Solutions to the Exercises of Section 5.4.

5.4.1. Suppose ϕ is an unbiased test of size α that is admissible within the class of unbiased tests. We are to show that ϕ is admissible. Suppose not. Then there is a test ϕ' that is better than ϕ; that is, $\mathrm{E}_\theta\phi'(X) \le \mathrm{E}_\theta\phi(X)$ for every $\theta \in \Theta_0$, and $\mathrm{E}_\theta\phi'(X) \ge \mathrm{E}_\theta\phi(X)$ for every $\theta \in \Theta_1$. But then since $\mathrm{E}_\theta\phi'(X) \le \mathrm{E}_\theta\phi(X) \le \alpha$ for $\theta \in \Theta_0$, and $\mathrm{E}_\theta\phi'(X) \ge \mathrm{E}_\theta\phi(X) \ge \alpha$ for $\theta \in \Theta_1$. Thus, ϕ' is unbiased so that ϕ is not admissible within the unbiased tests. This contradiction completes the proof.

5.4.2. We are given $f(\mathbf{t}|\theta) = c(\theta)h(\mathbf{t})\exp\{\sum_{i=1}^k \theta_i t_i\}$ and $0 \le \phi(\mathbf{t}) \le 1$. Thus,

$$\begin{aligned}
\frac{\partial}{\partial\theta_1}\mathrm{E}_\theta\phi(\mathbf{T}) &= \frac{\partial}{\partial\theta_1}c(\theta)\int \phi(\mathbf{t})h(\mathbf{t})\exp\{\theta'/bft\}\,d\nu(\mathbf{t}) \\
&= (\frac{\partial c(\theta)}{\partial\theta_1}\int \phi(\mathbf{t})h(\mathbf{t})\exp\{\theta'\mathbf{t}\}\,d\nu(\mathbf{t}) + c(\theta)\int \phi(\mathbf{t})h(\mathbf{t})\exp\{\theta'\mathbf{t}\}t_1\,d\nu(\mathbf{t}) \\
&= (\frac{\partial \log c(\theta)}{\partial\theta_1}\mathrm{E}_\theta\phi(\mathbf{T}) + \mathrm{E}_\theta\phi(\mathbf{T})T_1
\end{aligned}$$

We we be finished when we show $\partial c(\theta)/\partial\theta_1 = -\mathrm{E}_\theta T_1$. This is found in Exercise 3.5.2. It also follows from the above equation with $\phi(\mathbf{t}) \equiv 1$.

5.4.3 (a) Let $\vartheta = \lambda - \mu$. We are to test the hypothesis $H_0 : \vartheta \le 1$ vs $H_1 : \vartheta > 1$. Since the joint density of X, Y in terms of ϑ, μ is proportional to $\exp\{-x\vartheta - (x+y)\mu\}\mathrm{I}(x > 0, y > 0)$, we should make the change of variables, $Z = X + Y$ for Y. The joint density of X, Z is proportional to $\exp\{-x\vartheta - z\mu\}\mathrm{I}(x > 0, z - x > 0)$. The posterior density of X given $Z = z$ is proportional to $\exp\{-x\vartheta\}\mathrm{I}(0 < x < z)$, and so is equal to

$$f_{X|Z}(x|z,\vartheta) = \begin{cases} \vartheta(e^{-x\vartheta}/(1-e^{-z\vartheta}))\mathrm{I}(0<x<z) & \text{for } \vartheta \ne 0 \\ (1/z)\mathrm{I}(0<x<z) & \text{for } \vartheta = 0 \end{cases}$$

The UMP test of size α of H_0 vs H_1 conditional on $Z = z$ rejects H_0 if $X < c(z)$, where $c(z)$ is chosen so that

$$\alpha = \int_0^{c(z)} f_{X|Z}(x|z,\vartheta = 1)\,dx = \frac{1 - e^{c(z)}}{1 - e^z}.$$

Solving this for c gives $c(z) = -\log(1 - \alpha(1 - e^{-z}))$. Hence the UMP unbiased test of size $\alpha = .2$ for testing H_0 vs H_1 rejects H_0 if $X < -\log(1 - .2(1 - e^{-(X+Y)}))$, or equivalently, if $1 - e^{-X} < .2(1 - e^{-(X+Y)})$.

(b) In testing $H_0 : \vartheta = 0$ vs $H_1 : \vartheta \ne 0$, we must find the best unbiased test conditional on $Z = X + Y$. This is found using equations (5.73) which reduce, since the conditional distribution of X given $Z = z$ under H_0 is uniform on $(0, z)$, to

$$z_2(z) - z_1(z) = (1-\alpha)z$$
$$(1-\alpha)z^2 = z_2^2 - z_1^2$$

This gives $z_1(z) = (\alpha/2)z$ and $z_2(z) = (1 - (\alpha/2))z$, so the UMP unbiased test of size $\alpha = .2$ may be stated simply as: Reject H_0 if $X/(X+Y) < .1$ or $X/(X+Y) > .9$.

