > c|\theta = 0) = \alpha$.

(b) For testing $H_0' : \theta = 0$ against $H_1' : \theta \neq 0$, the problem is invariant in addition under the transformation $g(x,y) = (y,x)$, with $\bar{g}(\lambda,\mu) = (\mu,\lambda)$ and $|Z|$ is a maximal invariant. If $f_{|Z|}(z|\theta)/f_{|Z|}(z|0)$ is nondecreasing in z, then a UMP invariant size α test exists of the form: reject H_o' if $|Z| > c$ where c is chosen so that $\mathrm{P}(|Z| > c|\theta = 0) = \alpha$.

5.6.12. We are given $\Theta = \{-1, 1\}$, $\mathcal{A}$ is the real line and $L(\theta,a) = (\theta - a)^2$. The vector (X_1, X_2) of independent Poisson random variables has mean $(1,2)$ if $\theta = -1$ and mean $(2,1)$ if $\theta = 1$. Since $g(X_1,X_2) = (X_2,X_1)$ is independent Poisson with mean $(2,1)$ if $\theta = -1$ and mean $(1,2)$ if $\theta = 1$, we have $\bar{g}(\theta) = -\theta$. Furthermore, $L(\theta,a) = (\theta-a)^2 = L(\bar{g}(\theta), \tilde{g}(a)) = (-\theta - \tilde{g}(a))^2$ provided $\tilde{g}(a) = -a$.

A maximal invariant satisfies $T(X_1,X_2) = (X_2,X_1)$ and takes a different value on each orbit. The orbits are the singleton sets $\{(x,x)\}$, for $x = 0, 1, \ldots$, and the pairs $\{(x,y),(y,x)\}$ for $x \neq y$. The order statistics, $(X_{(1)}, X_{(2)})$, where $X_{(1)} = \min(X_1,X_2)$ and $X_{(2)} = \max(X_1,X_2)$, form a maximal invariant. We solve the problem conditionally given the order statistics. Since the loss is convex in a for each of these conditional problems, we may restrict attention to the nonrandomized rules.

A nonrandomized decision rule is invariant (equivariant) if $d(x,y) = -d(y,x)$; in particular $d(x,x) = 0$ for all $x = 0, 1, \ldots$. Thus the best invariant rule in the conditional problem is trivially zero on all singleton orbits. Suppose now we are given $(X_{(1)}, X_{(2)}) = (x,y)$ where $x < y$. We choose the value of $d(x,y) = z$, and hence $d(y,x) = -z$, to minimize the conditional risk. Since this risk does not depend on θ, we may use $\theta = 1$ for the computations.

$$\begin{aligned}\mathrm{P}_1((X_1,X_2) = (x,y)|(X_{(1)},X_{(2)}) = (x,y)) &= \frac{\mathrm{P}_1(X_1 = x)\mathrm{P}_1(X_2 = y)}{\mathrm{P}_1(X_1 = x)\mathrm{P}_1(X_2 = y) + \mathrm{P}_1(X_1 = y)\mathrm{P}_1(X_2 = x)} \\ &= \frac{(e^{-2}2^x/x!)\cdot(e^{-1}/y!)}{(e^{-2}2^x/x!)\cdot(e^{-1}/y!) + (e^{-1}/x!)\cdot(e^{-2}2^y/y!)} \\ &= \frac{2^x}{2^x + 2^y}.\end{aligned}$$

Hence, the conditional risk is

$$\mathrm{E}_1\{(1-d(X_1,X_2))^2|(X_{(1)},X_{(2)})=(x,y)\}=(1-z)^2\frac{2^x}{2^x+2^y}+(1+z)^2\frac{2^y}{2^x+2^y}.$$

The value of z that minimizes this is easily found to be $z=d(x,y)=(2^x-2^y)/(2^x+2^y)$. This same formula gives 0 for $x=y$ and its negative for $x>y$. Hence, $d(x_1,x_2)=(2_1^x-2_2^x)/(2_1^x+2_2^x)$ is the best invariant rule.