(c) This time, we let $\vartheta = \lambda - 2\mu$ and we are to test $H_0 : \vartheta \ge 0$ vs $H_1 : \vartheta < 0$. The joint density of X, Y is proportional to $\exp\{-x\vartheta - (2x+y)\mu\}\mathrm{I}(x > 0, y > 0)$, so we make the change of variable $Z = 2X + y$ for Y, giving a joint density proportional to $\exp\{-x\vartheta - z\mu\}\mathrm{I}(0 < x < z/2)$. From this the conditional density of X given Z is found to be

$$f_{X|Z}(x|z,\vartheta) = \begin{cases} (\vartheta e^{-x\vartheta}/(1-e^{-z\vartheta/2}))\mathrm{I}(0<x<z/2) & \text{for } \vartheta \ne 0 \\ (2/z)\mathrm{I}(0<x<z/2) & \text{for } \vartheta = 0 \end{cases}$$

The conditional UMP test rejects H_0 if $X > c(z)$ where $c(z)$ is chosen to have conditional size α when $\vartheta = 0$. This is the uniform distribution on $(0, z/2)$, so the UMP unbiased test of size $\alpha = .2$ rejects H_0 if $X/(2X+Y) > .8$.

5.4.4. (a) The joint density of T_1 and T_2 is

$$f_{T_1,T_2}((t_1,t_2|\theta_1,\theta_2) = n(n-1)\frac{(t_2-t_1)^{n-2}}{(\theta_2-\theta_1)^n}\mathrm{I}(\theta_1 < t_1 < t_2 < \theta_2).$$

The marginal density of T_2 is

$$f_{T_2}(t_2|\theta_1,\theta_2) = n\frac{(t_2-\theta_1)^{n-1}}{(\theta_2-\theta_1)^n}\mathrm{I}(\theta_1 < t_2 < \theta_2).$$

The conditional density of T_1 given $T_2 = t_2$ is the ratio of these,

$$f_{T_1|T_2=t_2}(t_1|\theta_1) = (n-1)\frac{(t_2-t_1)^{n-2}}{(t_2-\theta_1)^{n-1}}I(\theta_1 < t_1 < t_2),$$

exactly the density of the minimum of a sample of size $n-1$ from the uniform distribution on (θ_1, t_2).

(b) If $t_2 > 0$, the essentially unique UMP size α test of $H_0 : \theta_1 \le 0$ against $H_1 : \theta_1 > 0$ based on this distribution is

$$\phi(t_1|t_2) = \begin{cases} 1 & \text{if } c(t_2) < t_1 < t_2 \\ 0 & \text{if } t_1 < c(t_2) \end{cases}$$

where $c(t_2)$ is chosen so that the conditional size of the test is α. It is easily found that $c(t_2) = (1-\sqrt[n]{\alpha})t_2$. If $t_2 < 0$, we know H_0 is true. Let us accept H_0 in this case even though this is not a test of size α (if $\alpha > 0$).

(c) The combined test, call it ϕ_0, rejects H_0 if and only if $T_2 > 0$ and $(1-\sqrt[n]{\alpha})T_2 < T_1 < T_2$. The power function of every test is continuous on the parameter space, $\Theta = \{\theta_1 < \theta_2\}$. Therefore, by Theorem 1, it is sufficient to show that ϕ_0 is UMP α-similar and has size α. Under H_0, T_2 is a complete sufficient statistic for θ_2 so by Theorem 2, every similar test has Neyman structure. Then by the argument given at the top of page 229, ϕ_0 is UMP α-similar, no matter how it was defined for $T_2 \le 0$. We have defined it for $T_2 \le 0$ to be zero, so that, for $\theta_1 < 0$,

$$\mathrm{E}_{(\theta_1,\theta_2)}\phi_0(\mathbf{T}) = \mathrm{E}_{(\theta_1,\theta_2)}(\mathrm{E}_{\theta_1}(\phi_0(\mathbf{T}|T_2)) \le \alpha.$$

Thus ϕ_0 is a size α test, and hence is UMP unbiased. (The corresponding test that rejects H_0 with probabilty α whenever $T_2 \le 0$ is also UMP unbiased, but it would not be admissible since ϕ_0 would have smaller risk when $\theta_1 < 0$.)

5.4.5. The vector (X, Y) has a two-parameter exponential family of distributions with an exponent of the form $\exp\{x \log\theta_1 + y \log\theta_2\}$. For both parts (a) and (b), the boundary between H_0 and H_1 is the set $\{\vartheta = 0\}$ where $\vartheta = \theta_1 - \theta_2$. To change the exponential family to the proper form, we make the change of variable $Z = X + Y$ for Y. Then the joint density of X and Z is

$$f_{X,Z}(x,z|\theta_1,\theta_2) = (1-\theta_1)(1-\theta_2)\exp\{(x(\log\theta_1 - \log\theta_2) + z\log\theta_2\} \quad \text{for } z = 0,1,2,\ldots \text{ and } x = 0,\ldots,z,$$

and for $z = 0, 1, 2, \ldots$, the conditional density of X given $Z = z$ when $\vartheta = 0$ is uniform:

$$f_{X|Z=z}(x|\vartheta = 0) = \frac{1}{z+1} \qquad \text{for } x = 0,\ldots,z.$$

the uniform distribution on the integers from 0 to z.

(a) To test $H_0 : \vartheta \le 0$ against $H_1 : \vartheta > 0$, we reject H_0 when x is large. But since the distribution is discrete, we must randomize to obtain size $\alpha = 0.2$. Let q and r be the quotient and remainder when dividing $z+1$ by 5 ($z+1 = 5q+r$). Then it is easy to see that the test,

$$\phi(x,z) = \begin{cases} 1 & \text{for } x > z-q \\ r/5 & \text{for } x = z-q \\ 0 & \text{for } x < z-q, \end{cases}$$

has conditional size 0.2 for all z. Therefore, this test is UMP unbiased.

(b) To test $H_0 : \vartheta = 0$ against $H_1 : \vartheta \neq 0$, We reject H_0 if x is large or small. To obtain an unbiased test, we may use an equal tailed test since the conditional distribution of X given Z is symmetric under H_0. To obtain size 0.1 on each tail, we write $z+1 = 10q + r$ and use a similar calculation to the one of part (a). We arrive at the test

$$\phi(x,z) = \begin{cases} 0 & \text{for } q < x < z-q \\ r/10 & \text{for } x = q \text{ or } x = z-q \\ 1 & \text{for } x < q \text{ or } x > z-q \end{cases}$$

Since the vector (X,Y) has a two-parameter exponential family of distributions with an exponent of the form $\exp\{x\log\theta_1 + y\log\theta_2\}$, there exist UMP unbiased tests of $H_0 : \varphi(\theta_1,\theta_2) = 0$ for any φ of the form $\varphi(\theta_1,\theta_2) = a\log\theta_1 + b\log\theta_2 + c$ for any size α. Equivalently, we may find a UMP unbiased test of any hypothesis of the form $H_0 : \theta_1^a = c\theta_2^b$.

5.4.6. The sufficient statistics, $T_1 = \sum_1^m X_i$ and $T_2 = \sum_1^n Y_j$, form a 2-parameter exponential family of the form (5.63) with $\theta_1 = \mu$ and $\theta_2 = \eta$. The general theory implies there exists a UMP unbiased test of $H_0 : \mu \leq \eta$. To find it, we make the transformation $U_1 = (T_1/m) - (T_2/n)$ and $U_2 = T_1 + T_2$. This is useful because U_1 and U_2 are independent: the covariance is $\mathrm{Cov}(U_1,U_2) = (1/m)\mathrm{Var}(T_1) + ((1/m) - (1/n))\mathrm{Cov}(T_1,T_2) - (1/n)\mathrm{Var}(T_2) = 1 + 0 - 1 = 0$. Moreover, U_1 and U_2 have an exponential family of distributions:

$$\exp\{\theta_1 T_1 + \theta_2 T_2\} = \exp\{\vartheta_1 U_1 + \vartheta_2 U_2\}$$

where $\vartheta_1 = mn(\theta_1 - \theta_2)/(m+n)$ and $\vartheta_2 = (m\theta_1 + n\theta_2)/(m+n)$. The null hypothesis is now $H_0 : \vartheta_1 \leq 0$. The UMP unbiased test of H_0 is the UMP unbiased test conditional on U_2. But since U_1 and U_2 are independent, this reduces to the test that rejects H_0 when U_1 is too large. Since when $\vartheta_1 = 0$, U_1 has a normal distribution with mean 0 and variance $(1/m) + (1/n)$, this test rejects H_0 when $U_1 > z_\alpha\sqrt{(1/m) + (1/n)}$, where z_α is the upper α-cutoff point of the standard normal distribution.

5.4.7. The sufficient statistics, $T_1 = \sum_1^m X_i$, $T_2 = \sum_1^n Y_j$ and $T_3 = \sum_1^m X_i^2 + \sum_1^n Y_j^2$, form a 3-parameter exponential family of the form (5.63) with $\theta_1 = \mu/\sigma^2$, $\theta_2 = \eta/\sigma^2$ and $\theta_3 = -1/\sigma^2$. The hypothesis $\mu = \eta$ is equivalent to the hypothesis $\theta_1 = \theta_2$. Thus we are assured of the existence of a UMP unbiased test of $\mu = \eta$. To find it, we first make the change of variables

$$U_1 = \overline{X} - \overline{Y} = (T_1/m) - (T_2/n), \quad U_2 = T_1 + T_2 \quad \text{and} \quad U_3 = T_3.$$

Solving for T_1, T_2 and T_3 in terms of U_1, U_2 and U_3, we find

$$T_1 = \frac{m}{m+n}(U_2 + nU_1), \quad T_2 = \frac{n}{m+n}(U_2 - mU_1) \quad \text{and} \quad T_3 = U_3.$$

Since this is a linear transformation, U_1, U_2 and U_3 also have a 3-parameter exponential family, and we may solve

$$\exp\{\theta_1 T_1 + \theta_2 T_2 + \theta_3 T_3\} = \exp\{\vartheta_1 U_1 + \vartheta_2 U_2 + \vartheta_3 U_3\}$$

for the natural parameters,

$$\vartheta_1 = mn(\theta_1 - \theta_2)/(m+n) \quad \vartheta_2 = (m\theta_1 + n\theta_2)/(m+n) \quad \text{and} \quad \vartheta_3 = \theta_3.$$

The hypothesis $\mu = \eta$ now becomes $\vartheta_1 = 0$. To show that the usual t-test is UMP unbiased, we must show it is UMP unbiased in each of the conditional problems given U_2 and U_3. When the null hypothesis is satisfied, the t-statistic,

$$T = \frac{\overline{X} - \overline{Y}}{s\sqrt{(1/m) + (1/n)}}$$

has a t_{m+n-2}-distribution. Thus, T has a distribution independent of the parameters ϑ_2 and ϑ_3 when $\vartheta_1 = 0$. For this reason, T is stochastically independent of the sufficient statistics, U_2 and U_3, when

$\vartheta_1 = 0$. (This is known as Basu's Theorem and may be found in Exercise 7.4.7.) Therefore, since the t-test is unconditionally unbiased, it is unbiased in each of the conditional problems (i.e. satisfies the analog of (5.73)). To show it is UMP unbiased in each of these conditional problems, we must show it has the form, analogous to (5.70),

$$\text{Reject } H_0 \text{ if } \quad U_1 \le z_1(U_2, U_3) \quad \text{or} \quad U_1 > z_2(U_2, U_3)$$

for some functions z_1 and z_2. To do this, we write T in terms of U_1, U_2 and U_3. First note

$$\begin{aligned} U_2^2 + mnU_1^2 &= (m\overline{X} + n\overline{Y})^2 + mn(\overline{X} - \overline{Y})^2 \\ &= m^2\overline{X}^2 + 2mn\overline{X}\overline{Y} + n^2\overline{Y}^2 + mn\overline{X}^2 - 2mn\overline{X}\overline{Y} + mn\overline{Y}^2 \\ &= (m+n)(m\overline{X}^2 + n\overline{Y}^2). \end{aligned}$$

Then note

$$\begin{aligned} s^2 &= \frac{1}{m+n-2}\left[\sum (X_i - \overline{X})^2 + \sum (Y_j - \overline{Y})^2\right] \\ &= \frac{1}{m+n-2}\left[\sum X_i^2 + \sum Y_j^2 - m\overline{X}^2 - n\overline{Y}^2\right] \\ &= \frac{1}{m+n-2}\left[U_3 - \frac{1}{m+n}(U_2^2 + mnU_1^2)\right]. \end{aligned}$$

The t-statistic may be written in the form

$$T = \frac{\overline{X} - \overline{Y}}{s\sqrt{(1/m) + (1/n)}} = \frac{U_1\sqrt{m+n-2}}{\sqrt{\frac{m+n}{mn}U_3 - \frac{1}{mn}U_2^2 - U_1^2}}.$$

The t-test rejects H_0 when $|T| \ge t_{m+n-2;\alpha/2}$. Taking squares on both sides and multiplying by the denominator, we see this is equivalent to rejecting H_0 when

$$U_1^2(m+n-2) \ge t_{m+n-2;\alpha/2}^2\left(\frac{m+n}{mn}U_3 - \frac{1}{mn}U_2^2 - U_1^2\right),$$

or when

$$U_1^2(m+n-2+t_{m+n-2;\alpha/2}^2) \ge t_{m+n-2;\alpha/2}^2\left(\frac{m+n}{mn}U_3 - \frac{1}{mn}U_2^2\right).$$

Taking square roots of both sides, we see this is of the desired form with

$$z_2 = -z_1 = t_{m+n-2;\alpha/2}\sqrt{\left(\frac{m+n}{mn}U_3 - \frac{1}{mn}U_2^2\right)/(m+n-2+t_{m+n-2;\alpha/2}^2)}.$$